Robust Optimization for Dynamic Systems

Boris Houska

Outline of the Talk

- Uncertain Nonlinear Dynamic Systems
- Computation of Robust Positive Invariant Tubes
- Robust Optimization of Dynamic Systems
- Application: Robust Control of a Tubular Reactor
- Robust Optimization of Periodic Systems
- Open-Loop Stable Orbits of an Inverted Spring Pendulum
- Conclusions and Outlook

Uncertain Nonlinear Dynamic System

Notation:

- General uncertain system:

$$\forall \tau \in [t_1, t_2] : \quad \dot{x}(\tau) \quad = \quad f(\,\tau, x(\tau), w(\tau)\,)\ , \quad x(t_1) \ = \ x_1\ .$$

- Knowlegde about the uncertainty: $(x_1, w) \in \mathcal{W}$.

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Definition of a Solution-Tube:

- The solution $X : [t_1, t_2] \to \Pi(\mathbb{R}^{n_x})$ is defined as:

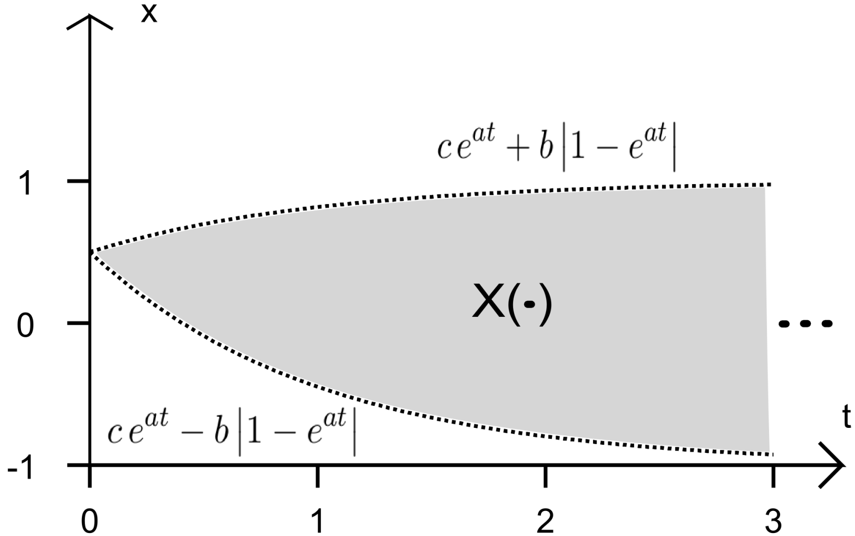
$$X(t) \;:=\; \left\{ x(t) \in \mathbb{R}^{n_x} \;\middle|\; \begin{array}{l} \exists x(\cdot), w(\cdot) : \\ \dot{x}(\tau) \;=\; f(\, \tau, \, x(\tau), \, w(\tau)\,) \\ (x(t_1), w) \in \mathcal{W} \quad \forall \tau \in [t_1, t] \end{array} \right\} \;.$$

Example 1

Scalar Linear System:

- $\dot{x} = ax + bw$ with $a, b, c \in \mathbb{R}$.
- $\mathcal{W} := \{ (x_1, w) \mid \forall t \in \mathbb{R}: -1 \leq w(t) \leq 1 , x_1 = c \}$.

Visualization:



- We use $a = -1$, $b = 1$, and $c = \frac{1}{2}$.

Example 2

Linear System with L_2-bounded Uncertainty:

- $f(t,x,w) = Ax + Bw$
- $\mathcal{W} \;:=\; \left\{ (x_1, w) \;\middle|\; x_1^2 + \int_{-\infty}^{\infty} \|w(\tau)\|_2^2 \, \mathrm{d}\tau \;\leq\; 1 \right\}$.

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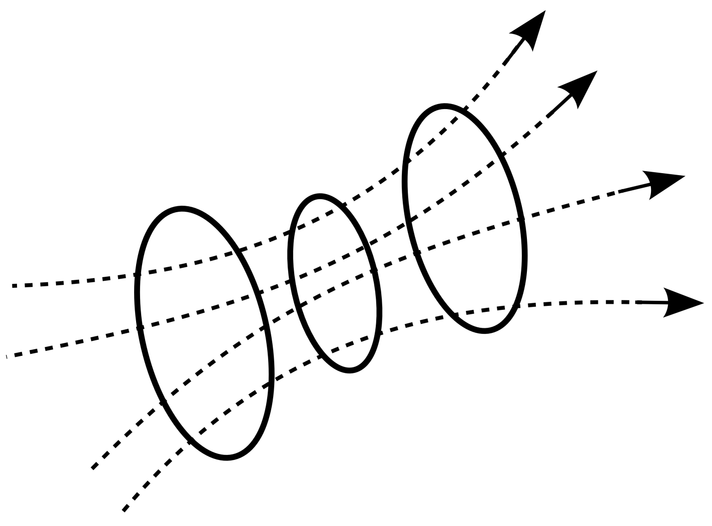

Solution:

- $X(t) \;=\; \mathcal{E}(\,Q(t)\,)$ with

$$\dot{Q} \;=\; AQ + QA^T + BB^T$$

and $Q(t_1) = I$.

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Definition:

- $$T(t_2, t_1)[X_1] \;:=\; \left\{ y \;\middle|\; \begin{array}{l} \exists (x, w) : \\ \dot{x}(\tau) = f(\tau, x(\tau), w(\tau)) \\ w(\tau) \in W(\tau) \;\; \forall \tau \in [t_1, t_2] \\ x(t_1) \in X_1\,,\;\; x(t_2) \;=\; y \end{array} \right\}\;.$$

Associativity of the Set Propagation Operator

Associativity:

- $(\,T(t_4,t_3)\circ T(t_3,t_2)\,)\circ T(t_2,t_1) \;=\; T(t_4,t_3)\circ(\,T(t_3,t_2)\circ T(t_2,t_1)\,)\;.$
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Infinitesimal Set Generation (Euler's method):

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Formal Definition of a Set Valued ODE:

- $\forall\tau\in[t_1,t_2]:\quad X(\tau^+) \;=\; F(\tau,X(\tau),W(\tau))$

$$\Longleftrightarrow$$

$$\forall\tau\in[t_1,t_2]:\quad X(\tau) \;:=\; T(\tau,t_1)[X(t_1)].$$

Aim of the Talk

- Learn how to formulate and solve optimal control problems of the following form

$$\min_{u(\cdot),p,T_\mathrm{e},X(\cdot)} \quad \int_0^{T_\mathrm{e}} \mathcal{L}(\tau, u(\tau), X(\tau), W(\tau))\,\mathrm{d}\tau \;+\; M(\,p, T_\mathrm{e}, X(T_\mathrm{e})\,)$$

$$\text{s.t.} \quad \left\{ \begin{array}{rcl} X(\tau^+) & = & F(\tau, u(\tau), p, X(\tau), W(\tau)) \\ X(0) & = & X_0 \\ 0 & \geq & H(\tau, u(\tau), p, X(\tau), W(\tau)) \end{array} \right.$$

(for all $\tau \in [0, T_\mathrm{e}]$).

- Generalizations: Other boundary conditions (e.g. periodicity), etc.

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Numerical Problem:

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- $X \subseteq Y \;\Rightarrow\; T(t_2, t_1)[X] \subseteq T(t_2, t_1)[Y]$.

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Definition:

- A function $\mathbb{X} : [t_1, t_2] \to \Pi(\mathbb{R}^{n_x})$ is called a robust positive invariant tube if

$$\forall \tau \in [t_1, t_2] : \quad \mathbb{X}(\tau^+) \supseteq F(\tau, \mathbb{X}(\tau), W(\tau)) \; .$$

Computational Methods for Linear Systems

Aim:

- Provide efficient methods for the computation of robust positive invariant tubes.
- Canditate sets: ellipsoids, polytopes, zonotopes, ...

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Strategy:

- Start with analysis of linear system of the form

$$\forall \tau \in \mathbb{R} : \qquad \dot{x}(\tau) = A(\tau)x(\tau) + B(\tau)w(\tau)$$

and generalize for nonlinear systems later.

Conservation Laws of Linear Uncertainty Propagation

Proposition:

- Let A and B be L_1-integrable, $X(t) := T(t,t_1)[X_1]$.
- If X_1 and $W(\tau)$ ($\tau \in [t_1,t]$) compact $\Rightarrow$ $X(t)$ compact.
- If X_1 and $W(\tau)$ ($\tau \in [t_1,t]$) convex $\Rightarrow$ $X(t)$ convex.
- If X_1 and $W(\tau)$ ($\tau \in [t_1,t]$) *-sym. $\Rightarrow$ $X(t)$ *-sym.

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Idea: Employ Techniques from Convex Analysis

- Given a compact and convex set $\mathcal{F} \subseteq \mathbb{R}^{n_x}$.
- Search lifted outer approximation $\mathcal{F} = \bigcap_{\lambda \in \mathbb{D}^+} \mathcal{F}^+(\lambda)$.

Example: Lifted Approximation of a Polytope

Problem:

- Given polytope $\mathcal{F} := \{ x \in \mathbb{R}^n \mid Ax \leq b \}$, $A \in \mathbb{R}^{m \times n}$.
- Construct a polytope with $l < m$ facets which approximates $\mathcal{F}$ from outside.

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Solution:

- Take any $(m \times l)$-matrix $\Lambda \in \mathbb{D}^+ := \{\, \Lambda \mid \Lambda_{i,j} \geq 0 \,\}$.
- Choose $\mathcal{F}^+(\Lambda) := \{\, x \mid C(\Lambda)x \leq d(\Lambda) \,\}$ with

$$C(\Lambda) := \Lambda^T A \quad \text{and} \quad d(\Lambda) := \Lambda^T b$$

- Then we have $\mathcal{F} = \bigcap_{\lambda \in \mathbb{D}^+} \mathcal{F}^+(\lambda)$.

Construction of Lifted Outer Approximations

General Technique:

- Support function of convex and compact set $\mathcal{F}$:

$$V(c) \quad := \quad \max_x \; c^T x \quad \text{s.t.} \quad x \in \mathcal{F} \; .$$

- Represent V by dual problem $V(c) \; = \; \inf_{\lambda \in \mathbb{D}^+} \; D(c, \lambda)$.

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- Define a parameterized outer approximation as

$$\forall \lambda \in \mathbb{D}^+ : \;\; \mathcal{F}^+(\lambda) \; := \; \bigcap_{c \in \mathbb{R}^n} \left\{ x \in \mathbb{R}^n \mid c^T x \; \leq \; D(c, \lambda) \right\} \, .$$

- No duality gap $\;\; \Longrightarrow \;\; \mathcal{F} \; = \; \bigcap_{\lambda \in \mathbb{D}^+} \mathcal{F}^+(\lambda) \, .$

Example: Sums of Ellipsoids

Problem:

- Find $Q \in \mathbb{S}^n_{++}$ with $\sum_{i=1}^{N} \mathcal{E}(Q_i) \subseteq \mathcal{E}(Q)$, $Q_i \in \mathbb{S}^n_{+}$ given.

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Motivation:

- Assume $x \in \mathcal{E}(Q_x)$ and $w \in \mathcal{E}(Q_w)$.
- Then $x^+ = G_x x + G_w w \in \mathcal{E}\left(G_x Q_x G_x^T\right) + \mathcal{E}\left(G_w Q_w G_w^T\right)$.

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Solution (Step 1):

$$\begin{aligned} V(c) &= \max_{x_1,\ldots,x_N} \; c^T\left(\sum_{i=1}^N x_i\right) \text{ s.t. } x_i^T Q_i^{-1} x_i \le 1 \\ &= \inf_{\lambda \ge 0} \; \sum_{i=1}^N \frac{c^T Q_i c}{4\lambda_i} + \sum_{i=1}^N \lambda_i \; . \end{aligned}$$

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Solution (Step 2):

$$V(c) = \inf_{\lambda \geq 0} \sum_{i=1}^N \frac{c^T Q_i c}{4\lambda_i} + \sum_{i=1}^N \lambda_i$$

$$\mathbb{D}^+ = \left\{ \lambda \in \mathbb{R}^N \;\middle|\; \sum_{i=1}^N \lambda_i = 1 \right\}$$

$$\forall \lambda \in \mathbb{D}^+ : \quad \mathcal{F}^+(\lambda) = \mathcal{E}\left(\sum_{i=1}^N \frac{1}{\lambda_i} Q_i \right) \supseteq \sum_{i=1}^N \mathcal{E}(Q_i) \; .$$

Back to Continous Time Systems...

Uncertain Linear System:

- $\dot{x}(t) \; = \; A(t)x(t) + B(t)w(t)$ with $x(0) = 0$
- Uncertainty assumption $\|w(t)\|_\infty \leq 1$ for all $t \in [0, T]$.

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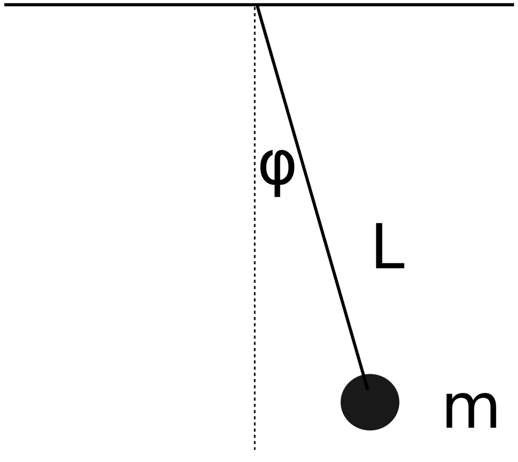

Linearized Pendulum:

$$\begin{aligned} \dot{\varphi}(t) &= \omega(t) \\ \dot{\omega}(t) &= -\frac{g}{L}\varphi(t) + \frac{w(t)}{mL} \end{aligned}$$

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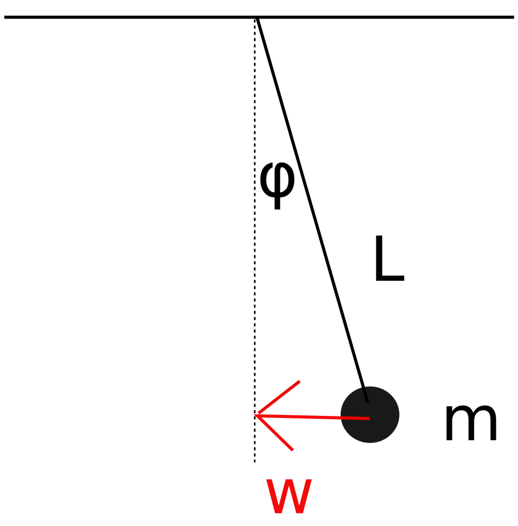

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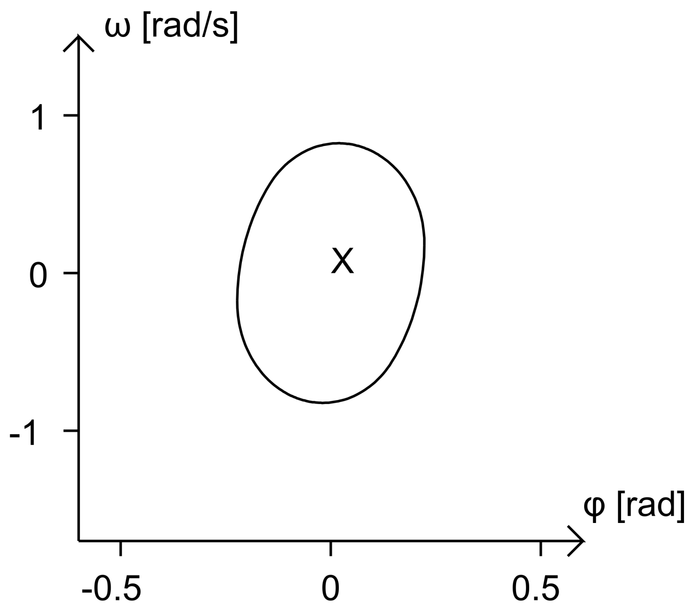
The horizontal force w is unknown.

Computation of the Uncertainty Tube



Example:

- The set $X(t)$ for the linearized pendulum.

t [s]	L [m]	g [m/s^2]	m [kg]
1.2	1	9.81	1

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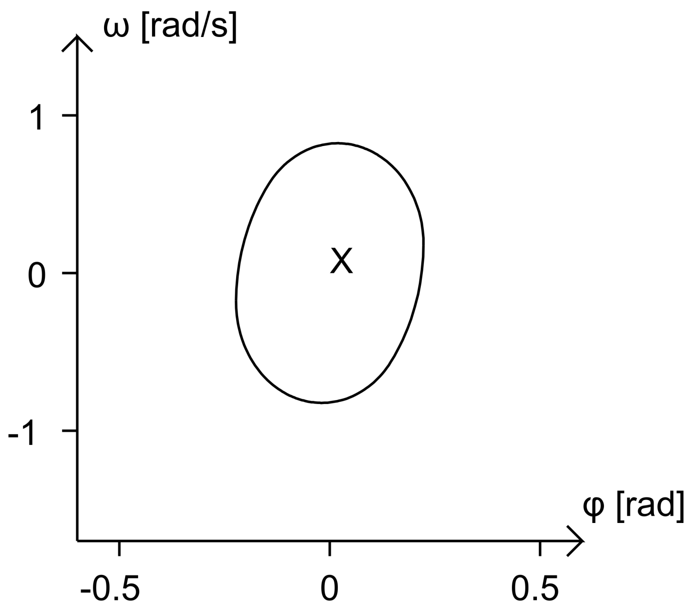

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Question: How can we compute the set $X(t)$?

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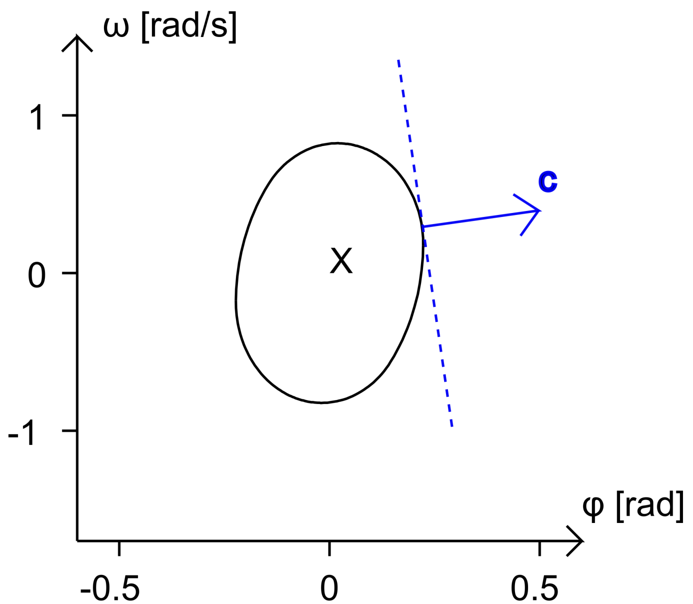

Solution: Solve the LP

$$\max_{x(\cdot),w(\cdot)} \; c^T x(t)$$

$$\text{s.t.} \begin{cases} \dot{x}(\tau) = Ax(\tau) + Bw(\tau) \\ x(0) \; = \; 0 \\ \|w(\tau)\|_\infty \leq 1 \quad \tau \in [0,t]. \end{cases}$$

for all directions c.

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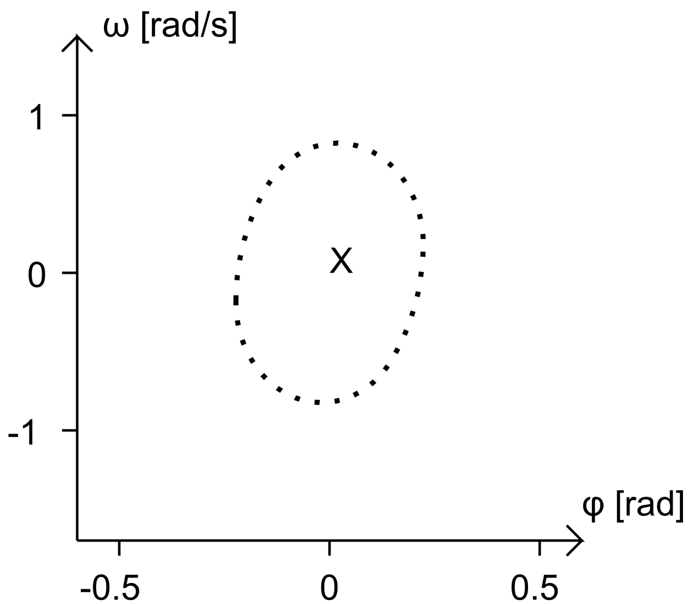
Computation of the Uncertainty Tube



CPU-time:

- Solve 100 LP's.
- Need approximately 1s for computing $X(t)$.

Problem: Computing $X(t)$ exactly takes very long if $n_x \gg 2$.

Strategy: Search for cheaper lifted outer approximations.

Example: Ellipsoidal Method

Assumption:

- $\Delta^n \;:=\; \{\, \lambda \in \mathbb{R}^n_{++} \mid \sum_{i=1}^n \lambda_i \;=\; 1 \,\}$
- $W(\tau) \;:=\; \{\, w \in \mathbb{R}^{n_w} \mid \forall\, \lambda \in \Delta^n : \;\; w \;\in\; \mathcal{E}(\,\Omega_\tau(\lambda)\,) \;\}$
- $\Omega_\tau : \mathbb{R}^n_{++} \to \mathbb{S}^{n_w}_+$ anti-homogeneous, i.e. $\Omega_\tau(\alpha\lambda) \;=\; \frac{1}{\alpha}\Omega_\tau(\lambda)$.

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Theorem:

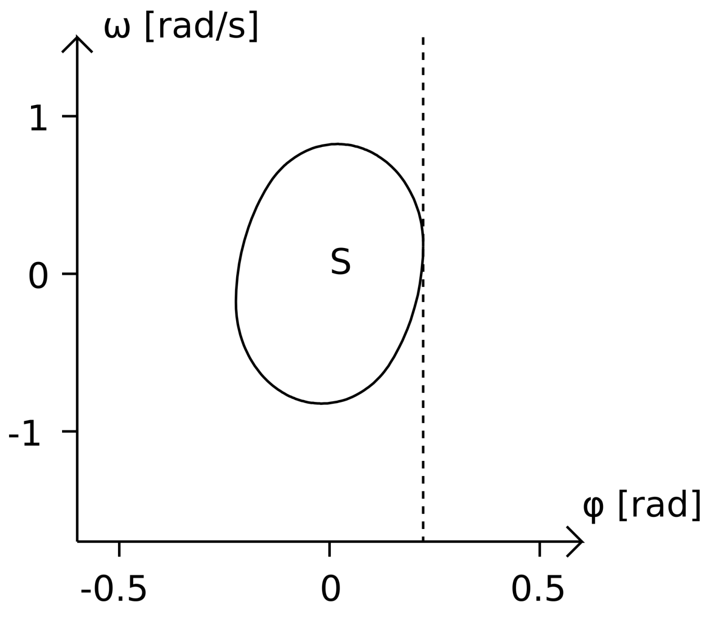
Let $Q : [t_1, t_2] \to \mathbb{S}^{n_x}_+$ and $\kappa : [t_1, t_2] \to \mathbb{R}^n_{++}$ be any functions satisfying

$$\dot{Q}(\tau) = A(\tau)Q(\tau) + Q(\tau)A(\tau)^T + \sum_{i=1}^n \kappa_i(\tau)Q(\tau) + B(\tau)\Omega_\tau(\kappa(\tau))B(\tau)^T \ .$$

Then $\mathbb{X}(\cdot) \ := \ \mathcal{E}(\, Q(\cdot) \,)$ is a robust positive invariant tube on $[t_1, t_2]$.

Ellipsoidal Outer Approximations

Visualization:



Application:

- Above Theorem allows to compute family of ellipoidal outer approximations which can e.g. be optimized for given directions c.

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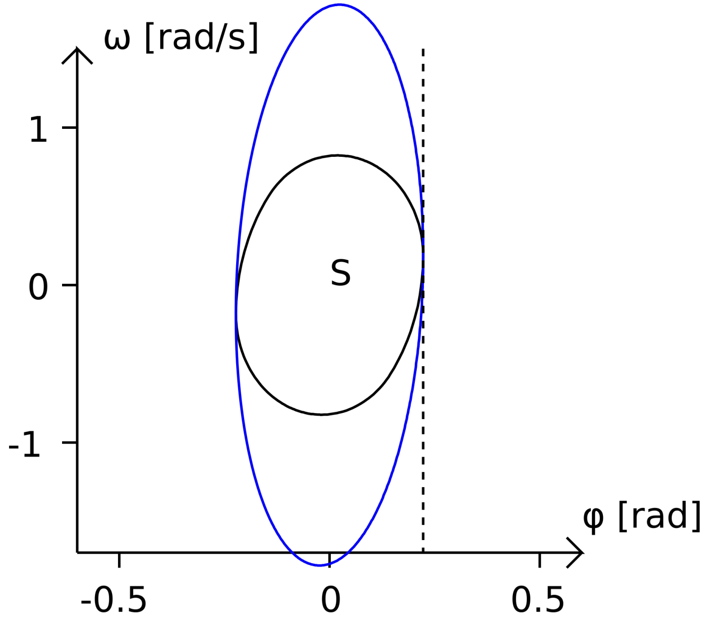

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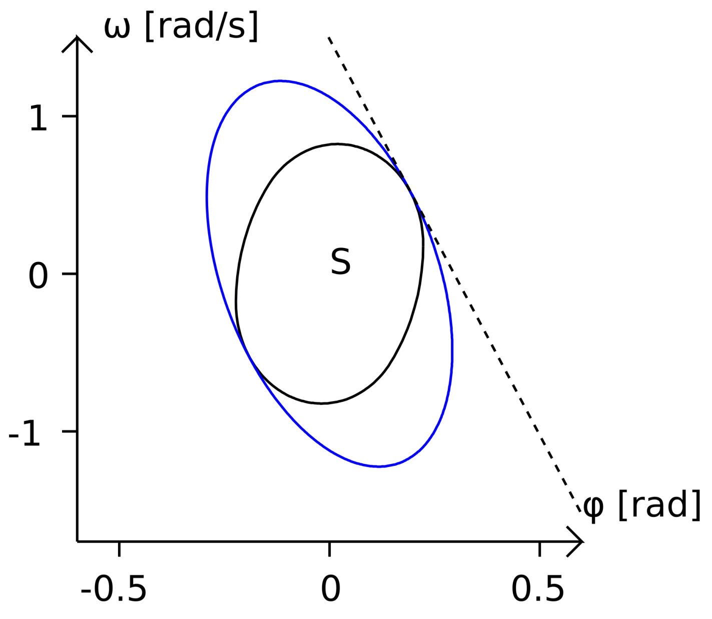

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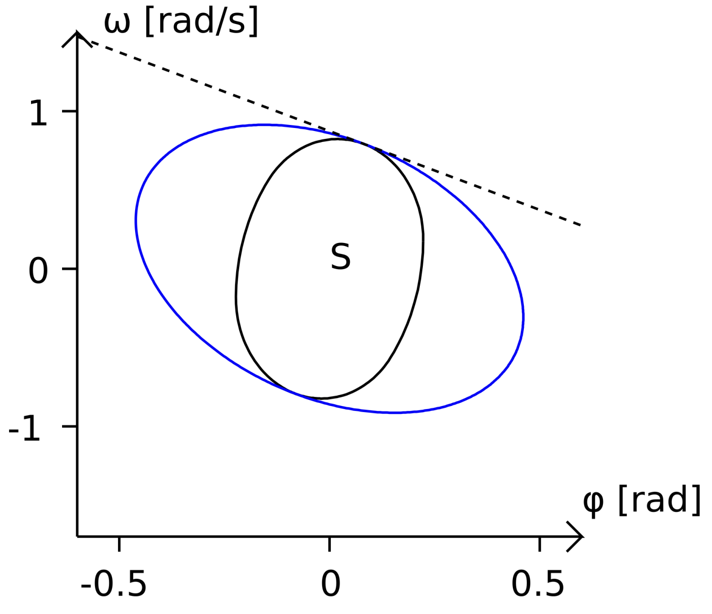

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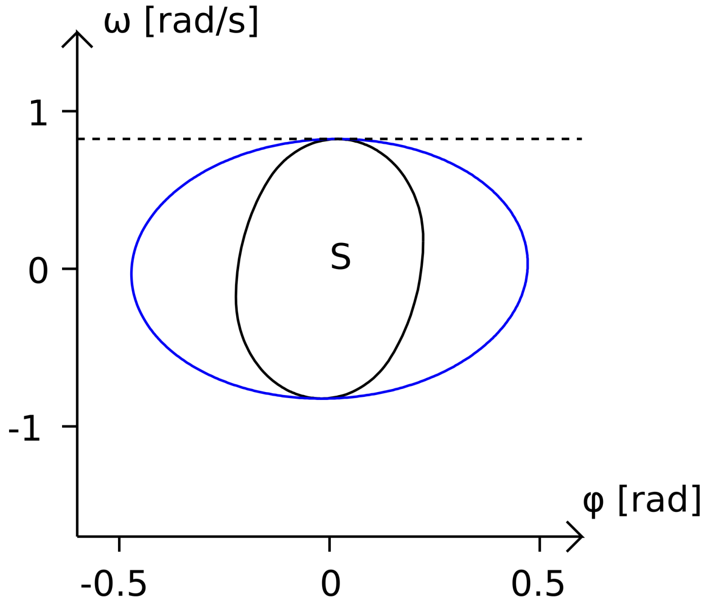

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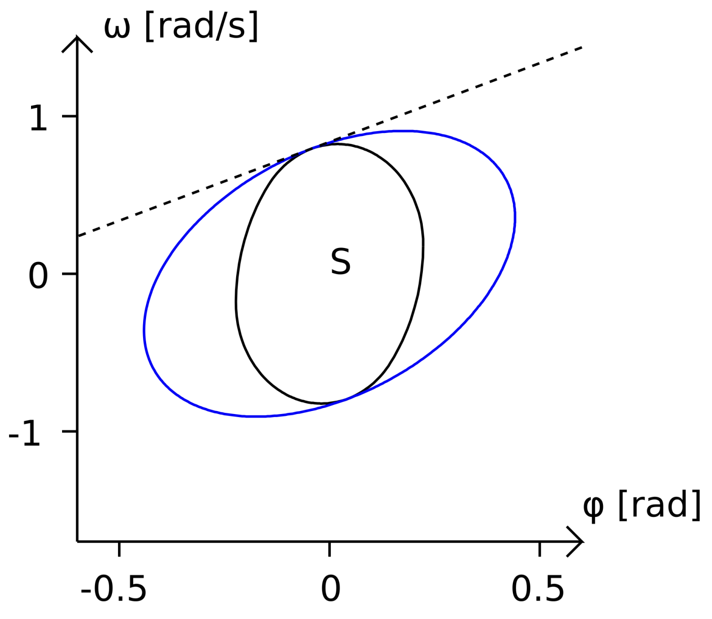

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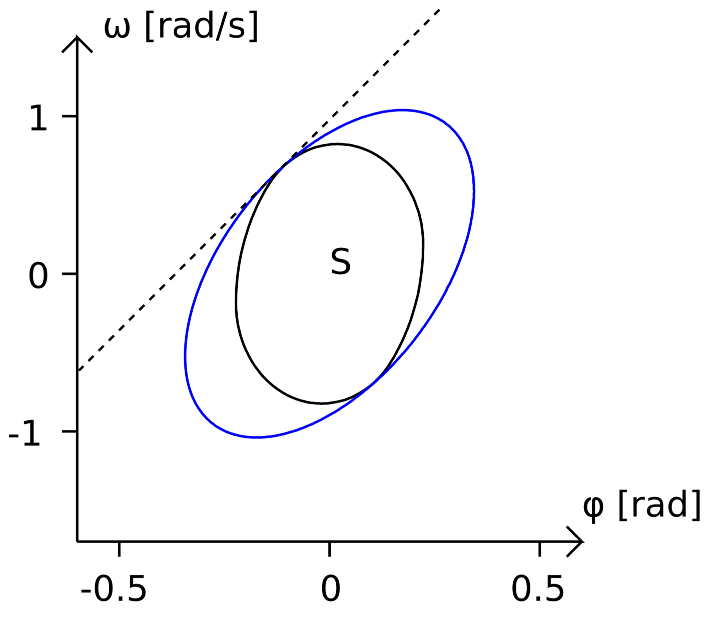

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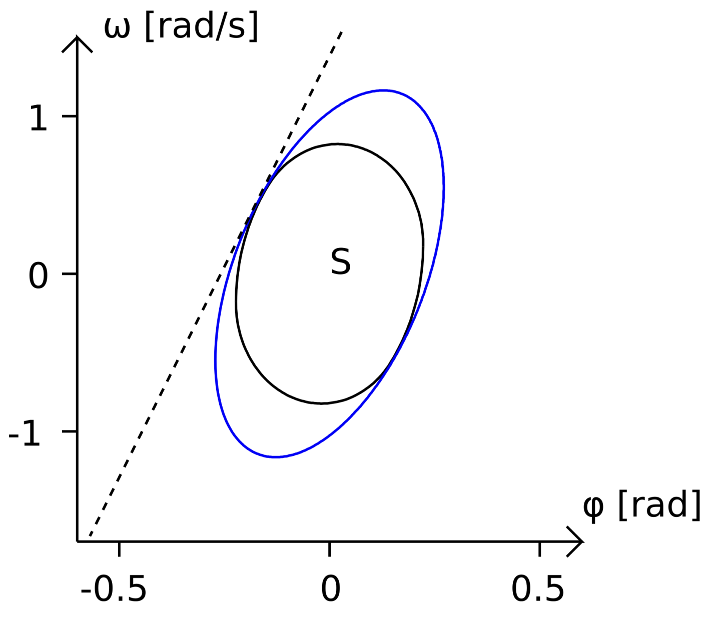

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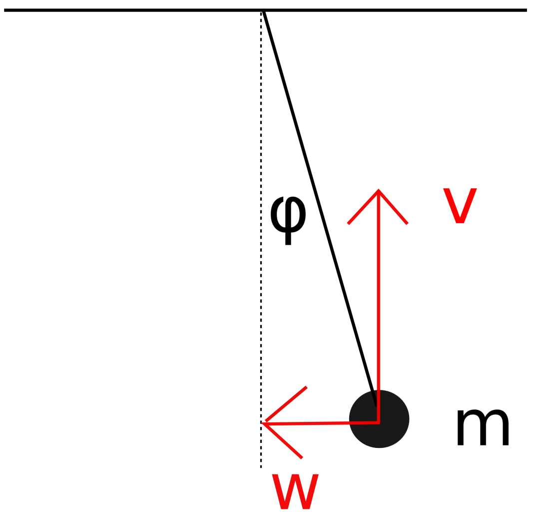
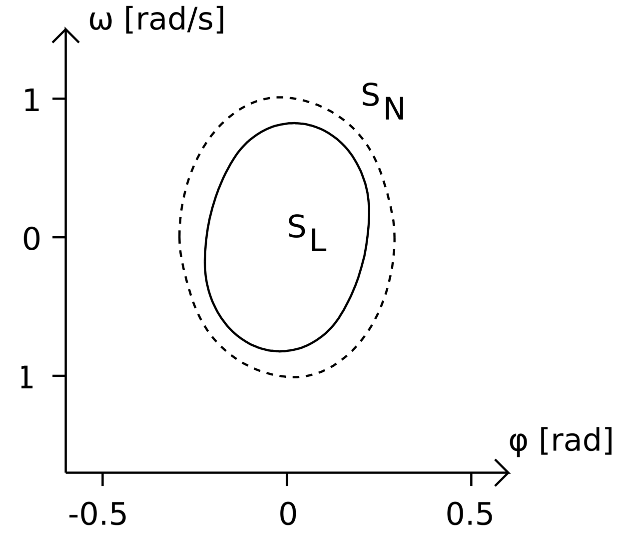
What about Nonlinear Dynamics?

Example: Pendulum with uncertain forces $|w(t)| \leq 1\,$N, $|v(t)| \leq 3\,$N.





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\end{aligned}
$$

Strategy for Nonlinear Dynamics

General Strategy:

- Define a central path $q : [t_1, \infty) \to \mathbb{R}^{n_x}$ as

$$\forall \tau \in [t_1, \infty) : \qquad \dot{q}(\tau) \; = \; \varphi(\tau, q(\tau)) \; := \; f(\tau, q(\tau), 0) \; , \quad q(t_1) \; = \; q_1 \; .$$

- Decompose the system into linear and nonlinear parts

$$\dot{x}(\tau) = d(\tau) + A(\tau)\left(x(\tau) - q(\tau)\right) + B(\tau)w(\tau) + f_{\mathsf{n}}(\tau, q(\tau), x(\tau), w(\tau)) .$$

- Employ “lifted” overestimate for influence of the nonlinear terms f_{n}.

Strategy for Nonlinear Right-Hand Sides

Assumption (Ellipsoidal Method):

- There exists an explicit nonlinearity estimate Ω_{N} with

$$\forall \lambda \in \Delta^{m-n} : \quad f_{\mathsf{n}}\left(\tau, q(\tau), x(\tau), w(\tau)\right) \in \mathcal{E}\left(\ \Omega_{\mathsf{N}}(\tau, q(\tau), Q, \lambda)\ \right)$$

  for all $x(\tau) \in \mathcal{E}(Q, q(\tau))$, $w \in \mathcal{W}$, $\tau \in [t_1, \infty)$, and for all $Q \in \mathbb{S}_+^{n_x}$.

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Strategy:

- Define $\Omega_{\mathrm{total}}(\tau, q, Q, \kappa) \;:=\; B(\tau)\Omega_\tau(\kappa^1)B(\tau)^T + \Omega_{\mathsf{N}}(\tau, q, Q, \kappa^2)$.
- Use $\Phi(\tau, q, Q, \kappa) \;:=\; A(\tau)Q + QA(\tau)^T + \sum_{i=1}^{m} \kappa_i Q + \Omega_{\mathrm{total}}(\tau, q, Q, \kappa)$.
- Now, if κ and Q are any functions which satisfy

$$\dot{Q}(\tau) \;=\; \Phi(\tau, q(\tau), Q(\tau), \kappa(\tau))$$

  then $\mathbb{X}(\cdot) := \mathcal{E}(Q(\cdot), q(\cdot))$ is a robust positive invariant tube.

Examples for Nonlinearity Estimates

Example 1: Consider $f_{\mathsf{n},i}(\tau, q, x, w) \;=\; x^T C x$ with $C \succeq 0$.

1. $\Omega_{\mathsf{N}}(\,\tau,\; q(\tau),\; Q,\; \lambda\,) \;:=\; [\,\mathrm{Tr}\,(QC)\,]^2\;\mathrm{diag}\,(\lambda)^{-1}$.
2. $\Omega_{\mathsf{N}}(\,\tau,\; q(\tau),\; Q,\; \lambda\,) \;:=\; \left[\sigma_{\mathsf{max}}\;\left(Q^{\frac{1}{2}}\,C\,Q^{\frac{1}{2}}\right)\right]^2 \mathrm{diag}\,(\lambda)^{-1}$.

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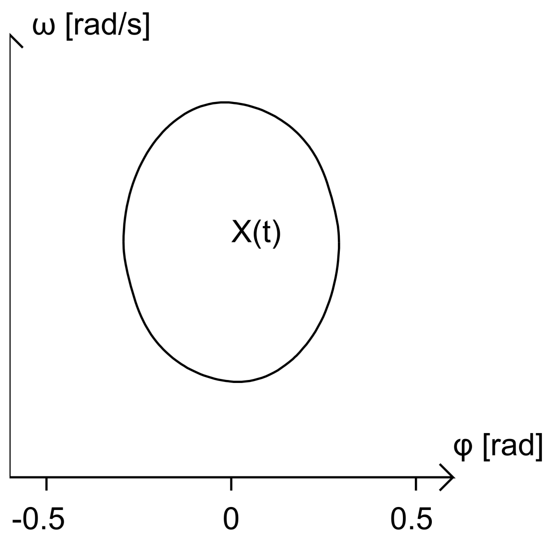
Example 2: $f_2(x,w) = -\frac{g}{L}\sin(x_1) + \frac{\cos(x_1)w_1 + 3\sin(x_1)w_2}{mL}$.

- On the domain $|x_1| \;\leq\; \frac{\pi}{2}$ may use

$$f_{\mathsf{nonlinear},2}(x,w) \;\leq\; \chi(Q) \;:=\; \frac{g}{L}\left[\sqrt{Q_{11}} - \sin(\sqrt{Q_{11}})\right] + \frac{1-\cos(\sqrt{Q_{11}}) + 3\sin(\sqrt{Q_{11}})}{mL}$$

leading to the nonlinearity estimate $\Omega_{\mathsf{N}} \;:=\; \frac{1}{\lambda}\begin{pmatrix} 0 & 0 \\ 0 & \chi(Q)^2 \end{pmatrix}$.

Ellipsoidal Outer Approximations



Visualization for the Pendulum:

- Use theorem to generate ellipsoidal outer approximation of the set $X(t)$.
- The dual control input κ can be optimized for different directions.

$$\Omega_{\mathrm{N}}(\,t,\; q(t),\; Q,\; \kappa\,) \;:=\; \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\kappa}\left[\frac{g}{L}\left[\sqrt{Q_{11}} - \sin(\sqrt{Q_{11}})\right] + \frac{1-\cos(\sqrt{Q_{11}})+3\sin(\sqrt{Q_{11}})}{mL}\right]^2 \end{pmatrix}.$$

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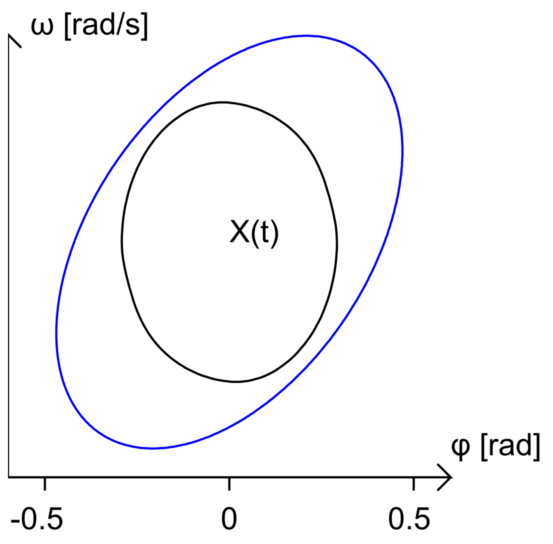

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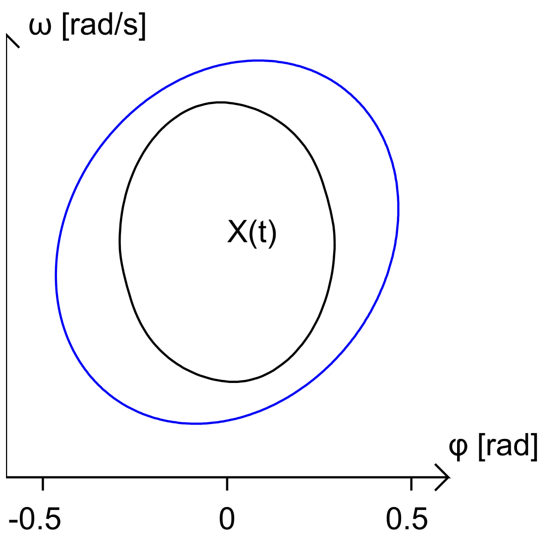

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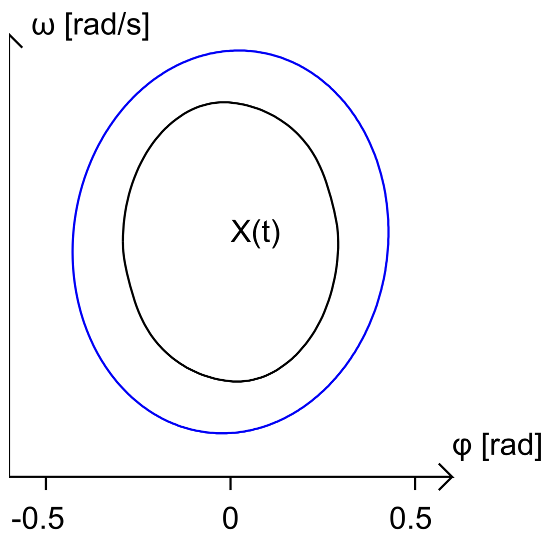

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Formulation of Robust Optimal Control Problems

Problem Formulation:

- Let us start with a robust optimal control problem of the form:

$$\min_{u(\cdot),p,T_\mathrm{e},X(\cdot)} \quad M(\,p,T_\mathrm{e},X(T_\mathrm{e})\,)$$

$$\text{s.t.} \quad \left\{ \begin{array}{rcl} X(\tau^+) & = & F(\tau,u(\tau),p,X(\tau),W(\tau)) \\ X(0) & = & X_0 \\ 0 & \geq & H(\tau,u(\tau),p,X(\tau),W(\tau)) \end{array} \right.$$

- **Aim:** Solve the above problem in a conservative approximation.

Definition of Monotonicity

Definition:

- A function $Z : \Pi(\mathbb{R}^{n_x}) \to \mathbb{R}$ is monotonically increasing, if for any sets $X, Y \subseteq \mathbb{R}^{n_x}$ with $X \subseteq Y$, we have that $Z(X) \leq Z(Y)$.

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Assumptions:

1. The Mayer term $M(\,p, T_{\mathrm{e}}, X(T_{\mathrm{e}})\,)$ is mononically increasing with respect to the variable $X(T_{\mathrm{e}})$.
2. The constraint function $H(\tau, u(\tau), p, X(\tau), W(\tau))$ is componentwise mononically increasing in $X(\tau)$.

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Question:

- Are the above assumption reasonable?

Example: Robust Counterpart Formulations

Robust Mayer Term:

- Assume nominal Mayer term $m : \mathbb{R}^{n_p} \times \mathbb{R}_+ \times \mathbb{R}^{n_x} \to \mathbb{R}$ is given.
- Define $M(p, T_\mathrm{e}, X(T_\mathrm{e})) := \sup_{x \in X(T_\mathrm{e})} m(p, T_\mathrm{e}, x)$.
- M is monotonically increasing in $X(T_\mathrm{e})$.

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Robust Constraint Function:

- Analogous formulation of a constraint:

$$H_i(\tau, u(\tau), p, X(\tau), W(\tau)) \;:=\; \sup_{\substack{x \in X(\tau) \\ w \in W(\tau)}} \; h_i(\, \tau,\, u(\tau),\, p,\, x,\, w\,)\ .$$

Example: Robustness Design Criteria

Minimize Maximum Distance of Two Points in the Terminal Set:

- $M(\, p, T_{\mathrm{e}}, X(T_{\mathrm{e}})\,) \;:=\; \text{diag}(X(T_{\mathrm{e}})) \;:=\; \sup\limits_{x,y \in X(T_{\mathrm{e}})} \|x - y\| \;.$
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Minimize the Volume of the Terminal Set:

- $M(\, p, T_{\mathrm{e}}, X(T_{\mathrm{e}})\,) \;:=\; \int_{X(T_{\mathrm{e}})} 1 \;\mathrm{d}x \;.$
- M is monotonically increasing in $X(T_{\mathrm{e}})$.

Ellipsoidal Approximation Assumption

Assumption:

- We have functions φ, Φ, q_0, and Q_0 such that: for any function $\kappa : [0, T_\mathrm{e}] \to \mathbb{R}^m_{++}$, and any vector $\kappa_0 \in \mathbb{R}^{n_0}_{++}$, which admit solutions $q : [0, T_\mathrm{e}] \to \mathbb{R}^{n_x}$ and $Q : [0, T_\mathrm{e}] \to \mathbb{S}^{n_x}_{+}$ of the coupled differential equation

$$
\begin{array}{rclrcl}
\dot{q}(\tau) & = & \varphi(\tau, u(\tau), p, q(\tau), Q(\tau), \kappa(\tau)) & q(0) & = & q_0(\kappa_0) \\
\dot{Q}(\tau) & = & \Phi(\tau, u(\tau), p, q(\tau), Q(\tau), \kappa(\tau)) & Q(0) & = & Q_0(\kappa_0)\ ,
\end{array}
$$

($\forall \tau \in [0, T_\mathrm{e}]$) the set valued function $\mathbb{X}(\cdot) := \mathcal{E}(Q(\cdot), q(\cdot))$ is a robust positive invariant tube on the interval $[0, T_\mathrm{e}]$ for which the condition $X_0 \subseteq \mathbb{X}(0)$ is also satisfied.

Ellipsoidal Approximation Strategy

- Consider an auxiliary problem of the form

$$\inf_{\xi(\cdot),\zeta(\cdot),\pi,T_{\mathrm{e}}} \quad M(\,p,\,T_{\mathrm{e}},\,\mathcal{E}(Q(T_{\mathrm{e}}),q(T_{\mathrm{e}}))\,)$$

$$\text{s.t.} \left\{ \begin{array}{rcll} \dot{q}(\tau) & = & \varphi(\tau,u(\tau),p,q(\tau),Q(\tau),\kappa(\tau)) & q(0)=q_0(\kappa_0) \\ \dot{Q}(\tau) & = & \Phi(\tau,u(\tau),p,q(\tau),Q(\tau),\kappa(\tau)) & Q(0)=Q_0(\kappa_0) \\ 0 & \geq & H(\tau,u(\tau),\,\mathcal{E}(Q(\tau),q(\tau)),\,W(\tau)) & \forall\,\tau\in[0,T_{\mathrm{e}}]\,. \end{array} \right. ,$$

- If H is componentwise monotonically increasing: every feasible input (u,p) corresponds to a feasible point of the original problem.
- If additionally M is monotonically increasing: objective value of the above problem is an upper bound on exact objective value.

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Nonlinear Jacketed Tubular Reactor (joint work with Filip Logist)

Model Equations:

- Consider tubular reactor with length L and perfect radial mixing.
- Assume: const. density & heat capacity of fluid, Arrhenius law.
- States: concentration and temperature

$$C(z) = C_{\mathsf{F}}(1 - x_1(z)) \quad T(z) = T_{\mathsf{F}}(1 + x_2(z)) - 273.15^\circ\mathsf{C}$$

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- Steady state depends on the spatial coordinate $z \in [0, L]$:

$$\begin{aligned} \partial_z\, x_1(z) &= (\alpha/v)(1 - x_1(z))e^{\frac{\gamma x_2(z)}{1+x_2(z)}} \\ \partial_z\, x_2(z) &= (\alpha\delta/v)(1 - x_1(z))e^{\frac{\gamma x_2(z)}{1+x_2(z)}} + (\beta(z)/v)(u(z) - x_2(z)) \end{aligned}$$

- Jacket temperature u is controlled, heat transfer β is uncertain.

Nonlinear Jacketed Tubular Reactor (joint work with Filip Logist)

Aim: Maximize conversion, i.e minimize $m := C_{\mathsf{F}}(1 - x_1(L))$ s.t.:

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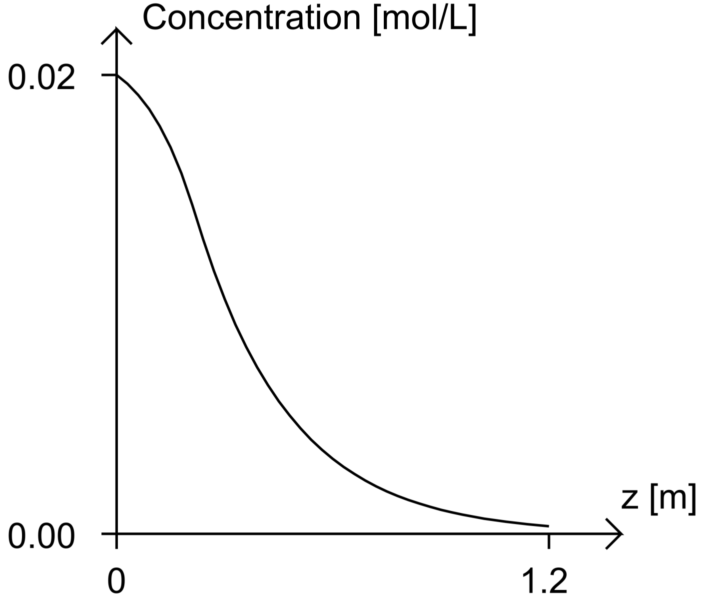
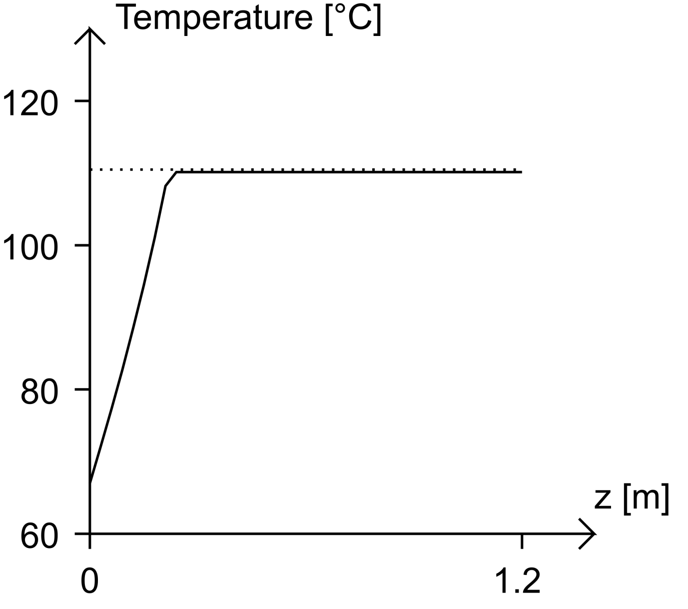
- control bounds, and maximum temperature constraint $T(z) \leq T_{\mathsf{max}}$.





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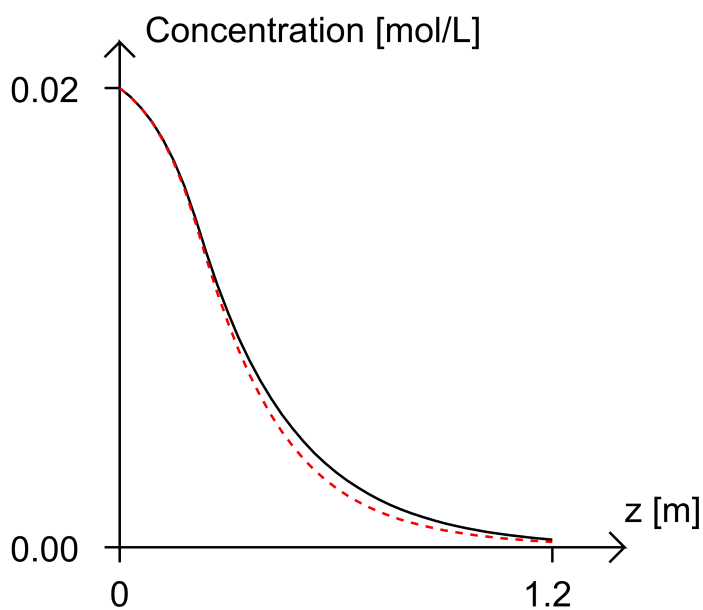

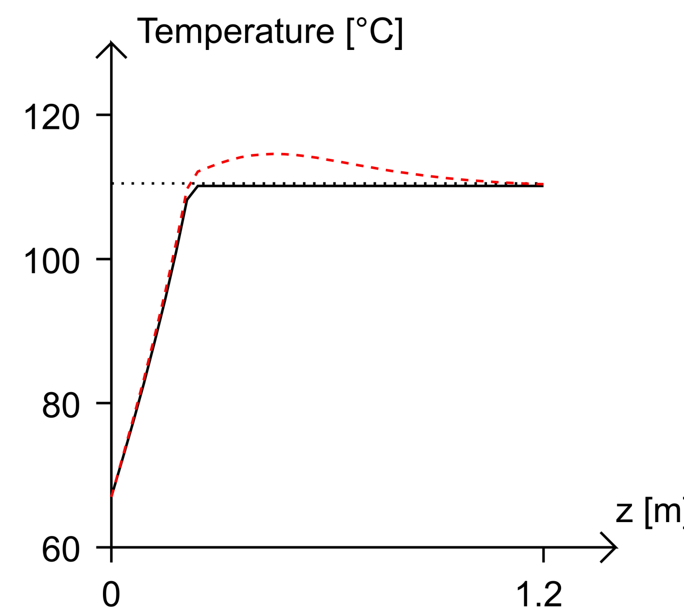

Problem: 6% uncertainty in $\beta(z)$ leads to contraint violation.

How to Construct a Nonlinearity Estimate?

Aim:

- Find a nonlinearity estimate of the form

$$|f_i(q+\Delta x, w) - f_i(q,0) - \partial_x f_i(q,0)\Delta x - \partial_w f_i(q,0) w| \leq l_i(q,Q)$$

  for all w with $|w(t)| \leq 1$ and all $\Delta x \in \mathcal{E}(Q)$.

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- Here:

$$\begin{aligned} f_1(x, w) &= \frac{\alpha}{v}(1 - x_1)e^{\frac{\gamma x_2}{1+x_2}} \\ f_2(x, w) &= \frac{\alpha\delta}{v}(1 - x_1)e^{\frac{\gamma x_2}{1+x_2}} + \frac{\beta_{\text{nominal}}(1 + \Gamma w)}{v}(u - x_2) \end{aligned}$$

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Main Idea: Use that $\forall y \in \mathbb{R}: \quad e^y \ \leq \ 1 + y + \frac{y^2}{2} e^{|y|}$.

How to Construct a Nonlinearity Estimate?

... and work out the details:

$$
\begin{aligned}
j(q,Q) &:= \frac{\gamma}{(1+q_2)(1+q_2-\sqrt{Q_{22}})} \\
r_1(q,Q) &:= j(q,Q) + \frac{\sqrt{Q_{22}}}{2}\, j(q,Q)^2 \;\exp(\,\sqrt{Q_{22}}\;|j(q,Q)|\,) \\
r_2(q,Q) &:= \frac{j(q,Q)}{1+q_2} + \frac{j(q,Q)^2}{2}\;\exp(\,\sqrt{Q_{22}}\;|j(q,Q)|\,) \\
l_1(q,Q) &:= \frac{\alpha}{v}\exp\left(\frac{\gamma q_2}{1+q_2}\right)\left[r_1(q,Q)\,\sqrt{Q_{11}Q_{22}} + r_2(q,Q)\,Q_{22}\right] \\
l_2(q,Q) &:= \delta\, l_1(q,u,Q) + \frac{\Gamma\beta_{\text{nominal}}\sqrt{Q_{22}}}{v} \quad \Omega_{\mathsf{N}}(\lambda,q,Q) \;:=\; \mathsf{diag}(l_i(q,Q)/\lambda_i)
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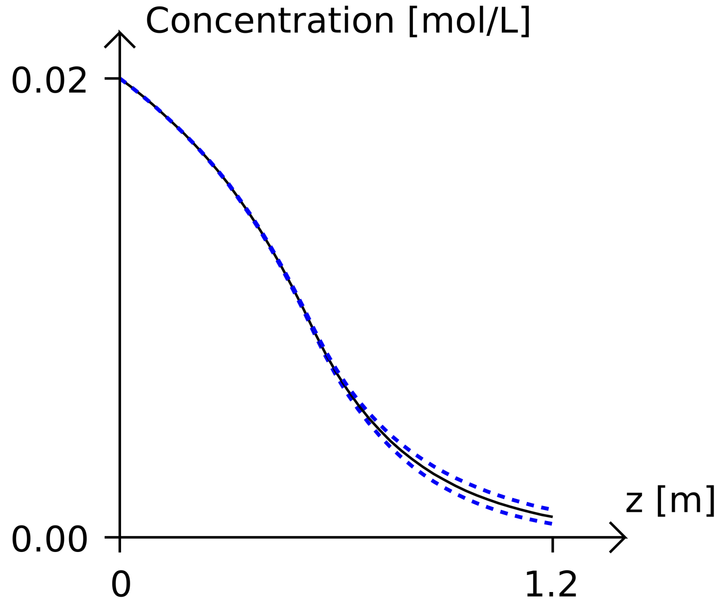
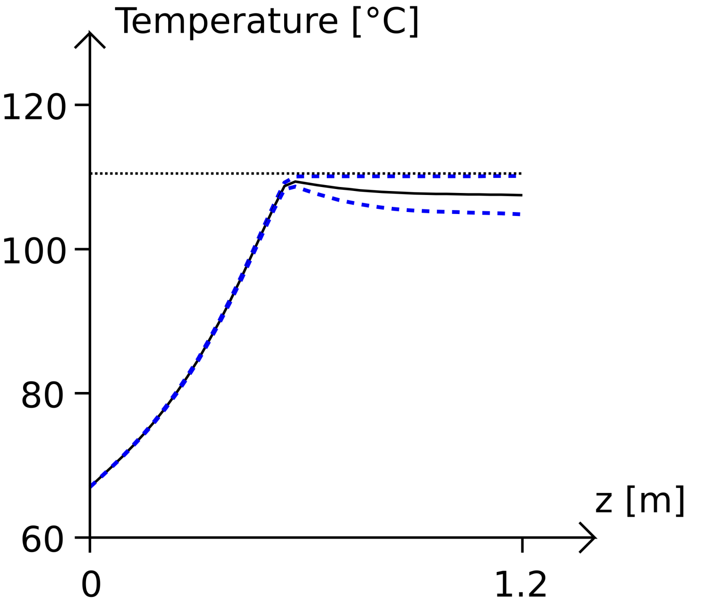
\end{aligned}
$$

- **This Nonlinearity Estimate enables us to robustly optimize the conversion such that the temperature constraint is satisfied for all possible uncertainties.**

Robust Control of the Jacketed Tubular Reactor

Optimized Robust Solution:





- Constraints are guaranteed to be robustly satisfied.
- Simulation with 6% uncertainty remains in ellipsoidal tube.

Robust Control of the Jacketed Tubular Reactor

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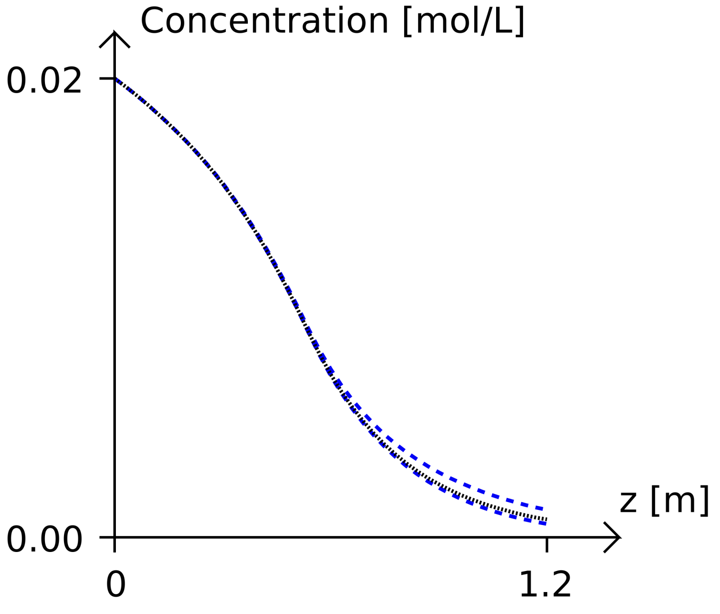

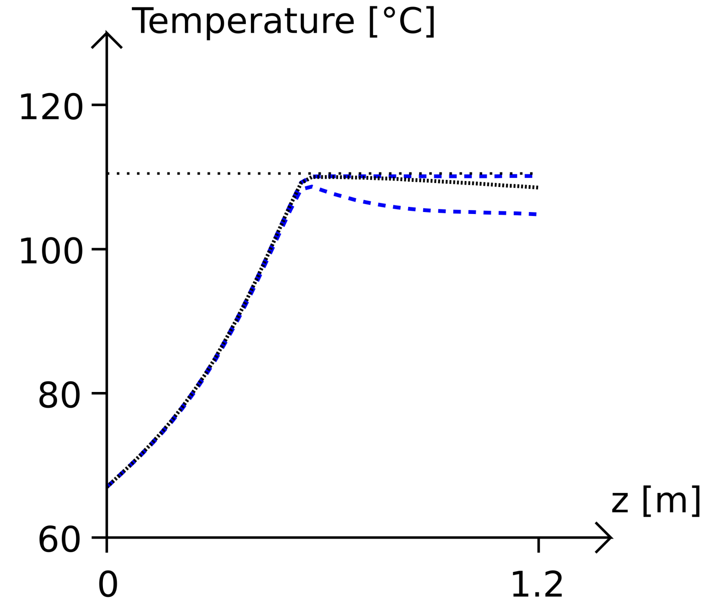

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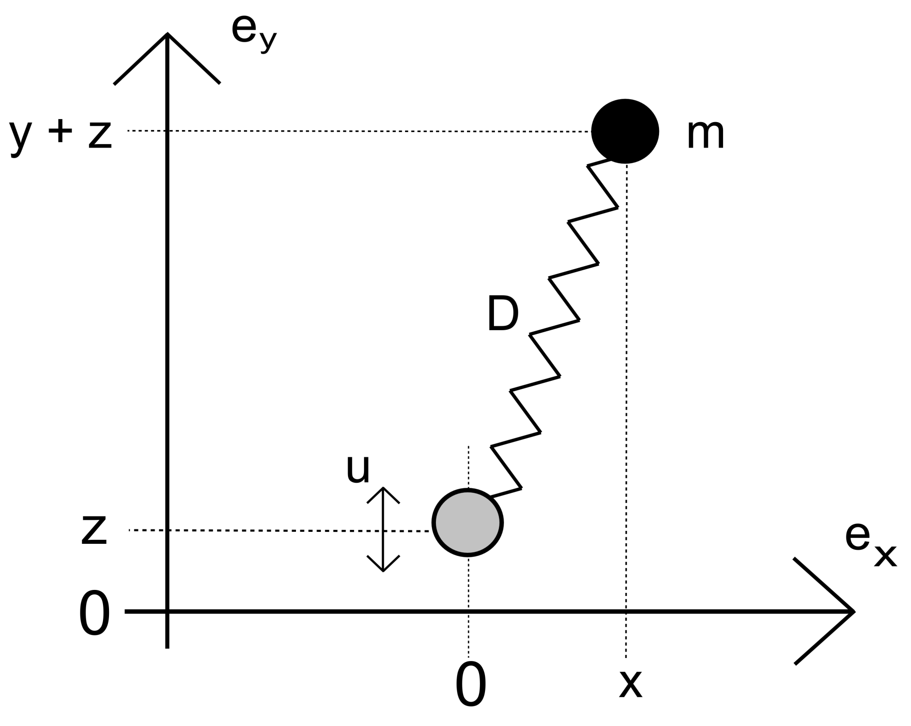
The Inverted Spring Pendulum



Parameter	Value
L	$1\,$m
m	$0.1\,$kg
D	$700\ \frac{\text{N}}{\text{m}}$
g	$9.81\ \frac{\text{m}}{\text{s}^2}$
b	$5\ \frac{1}{\text{s}}$
$\overline{w}$	$0.03\ \frac{\text{N}}{\text{kg}}$
$\overline{u}$	$200\ \frac{\text{N}}{\text{kg}}$
$\overline{v}_z$	$3.2\ \frac{\text{m}}{\text{s}}$

The Inverted Spring Pendulum

Model:

- Right-hand side function is given by

$$f(\xi, u, w) \;=\; \begin{pmatrix} v_x \\ v_y \\ -\frac{Dx}{m}\left(1-\frac{L}{\sqrt{x^2+y^2}}\right)-bv_x+w \\ -g+u-\frac{Dy}{m}\left(1-\frac{L}{\sqrt{x^2+y^2}}\right)-bv_y \\ v_z \\ u \end{pmatrix} .$$

Optimal Control Problem

Aim:

- Find an open-loop stable periodic orbit at the “inverted” position.

Objective:

- Minimize the time-average over the maximum displacement of the mass point in x-direction:

$$\mathcal{L}(\tau, u(\tau), T_{\mathrm{e}}, X(\tau), W(\tau)) \quad := \quad \max_{\xi \in X(\tau)} \frac{\left(e_x^T \xi\right)^2}{T_{\mathrm{e}}}$$

with $e_x^T \; := \; (\,1,\, 0,\, \ldots,\, 0\,)^T \in \mathbb{R}^6$.

Optimization Variables:

- Set valued function X, control input u, end time T_{e}.

Optimal Control Problem

Constraints:

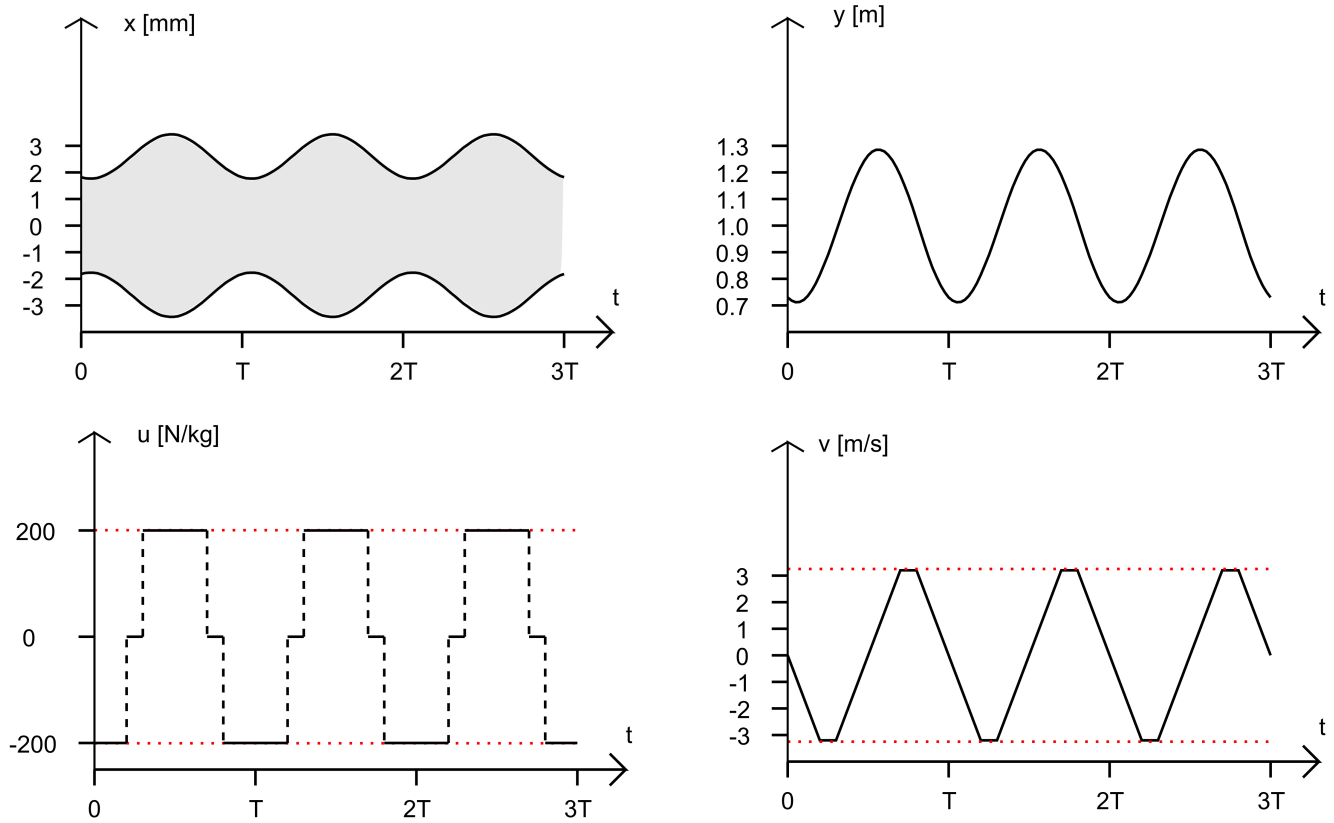
- Periodic propagation of the uncertainty tube

$$X(\tau^+) \;=\; F(\tau, u(\tau), p, X(\tau), W(\tau)) \quad \text{with} \quad X(0) = X(T_\mathrm{e})\;.$$

- Here: $W(\tau) \;=\; \{\, w \in \mathbb{R} \mid \underline{w} \;\leq\; w \;\leq\; \overline{w} \,\}$
- Control and State Constraint Function:

$$H(\tau, u(\tau), X(\tau)) \;:=\; \begin{pmatrix} u(\tau) - \overline{u} \\ -u(\tau) + \overline{u} \\ \max\limits_{\xi \in X(\tau)} \; e_{v_z}^T \xi - \overline{v}_z \\ \min\limits_{\xi \in X(\tau)} \; - e_{v_z}^T \xi + \overline{v}_z \end{pmatrix}\;.$$

Optimization Results Based on Ellipsoidal Technique:



Optimal period time: $T_e \approx 0.79$ms (≈ 80Hz).

Summary

- “State” of uncertain systems is a set-valued function.
- Lifted set approximation techniques
- Ellipsoidal robust positive invariant tubes have desirable complexity.
- Formulation and Solution Techniques for Robust Optimal Control Problems
- Nonlinear application: robust control of a tubular reactor
- Existence of periodic tubes based in Schauder’s fixed point theorem
- Open-loop stable orbits of an inverted pendulum