Exercises for Lecture Course on Numerical Optimization (NUMOPT)
Albert-Ludwigs-Universität Freiburg – Winter Term 2015-2016

Exercise 3: Duality Theory and Semidefinite Programming

(to be completed during exercise session on Nov 11, 2015 or sent by email to dimitris.kouzoupis@imtek.uni-freiburg.de before Nov 13, 2015)

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The goal of this exercise is on the one hand to train on the derivation of dual problems and on the other to explore the potential of Semidefinite programming by means of a practical example.

Exercise Tasks

1. **Lagrange duality and dual problems:**

   (a) Derive the dual of the following *logarithmic barrier* problem:

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & c^T x - \sum_{j=1}^{n} \log x_j \\ \text{s.t.} \quad & a^T x = b. \end{aligned}$$

   Remark: Problems using a logarithmic barrier as the one above will be at the core of interior point methods that we will analyze later in this course.

   (2 points)

   (b) Consider the following *mixed-integer quadratic program* (MIQP):

$$\begin{aligned} \min_{x\in\{0,1\}^n} \quad & x^T Q x + q^T x \\ \text{s.t.} \quad & Ax \geq b, \end{aligned}$$

   where the optimization variables x_i are restricted to take values in $\{0, 1\}$. Solving mixed-integer problems is in general a challenging task, thus it is common practice to exploit continuous reformulations as the following:

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & x^T Q x + q^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x_i(1 - x_i) = 0 \qquad i = 0, \cdots, n-1. \end{aligned}$$

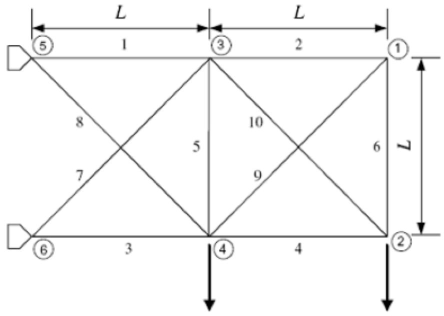
   i. Is this reformulation convex? Does strong duality hold?

   (1 point)

   ii. It is possible to compute a lower bound to the optimal solution by solving the (convex) dual problem (Lemma 4.2)? Derive the dual of the continuous reformulation.

   (2 points)

2. **Truss design:** Aim of this task is to design a truss topology with minimum compliance under the influence of external forces. Assume we have the following structure with 6 nodes and 10 bars



with external forces f_e. Under these forces, the nodes are displaced on the directions they are free to move until a certain equilibrium is reached. We denote the vector of displacements with u. Our goal is to find the optimal cross-sectional area x_i of each bar i that minimizes the compliance $f_e^\top u$ of the structure while respecting restrictions on the available material. The reaction forces that are caused by the external load depend linearly on u via the stiffness matrix $K(x)$, i.e., $f_r = -K(x)u$. On the other hand, at equilibrium, it should also hold $f_r = -f_e$. Taking also into account the constraints on the materials, we end up with the following optimization problem:

$$\underset{u,x}{\text{minimize}}\ f_e^\top u \tag{1a}$$

$$\text{subject to: } K(x)u = f_e \tag{1b}$$

$$0 \leq x_i \leq x_{\max} \tag{1c}$$

$$\sum_{i=1}^{m} l_i x_i \leq V_{\max} \tag{1d}$$

where m is the number of bars, here 10, $x_{\max}$ the maximum cross-sectional area of a bar and $V_{\max}$ the maximum allowed volume of the structure. The stiffness matrix depends linearly on the cross-sectional area of each bar via the relation $K(x) = \sum_{i=1}^{m} K_i x_i$.

At first glance, Problem (1) seems like a hard, highly nonlinear problem. However, after some manipulation, we can derive an equivalent convex problem in a form suitable for an SDP solver, namely:

$$\underset{x}{\text{minimize}}\ \alpha \tag{2a}$$

$$\text{subject to: } \begin{bmatrix} \alpha & f_e^\top \\ f_e & K(x) \end{bmatrix} \succeq 0 \tag{2b}$$

$$\text{Constraints (1c) and (1d)} \tag{2c}$$

(a) Show how Problem (1) can be transformed to the equivalent Problem (2). You will need to use transformations similar to the previous exercise as well as the Schur complement lemma. Keep in mind that matrix $K(x)$ is always invertible for stable structures.

(2 points)

(b) Solve Problem (2) with YALMIP. Use the provided functions to load the data, calculate the stiffness matrix (and its components) and plot the results. Use $x_{\max} = 200$ and $V_{\max} = 10^5$. Make sure you have an SDP solver installed, i.e., `SDPT3`. Interpret the results.

(3 points)

(c) Move the load to node 1, plot the new structure. You can also try to constrain some x_i to 0.

(1 point)

(d) **Extra:** Modify the code to allow for an arbitrary number of nodes within the same boundaries. Enforce equal cross-sections within each structure block to yield a pixel-like result for high number of nodes.

(4 bonus points)