AWESCO Winter School on Numerical Optimal Control with DAE - University of Freiburg

Exercise 4: Optimal Control with Single Shooting

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In this exercise we will start working with Optimal Control Problems (OCP). In particular, we will solve the following continuous-time infinite dimensional problem:

$$\begin{aligned} \min_{x,u} \quad & \int_0^T x(t)^2 + u(t)^2 \, dt \\ \text{s.t.} \quad & \dot{x} = (1+x)\, x + u \\ & |u(t)| \leq 0.075 \\ & x(0) = \bar{x}_0 \\ & x(T) = 0, \end{aligned}$$

where $u \in \mathbb{R}$ is the control input and $x \in \mathbb{R}$ is the state of the system, $T = 3$ and $\bar{x}_0 = 0.05$. The above formulation can be discretized by integrating the dynamics of the system over a fixed grid with $N+1$ nodes leading to the finite-dimensional discrete-time problem

$$\begin{aligned} \min_{\substack{x_0,\cdots,x_N \\ u_0,\cdots,u_{N-1}}} \quad & h \sum_{i=0}^{N-1} (x_i^2 + u_i^2) + x_N^2 \\ \text{s.t.} \quad & x_{i+1} = f(x_i, u_i), \quad i = 0, \cdots, N-1 \\ & |u_i| \leq 0.075, \qquad\qquad i = 0, \cdots, N-1, \\ & x_0 = \bar{x}_0 \\ & x_N = 0, \end{aligned} \tag{1}$$

where f describes the discretized dynamics obtained using an integration scheme, $h := \frac{T}{N}$, x_i and u_i refer to the evaluation of state and control trajectories, respectively, where x_i correspond to the discretization nodes:

$$x_i = x(ih), \quad i = 0, \cdots, N. \quad i = 0, \cdots, N-1.$$

It is possible to eliminate the state variables from (2) by exploiting the equality constraints obtaining a more compact form:

$$\begin{aligned} \min_{u_0,\cdots,u_{N-1}} \quad & \Phi(u) \\ \text{s.t.} \quad & |u_i| \leq 0.075, \qquad\qquad i = 0, \cdots, N-1, \\ & x_N(u) = 0, \end{aligned} \tag{2}$$

where $\Phi(u) := h(\bar{x}_0^2 + u_0^2 + f(\bar{x}_0, u_0)^2 + u_1^2 + \cdots)$. This approach is the so called *single shooting* scheme.

4.1 Implement a CasADi `Function` f that takes as argument the states x and input u and returns the ODE right-hand-side $\dot{x}$.

4.2 Divide the time horizon into $N = 30$ equidistant control intervals, then use the RK4 scheme to define the discrete-time dynamics as a CasADi function. This function should take $x(t_i)$ and u_i as inputs and return $x(t_{i+1})$. The key lines of the integrator implementation could look like this:

```
out   = f({X,U});
k1    = out{1};
% ...
X = X + h/6*(k1   + 2*k2   + 2*k3   + k4);
```

4.3 Formulate the direct single shooting NLP and solve it with IPOPT. Note that the NLP should have N degrees of freedom, so start by defining a variable $u \in \mathbb{R}^N$:

```
u = SX.sym('u',N);
```

The key lines of the NLP formulation could look like this:

```
X = X0;
for i = 1:N
    out = F({X,v(i)});
    X = out{1};
    J = J + X(1)^2 + u(i)^2;
end
```

4.4 Introduce the additional path constraints $x_i \geq 0.05,\ i = 15, \cdots, 17$. Change your scripts to solve the modified problem.

4.5 **Extra:** Replace the dynamics in the NLP from Task 4.4 with their linearization at the origin $x_0 = 0$. Compute the optimal solution and apply it to the original system. Are the path constraints satisfied? Is there a neighborhood of the origin where this linearized optimal control problem will provide a feasible solution?