Performance bounds and robust control for constrained systems

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Introduction

The optimal control problem:

$$V_\infty^*(x) := \min_\pi \mathbb{E}\left[\sum_{i=0}^{\infty} \alpha^i \left(x_i^\top Q x_i + u_i^\top R u_i\right)\right]$$

subject to:

$$\left.\begin{aligned} x_{i+1} &= Ax_i + Bu_i + Gw_i \\ (x_i, u_i) &\in Z, \;\; x_0 = x \end{aligned}\right\} \mathbb{P}\text{-a.s.}$$

Some assumptions:

- Disturbances: $w_i \in W$, i.i.d. and W compact.
- Constraints: Z polyhedral
- Control policy: $\pi := \{\mu_0(\cdot), \mu_1(\cdot), ...\}$, arbitrary functions
- Measurements: states are directly measurable

Introduction

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subject to:

$$\left.\begin{aligned} x_{i+1} &= Ax_i + Bu_i + Gw_i \\ (x_i, u_i) &\in Z,\ \ x_0 = x \end{aligned}\right\} \mathbb{P}\text{-a.s.}$$

Some goals for this system:

- Find a computable suboptimal policy.
- Find a performance bound on the best possible controller.

Overview talk

Finite horizon problems:

- Upper/sub-optimal approximation with linear policies
- Lower/super-optimal approximation with linear multipliers

Infinite horizon problems:

- Control using upper bound terminal cost
- Performance bounds using lower bound terminal cost

Goal: a computable bound in the form

$$V^{\ell}(x) \leq V_{\infty}^{*}(x) \leq V^{u}(x)$$

Finite–horizon problems

A finite horizon version:

$$V_N^*(x) := \min_\pi \mathbb{E}\left[\sum_{i=0}^{N-1} \alpha^i \left(x_i^\top Q x_i + u_i^\top R u_i\right) + x_N^\top P x_N\right]$$

subject to:

$$\left.\begin{aligned} x_{i+1} &= Ax_i + Bu_i + Gw_i \\ (x_i, u_i) &\in Z,\ x_0 = x \\ x_N &\in \mathcal{X}_f \end{aligned}\right\} \mathbb{P}\text{-a.s.}$$

Assumptions:

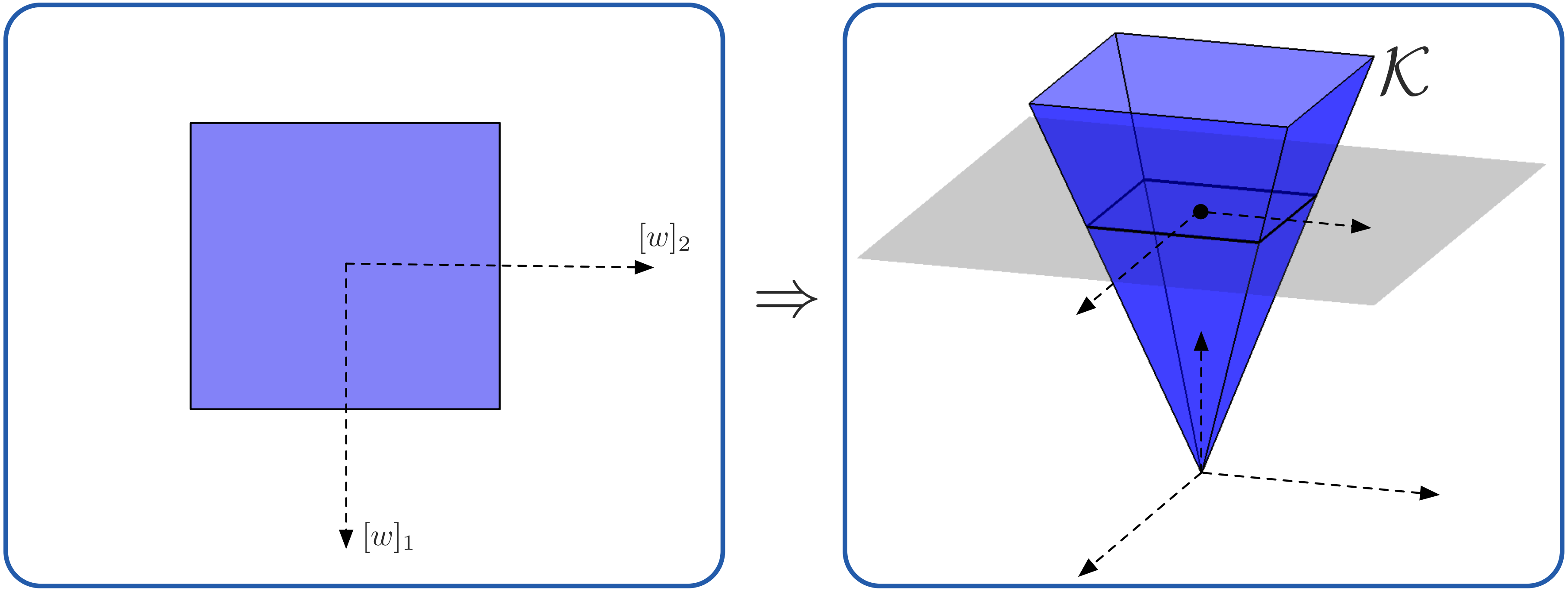
- All stage costs positive (semi)definite
- All constraints polyhedral

Finite-horizon problems

Uncertainty set:

$$\mathbf{w} := (1; w_0; \cdots; w_{N-1}), \quad \mathbb{E}[\mathbf{w}\mathbf{w}^\top] =: M$$

$$\mathbf{w} \in \mathcal{W} := \{\mathbf{w} \mid \mathbf{w} \succeq_{\mathcal{K}} 0,\ e_0^\top \mathbf{w} = 1\}$$



Finite–horizon problems

A finite horizon version:

$$
\begin{aligned}
\inf \quad & \mathbb{E}\left[\mathbf{u}^\top \mathcal{R}\mathbf{u} + \mathbf{x}^\top \mathcal{Q}\mathbf{x}\right] \\
\text{s.t.} \quad & \left.\begin{aligned}
& \mathbf{u} \in \mathcal{N},\ \mathbf{s} \in \mathcal{L}^2 \\
& \mathbf{x} = \mathcal{B}\mathbf{u} + \mathcal{C}\mathbf{w} \\
& F_u\mathbf{u} + F_x\mathbf{x} + \mathbf{s} = h \\
& \mathbf{s} \geq 0
\end{aligned}\right\} \quad \mathbb{P}\text{-a.s.}
\end{aligned}
$$

Decision variables:

- $\mathbf{u} \in \mathcal{N}$: *causal* mapping from measurements to inputs
- $\mathbf{s} \in \mathcal{L}^2$: Constraint slacks are a measurable random vector

Constraints must hold robustly for all $\mathbf{w} \in \mathcal{W}$

Finite–horizon problem (upper bound)

A suboptimal version:

$$
\begin{aligned}
\inf \quad & \mathbb{E}\left[\mathbf{u}^\top \mathcal{R}\mathbf{u} + \mathbf{x}^\top \mathcal{Q}\mathbf{x}\right] \\
\text{s.t.} \quad & \left.\begin{aligned}
& Q \in \mathcal{U},\ \mathbf{s} \in \mathcal{L}^2 \\
& \mathbf{x} = \mathcal{B}\mathbf{u} + \mathcal{C}\mathbf{w},\ \ \mathbf{u} = Q\mathcal{G}\mathbf{w} \\
& F_u\mathbf{u} + F_x\mathbf{x} + \mathbf{s} = h \\
& \mathbf{s} \geq 0
\end{aligned}\right\} \quad \mathbb{P}\text{-a.s.}
\end{aligned}
$$

Restrict the control policy to linear decision rules:

$$
u = \underbrace{\left(\begin{array}{c|ccc}
q_0 & Q_{0,0} & & \\
\vdots & \vdots & \ddots & \\
q_{N-1} & Q_{N-1,0} & \cdots & Q_{N-1,N-1}
\end{array}\right)}_{\in \mathcal{U}}
\begin{pmatrix} 1 \\ \eta_0 \\ \vdots \\ \eta_{N-1} \end{pmatrix} = Q\mathcal{G}\mathbf{w}
$$

Finite–horizon problem (upper bound)

An equivalent form:

$$V_N^u(x) := \inf \ \mathbb{E}\left[\mathbf{u}^\top \mathcal{R}\mathbf{u} + \mathbf{x}^\top \mathcal{Q}\mathbf{x}\right]$$
$$\text{s.t.} \quad Q \in \mathcal{U}, \quad S \succeq_{\mathcal{K}^*} 0 \text{ (column-wise)}$$
$$(F_x\mathcal{B} + F_u)Q\mathcal{G} + F_x\mathcal{C} + S^\top = h e_0^\top$$

Result is a conic program:

- QP if all sets polyhedral. Size scales with N^2
- Provides upper bound on value and implementable policy.

$$V_N^*(x) \le V_N^u(x)$$

Upper bound makes primal variables linear in uncertainty

Finite-horizon problem again

The original finite horizon version:

$$
\begin{aligned}
V_N^*(x) := \quad & \inf \quad \mathbb{E}\left[\mathbf{u}^\top \mathcal{R}\mathbf{u} + \mathbf{x}^\top \mathcal{Q}\mathbf{x}\right] \\
& \text{s.t.} \quad \left.\begin{aligned}
& \mathbf{u} \in \mathcal{N},\, \mathbf{s} \in \mathcal{L}^2 \\
& \mathbf{x} = \mathcal{B}\mathbf{u} + \mathcal{C}\mathbf{w} \\
& F_u\mathbf{u} + F_x\mathbf{x} + \mathbf{s} = h \\
& \mathbf{s} \geq 0
\end{aligned}\right\} \quad \mathbb{P}\text{-a.s.}
\end{aligned}
$$

Want to bound the degree of suboptimality caused by LDRs.

- Compute a *lower* bound on the value function:

$$V^\ell(x) \leq V_\infty^*(x) \leq V^u(x)$$

Finite–horizon problem (partial Lagrangian)

A partially dualized version:

$$\inf \sup_{\boldsymbol{\nu}\in\mathcal{L}^2} \quad \mathbb{E}\left[(\mathbf{u}^\top \mathcal{R}\mathbf{u} + \mathbf{x}^\top \mathcal{Q}\mathbf{x}) + \boldsymbol{\nu}^\top (F_u \mathbf{u} + F_x \mathbf{x} + \mathbf{s} - h)\right]$$

$$\text{s.t.} \quad \left.\begin{array}{l} \mathbf{u} \in \mathcal{N},\ \mathbf{s} \in \mathcal{L}^2 \\ \mathbf{x} = \mathcal{B}\mathbf{u} + \mathcal{C}\mathbf{w} \\ \mathbf{s} \geq 0 \end{array}\right\} \quad \mathbb{P}\text{-a.s.}$$

A lower bound makes multipliers linear in uncertainty

Finite–horizon problem (partial Lagrangian)

A partially dualized version:

$$
\begin{aligned}
\inf \ \sup_{\boldsymbol{\nu}\in\mathcal{L}^2} \quad & \mathbb{E}\left[(\mathbf{u}^\top \mathcal{R}\mathbf{u} + \mathbf{x}^\top \mathcal{Q}\mathbf{x}) + \boldsymbol{\nu}^\top (F_u\mathbf{u} + F_x\mathbf{x} + \mathbf{s} - h)\right] \\
\text{s.t.} \quad & \left.\begin{aligned} & \mathbf{u}\in\mathcal{N},\ \mathbf{s}\in\mathcal{L}^2 \\ & \mathbf{x} = \mathcal{B}\mathbf{u} + \mathcal{C}\mathbf{w} \\ & \mathbf{s}\geq 0 \end{aligned}\right\} \quad \mathbb{P}\text{-a.s.}
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
V_N^\ell(x) := \quad \inf \sup_{Y} \ & \mathbb{E}\left[(\mathbf{u}^\top \mathcal{R}\mathbf{u} + \mathbf{x}^\top \mathcal{Q}\mathbf{x}) + \boldsymbol{\nu}^\top (F_u\mathbf{u} + F_x\mathbf{x} + \mathbf{s} - h)\right] \\
\text{s.t.} \quad & \left.\begin{aligned} & \mathbf{u}\in\mathcal{N},\ \mathbf{s}\in\mathcal{L}^2 \\ & \mathbf{x} = \mathcal{B}\mathbf{u} + \mathcal{C}\mathbf{w} \\ & \mathbf{s}\geq 0,\ \ \boldsymbol{\nu} = Y\mathbf{w} \end{aligned}\right\} \quad \mathbb{P}\text{-a.s.}
\end{aligned}
$$

Finite–horizon problem (lower bound)

First deal with the inner maximization:

$$\begin{aligned}
&\sup_{Y} \mathbb{E}\left[\mathbf{w}^\top Y^\top \left[F_u\mathbf{u} + F_x\mathbf{x} + F_s\mathbf{s} - h\right]\right] \\
= \quad &\sup_{Y} \mathbb{E}\left[\mathrm{tr}\left\{\left[F_u\mathbf{u} + F_x\mathbf{x} + F_s\mathbf{s} - h\right]\mathbf{w}^\top Y^\top\right\}\right] \\
= \quad &\begin{cases} 0 & \text{if } \mathbb{E}\left[\left(F_u\mathbf{u} + F_x\mathbf{x} + F_s\mathbf{s} - h\right)\mathbf{w}^\top\right] = 0 \\ \infty & \text{otherwise.} \end{cases}
\end{aligned}$$

Restricting multipliers to linear form results in an exp. constraint.

- Eliminate states -> equivalent to:

$$(F_u + F_x\mathcal{B})\mathbb{E}[\mathbf{u}\mathbf{w}^\top] + \mathbb{E}[\mathbf{s}\mathbf{w}^\top] + (F_x\mathcal{C} - he_0^\top)\mathbb{E}[\mathbf{w}\mathbf{w}^\top] = 0$$

Finite-horizon problem (lower bound)

Then deal with expectations:

$$(F_u + F_x\mathcal{B})\mathbb{E}[\mathbf{u}\mathbf{w}^\top] + \mathbb{E}[\mathbf{s}\mathbf{w}^\top] + (F_x\mathcal{C} - he_0^\top)\mathbb{E}[\mathbf{w}\mathbf{w}^\top] = 0, \;\; \mathbf{s} \geq 0$$

① ② ③ ②

Some technical steps:

① For all $\mathbf{u} \in \mathcal{N}$ there exists $Q \in \mathcal{U}$ s.t. $\mathbb{E}[\mathbf{u}\mathbf{w}^\top] = QGM$

② For all $0 \leq \mathbf{s} \in \mathcal{L}^2$, there exists S s.t. $\mathbb{E}[\mathbf{s}\mathbf{w}^\top] = S^\top M$ and $MS \succeq_{\mathcal{K}} 0$

③ $\mathbb{E}[\mathbf{w}\mathbf{w}^\top] = M$, and right-multiply everything by M^{-1}

Finite-horizon problem (lower bound)

Then deal with expectations:

$$(F_u + F_x\mathcal{B})QG\cancel{M}_{①} + S^\top \cancel{M}_{②} + (F_x\mathcal{C} - he_0^\top)\cancel{M}_{③} = 0$$

Some technical steps:

① For all $\mathbf{u} \in \mathcal{N}$ there exists $Q \in \mathcal{U}$ s.t. $\mathbb{E}[\mathbf{u}\mathbf{w}^\top] = QGM$

② For all $0 \le \mathbf{s} \in \mathcal{L}^2$, there exists S s.t. $\mathbb{E}[\mathbf{s}\mathbf{w}^\top] = S^\top M$ and $MS \succeq_{\mathcal{K}} 0$

③ $\mathbb{E}[\mathbf{w}\mathbf{w}^\top] = M$, and right-multiply everything by M^{-1}

4) Explicit minimization over $\mathbf{u} \in \mathcal{N}$ results in $\mathbf{u}^* = QG\mathbf{w}$.

Finite-horizon problem (lower bound)

An equivalent form:

$$V_N^\ell(x) := \inf \; \mathbb{E}\left[\mathbf{u}^\top \mathcal{R}\mathbf{u} + \mathbf{x}^\top \mathcal{Q}\mathbf{x}\right]$$
$$\text{s.t.} \;\; Q \in \mathcal{U}, \quad MS \succeq_{\mathcal{K}} 0 \text{ (column-wise)}$$
$$(F_x\mathcal{B} + F_u)Q\mathcal{G} + F_x\mathcal{C} + S^\top = he_0^\top$$

Result is a conic program:

- QP if all sets polyhedral. Size scales with N^2
- Provides lower bound on value:

$$V_N^\ell(x) \le V_N^*(x)$$

Finite–horizon bounds

Upper bounding problem:

$$\begin{aligned} V_N^u(x) := \quad & \inf \ \mathbb{E}\left[\mathbf{u}^\top \mathcal{R}\mathbf{u} + \mathbf{x}^\top \mathcal{Q}\mathbf{x}\right] \\ & \text{s.t.} \quad Q \in \mathcal{U}, \quad S \succeq_{\mathcal{K}*} 0 \text{ (column-wise)} \\ & \qquad (F_x\mathcal{B} + F_u)Q\mathcal{G} + F_x\mathcal{C} + S^\top = he_0^\top \end{aligned}$$

Lower bounding problem:

$$\begin{aligned} V_N^\ell(x) := \quad & \inf \ \mathbb{E}\left[\mathbf{u}^\top \mathcal{R}\mathbf{u} + \mathbf{x}^\top \mathcal{Q}\mathbf{x}\right] \\ & \text{s.t.} \quad Q \in \mathcal{U}, \quad MS \succeq_{\mathcal{K}} 0 \text{ (column-wise)} \\ & \qquad (F_x\mathcal{B} + F_u)Q\mathcal{G} + F_x\mathcal{C} + S^\top = he_0^\top \end{aligned}$$

Infinite-horizon problems

An infinite horizon version:

$$V_\infty^*(x) := \min_\pi \mathbb{E}\left[\sum_{i=0}^{\infty} \alpha^i \left(x_i^\top Q x_i + u_i^\top R u_i\right)\right]$$

subject to:

$$\left.\begin{aligned} x_{i+1} &= Ax_i + Bu_i + Gw_i \\ (x_i, u_i) \in Z,\ x_0 &= x \end{aligned}\right\} \mathbb{P}\text{-a.s.}$$

Infinite-horizon problems

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A finite horizon version:

$$V_\infty^*(x) := \min_\pi \mathbb{E}\left[\sum_{i=0}^{N-1} \alpha^i \left(x_i^T Q x_i + u_i^T R u_i\right) + \alpha^N V_\infty^*(x_N)\right]$$

subject to:

$$\left.\begin{aligned} x_{i+1} &= Ax_i + Bu_i + Gw_i \\ (x_i, u_i) &\in Z, \;\; x_0 = x \end{aligned}\right\} \mathbb{P}\text{-a.s.}$$

Infinite-horizon problems

Upper bounds: $V_0^u \geq V_\infty^* \implies V_N^u \geq V_\infty^*$

$$V_N^u(x) := \min_{\pi_N} \mathbb{E}\left[\sum_{i=0}^{N-1} \alpha^i \left(x_i^T Q x_i + u_i^T R u_i\right) + \alpha^N V_0^u(x_N)\right]$$

subject to:

$$\left.\begin{aligned} x_{i+1} &= Ax_i + Bu_i + Gw_i \\ (x_i, u_i) &\in Z,\ x_0 = x \end{aligned}\right\} \mathbb{P}\text{-a.s.}$$

Infinite-horizon problems

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subject to:

$$\left.\begin{aligned} x_{i+1} &= Ax_i + Bu_i + Gw_i \\ (x_i, u_i) &\in Z, \;\; x_0 = x \end{aligned}\right\} \mathbb{P}\text{-a.s.}$$

Lower bounds: $V_0^l \leq V_\infty^* \implies V_N^l \leq V_\infty^*$

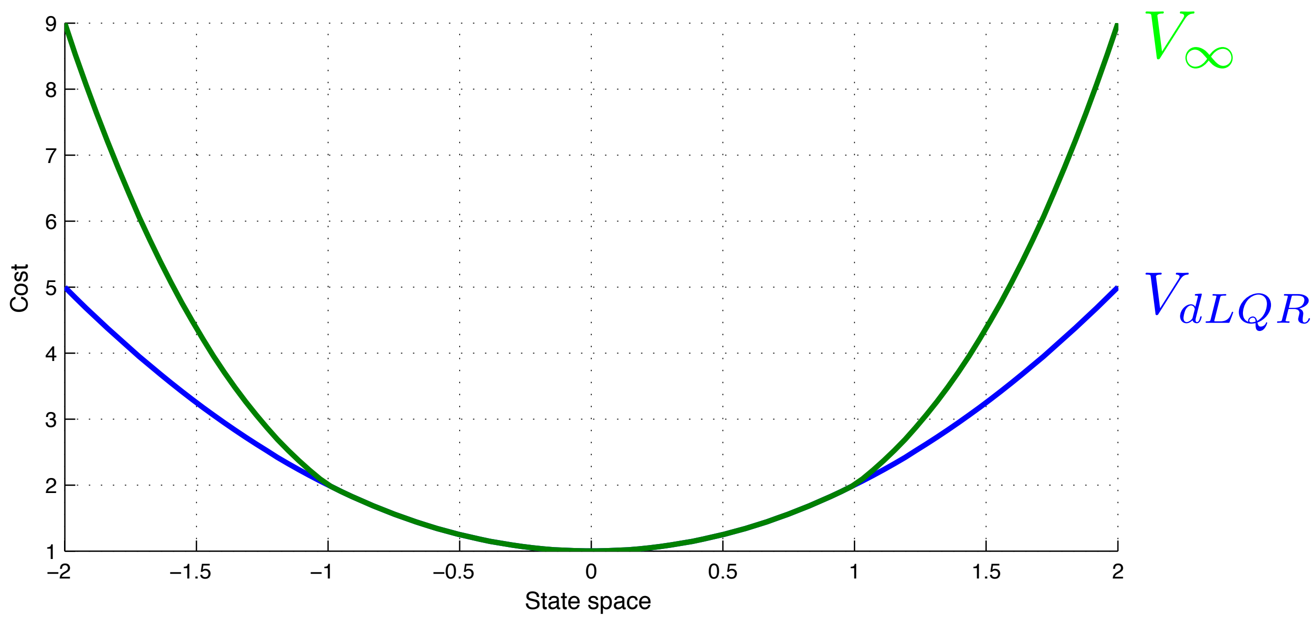
$$V_N^l(x) := \min_{\pi_N} \mathbb{E}\left[\sum_{i=0}^{N-1} \alpha^i \left(x_i^T Q x_i + u_i^T R u_i\right) + \alpha^N V_0^l(x_N)\right]$$

subject to:

$$\left.\begin{aligned} x_{i+1} &= Ax_i + Bu_i + Gw_i \\ (x_i, u_i) &\in Z, \;\; x_0 = x \end{aligned}\right\} \mathbb{P}\text{-a.s.}$$

Infinite-horizon problems

An initial lower bound:



Unconstrained case (discounted LQR): $V_{dLQR} = x^T P x + q$, where

$$P = Q + \bar{A}^T P \bar{A} - \bar{A}^T P \bar{B} \left(R + \bar{B}^T P \bar{B}\right)^{-1} \bar{B}^T P \bar{A}$$

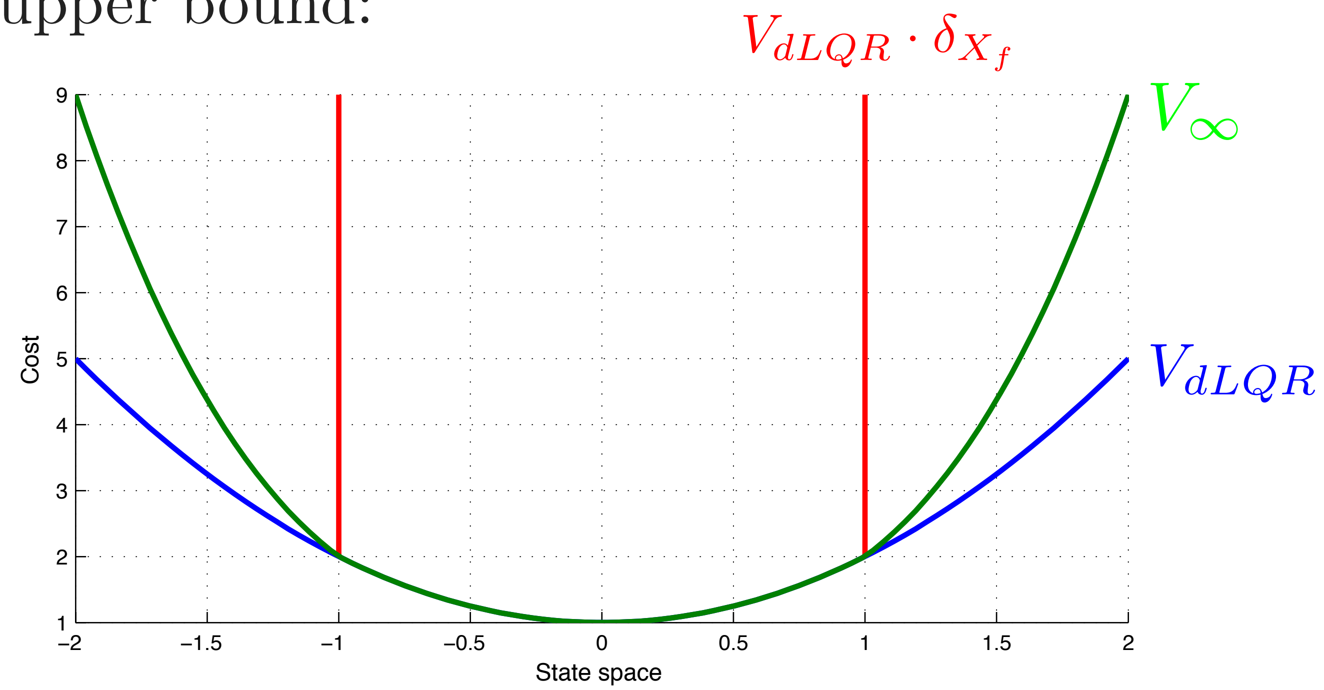
$$q = \frac{\operatorname{tr}\left(\bar{C} P \bar{C} M\right)}{1 - \alpha}$$

$$K = -(R + \bar{B}^T P \bar{B})^{-1} \bar{B}^T P \bar{A}$$

and $\bar{A} := \sqrt{\alpha} A$, $\bar{B} := \sqrt{\alpha} B$ and $C := \sqrt{\alpha} C$.

Infinite-horizon problems

An initial upper bound:



X_f robust positive invariant (RPI) for K

- $(A + BK)x + Cw \in X_f$ for all $x \in X_f$ and $w \in W$
- $(x, Kx) \in Z$ for all $x \in X_f$

Example (infinite-horizon)

- Two-state system:

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.2 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w$$

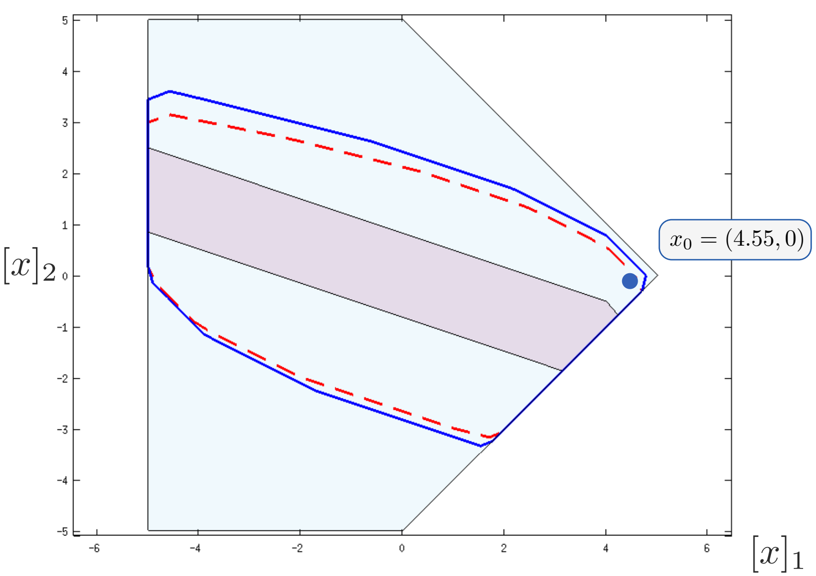
- Cost function:

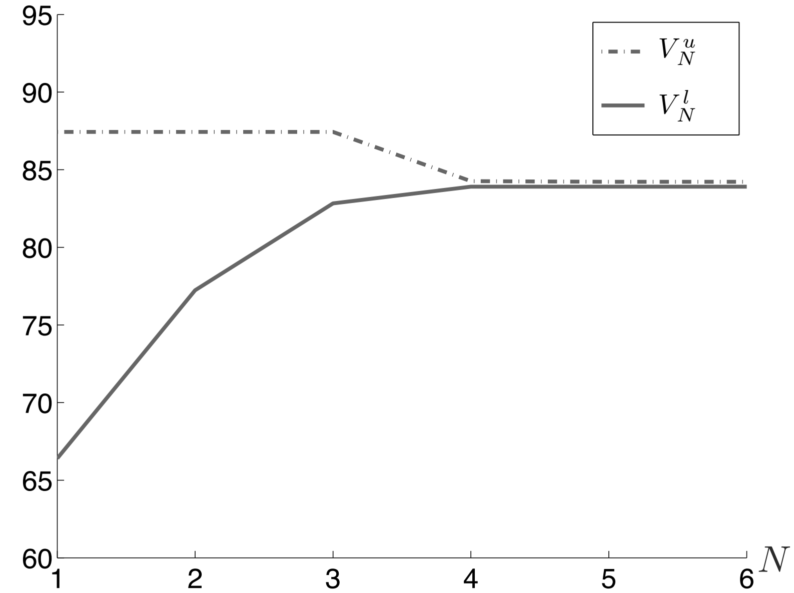
$Q = I,\ R = 1,\ P = \text{dLQR solution},\ \alpha = 0.95$

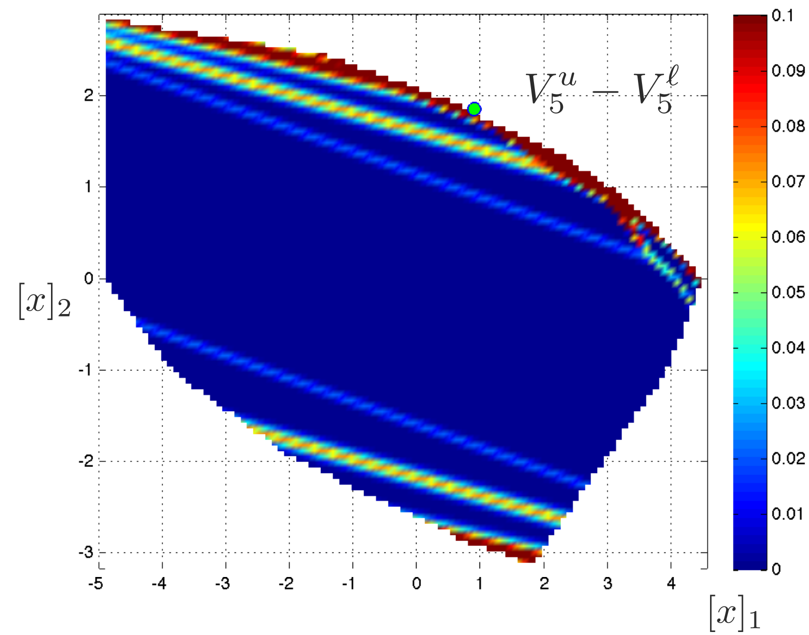
- Disturbance/input constraints:

$|u| \leq 1,\ w$ uniform on $[0, 2]$

- State constraints:







Thanks

References:

- P. Goulart, E. Kerrigan, and J. Maciejowski, "Optimization over state feedback policies for robust control with constraints," *Automatica*, April, 2006.
- P. Goulart, E. Kerrigan, T. Alamo, "Control of constrained discrete-time systems with bounded ℓ_2 gain", *IEEE-TAC*, May, 2009.
- M. Hadjiyiannis, P. Goulart, and D. Kuhn. "An efficient method to estimate the suboptimality of affine controllers," *IEEE-TAC*, Dec, 2011.
- B. Van Parys, P. Goulart, M. Morari, "Infinite-horizon performance bounds for uncertain constrained systems", *submitted to IEEE-CDC*, 2012.