

Exercise 7: Direct Single Shooting and Direct Multiple Shooting

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Consider the following simple OCP for controlling a Van-der-Pol oscillator:

$$\begin{aligned} & \underset{x,u}{\text{minimize}} && \int_0^T x_1(t)^2 + x_2(t)^2 + u(t)^2 dt \\ & \text{subject to} && \dot{x}_1 = (1 - x_2^2)x_1 - x_2 + u, & x_1(0) = 0 \\ & && \dot{x}_2 = x_1, & x_2(0) = 1 \\ & && -1 \leq u(t) \leq 1, \end{aligned} \tag{1}$$

where $T = 10$.

Tasks:

- 7.1 Implement a CasADi Function $f : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2 \times \mathbb{R}$ that takes the states x and input u and returns the ODE right-hand-side \dot{x} and the Lagrange objective term L . Anticipating the upcoming simplified calling syntax in CasADi, define the following helper function:

```
1 function varargout = easycall(fcn, varargin) % save as easycall.m
2     varargout = fcn(varargin);
3 end
```

- 7.2 Divide the time horizon into $N = 20$ equidistant intervals, $[t_k, t_{k+1}]$, $k = 0, \dots, N - 1$ and assume a constant control u_k on each interval. Then take $M = 4$ steps with a RK4 scheme to define the discrete-time dynamics as a Function $F : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2 \times \mathbb{R}$.

F should take $x(t_k)$ and u_k and return $x(t_{k+1})$ and $J_k = \int_{t_k}^{t_{k+1}} L(x, u_k)$, the contribution to the objective from interval k . The key rows of the integrator could look like:

```
1 for j=1:M
2     [k1, k1_q] = easycall(f, X, U);
3     [k2, k2_q] = easycall(f, X + DT/2 * k1, U);
4     [k3, k3_q] = easycall(f, X + DT/2 * k2, U);
5     [k4, k4_q] = easycall(f, X + DT * k3, U);
6     X = X + DT/6*(k1 + 2*k2 + 2*k3 + k4);
7     Q = Q + DT/6*(k1_q + 2*k2_q + 2*k3_q + k4_q);
8 end
```

Evaluate the integrator with $x(t_k) = [0.2, 0.3]$ and $u_k = 0.4$.

7.3 Formulate the direct single shooting NLP and solve it with IPOPT. Construct the NLP variables step-by-step starting with empty list:

```
1 w = {};
2 lbw = [];
3 ubw = [];
```

The key rows of the NLP formulation should look something along the lines of:

```
1 for k=0:N-1
2     % New NLP variable for the control
3     Uk = MX.sym(['U_' num2str(k)]);
4     w = {w{:}, Uk};
5     lbw = ...
6     ubw = ...
7
8     % Integrate till the end of the interval
9     [Xk, Jk] = easycall(F, Xk, Uk);
10    J = J + Jk;
11 end
```

To plot the results, you can use

```
1 tgrid = linspace(0, T, N+1);
2 hold on
3 plot(tgrid, x1_opt, '--')
4 plot(tgrid, x2_opt, '-')
5 stairs(tgrid, [u_opt; nan], '-.')
6 xlabel('t')
7 legend('x1', 'x2', 'u')
8 grid on
```

7.4 Modify the script to so that it implements the direct multiple shooting method. The control parametrization and the definition of the integrator should remain the same. Introduce NLP variables corresponding to the state for all discrete time points, including $k = 0$.

7.5 Compare the IPOPT output for both scripts. How did the change from direct single shooting to direct multiple shooting influence:

- The number of iterations
- The number of nonzeros in the Jacobian of the constraints
- The number of nonzeros in the Hessian of the Lagrangian

7.6 **Extra:** Introduce the additional constraint $x_1(t) \geq -0.25$. You only need to enforce this path constraint at the end of each control interval. Modify your scripts to solve the modified problem with direct multiple shooting (easy) and direct single shooting (more tricky).