AWESCO Winter School on Numerical Optimal Control with DAE - University of Freiburg

Exercise 6: Interior Point Methods

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While in the previous exercise we implemented an SQP solver, in the following we will consider interior point (IP) methods, a second possible approach for the solutions of NLPs. The main idea behind such a scheme is to solve the system of equations describing the first order optimality conditions with a quasi-Newton method.

For this exercise, we will use a different formulation of the NLP, where equality and inequality constraints are kept separate:

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t.} \quad & g(x) = 0 \\ & h(x) \leq 0, \end{aligned} \tag{1}$$

This is necessary in order to simplify the notation when describing how inequalities are treated by IP methods. In particular, consider the following system of nonlinear equations:

$$\nabla_x f(x) + \nabla_x g(x)\lambda + \nabla_x h(x)\nu = 0 \tag{2a}$$

$$g(x) = 0 \tag{2b}$$

$$h(x) + s = 0 \tag{2c}$$

$$\nu_i s_i(x) = \tau, \quad i = 1, \cdots, p \tag{2d}$$

where λ and ν are the multipliers associated with the equality and inequality respectively, p is the number of inequalities and slack variables $s \geq 0$ have been introduced. For $\tau = 0$ the above system corresponds to the first order optimality conditions of (3). However, due to the nonsmoothness of equations (2d), the Newton method cannot be applied directly.

To circumvent this problem, it is then possible to relax the complementarity condition by setting τ to some positive value. So called primal-dual interior point methods iteratively solve this relaxed set of equations while shrinking τ. For τ that tends to zero, a point satisfying the initial optimality conditions is then recovered.

In the following, we will implement a simple interior point methods using CasADi to generate the quantities needed to compute the Newton steps to solve equations (2b)-(2d). To this end, consider the following optimization problem:

$$\begin{aligned} \min_{x} \quad & f(x) := (x_1 - 4)^2 + (x_2 - 4)^2 \\ \text{s.t.} \quad & g(x) := \sin(x_1) - x_2^2 = 0 \\ & h(x) := x_1^2 + x_2^2 - 4 \leq 0, \end{aligned} \tag{3}$$

Tasks:

6.1 Using the MATLAB template, implement three CasADi functions `F`, `G` and `H` that return evaluations of f, g and h respectively. For the test point $x_1 = 2$, $x_2 = 3$, check that you obtain the same values as the ones given in the template.

6.2 Implement CasADi functions for the Jacobians and Hessians of f, g and h. Check again the correctness of your code with the numerical values provided for the test point.

6.3 Derive the form of the linear system associated with every Newton iteration. *Hint:* in order to do so, you have to compute a linearization of equations (2b)-(2d) with respect to (x, λ, ν, s). The Newton step $q := (\Delta x, \Delta\mu, \Delta\lambda, \Delta s)$ at iteration k is the solution to the system:

$$M(x^k, \lambda^k, \nu^k, s^k)q = r(x^k, \lambda^k, \nu^k, s^k) \tag{4}$$

6.4 Implement the Newton algorithm in your code by completing the M and r in the template.

6.5 In order for the dual solution to be meaningful we have to enforce positivity of the multipliers associated with inequality constraints ν and slack variables s. In order to do so, it is possible to rely on a line-search scheme that, at every iteration computes a trial step for ν and s. If the resulting dual solution is infeasible, a backtracking line-search is used to scale the Newton step until feasibility is recovered. Fill in the template in order to implement the line-search.

6.6 Run your implementation of the primal-dual interior point starting from the initial guess $x^0 = [-2,\ 4]^T$, $\lambda = 10$, $\mu = 10$ and $s = 10$ and observe the plots. What is the value of the inequality multiplier ν? Is the inequality constraint active? Does the solution change if you set the initial guess for the primal solution to $x^0 = [-2,\ -4]^T$? Is the inequality constraint active in this case?