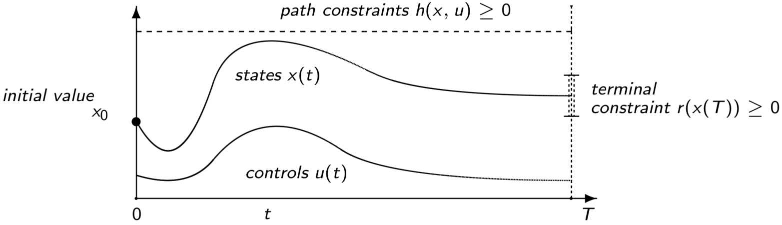
Optimal Control, MPC and MHE

Mario Zanon

Outline

Optimal Control Problem (OCP)



$$\underset{x(\cdot),u(\cdot),T}{\text{minimise}} \int_0^T \ell(x(\tau), u(\tau))\,\mathrm{d}\tau + V(x(T))$$

$$\begin{aligned} \text{subject to } \dot{x}(\tau) &= f(x(\tau), u(\tau)), & t \in [0, T], \quad & \text{(dynamic model)} \\ h(x(\tau), u(\tau)) &\geq 0, & t \in [0, T], \quad & \text{(path constraints)} \\ r(x(0), x(T)) &\geq 0. & & \text{(initial/terminal conditions)} \end{aligned}$$

OCP Solution Approacehs

Three families of approaches:

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- Hamilton-Jacobi-Bellman / Dynamic Programming

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OCP Solution Approacehs

Three families of approaches:

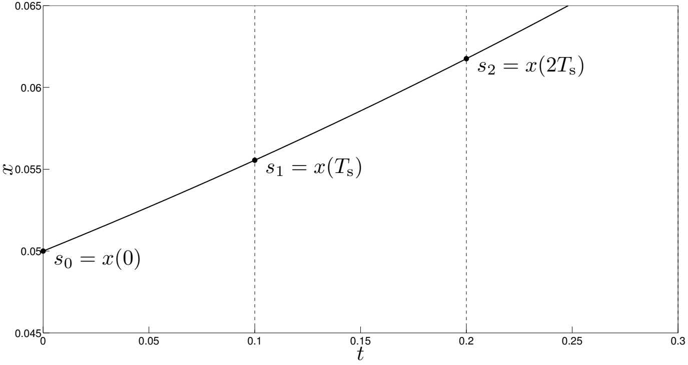
- Hamilton-Jacobi-Bellman / Dynamic Programming
- Indirect Methods (first optimise then discretise)
- **Direct Methods** (first discretise then optimise)
  - Direct single shooting
  - Direct multiple shooting
  - Direct collocation

Single vs. Multiple Shooting

Single vs. Multiple Shooting

- Single Shooting:

From $x(t_0)$ integrate the system on **the whole horizon**
$\rightarrow$ **continuous trajectory**



Single vs. Multiple Shooting

- Single Shooting:
  From $x(t_0)$ integrate the system on **the whole horizon**
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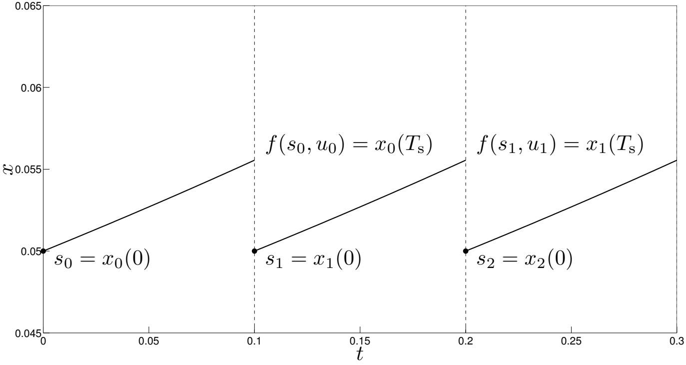
- Multiple Shooting:
  From $x(t_k)$ integrate the system on **each interval separately**
  $\rightarrow$ **discontinuous trajectory**



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  From $x(t_0)$ integrate the system on **the whole horizon**
  $\rightarrow$ **continuous trajectory**
- Multiple Shooting:
  From $x(t_k)$ integrate the system on **each interval separately**
  $\rightarrow$ **discontinuous trajectory**
- Colloctaion:
  For each point $x(t_k)$ satisfy the **collocation equations**
  $\rightarrow$ **discontinuous trajectory**

MS OCP Discretisation

The OCP becomes a nonlinear programming problem (NLP)

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The OCP becomes a nonlinear programming problem (NLP)

$$\underset{\substack{x_0,\dots,x_N,\\ u_0,\dots,u_{N-1}}}{\text{minimise}} \quad \sum_{k=0}^{N} \ell(x_k, u_k) + V(x_N)$$

$$\begin{aligned}
\text{subject to} \quad & x_{k+1} = f(x_k, u_k), && k = 0, \dots, N-1, && \text{(dynamic model)} \\
& h(x_k, u_k) \geq 0, && k = 0, \dots, N-1, && \text{(path constraints)} \\
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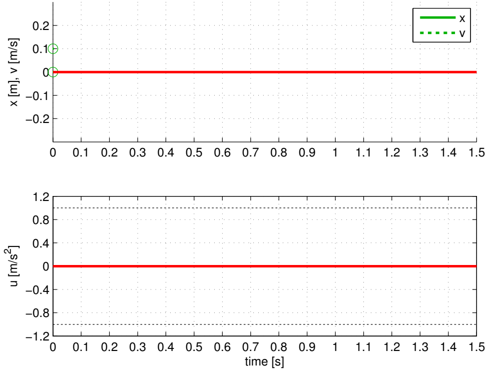
... and it can be seen as an **OCP for discrete time systems**

Model Predictive Control

Looking ahead in the future...

...design a control sequence minimizing the deviation from the reference



$$
\begin{aligned}
\min_{u,s} \quad & \|s_N - s_{ref}\|_P^2 + \sum_{k=0}^{N-1} \|s_k - s_{ref}\|_Q^2 + \|u_k - u_{ref}\|_R^2 && \rightarrow \textbf{deviation} \text{ from the reference} \\
\text{s.t.} \quad & s_{k+1} = f\left(s_k, u_k\right), \quad k = 0, \dots, N-1, && \rightarrow \textbf{model} \text{ of the system evolution} \\
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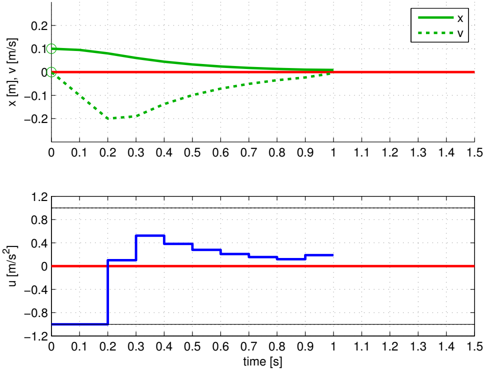

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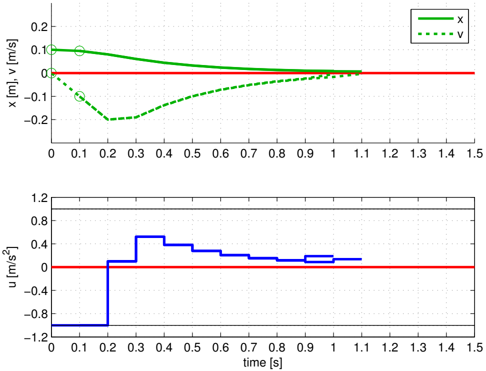

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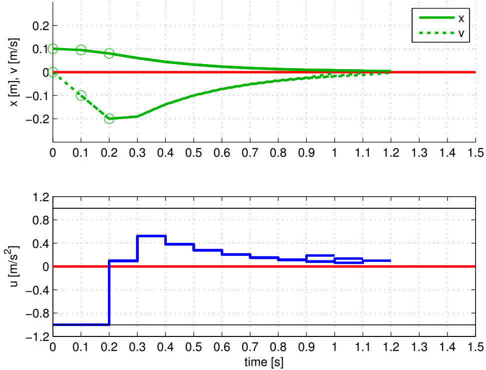

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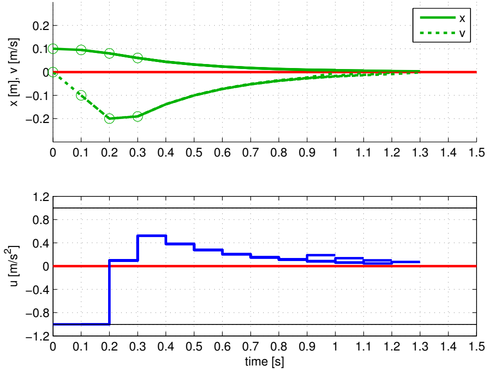

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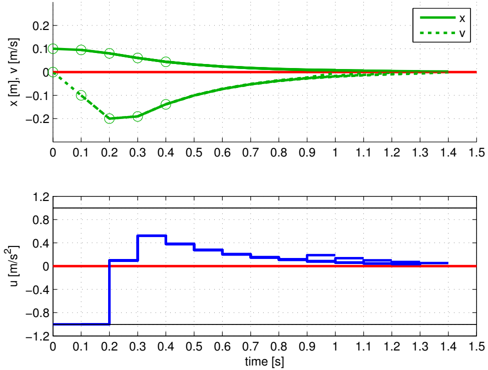

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Real-Time Dilemma and Real Time Iteration (RTI) Scheme

NLP cannot be solved instantaneously: better to apply

- suboptimal solution (almost) instantaneously?
- optimal solution after a longer time delay?

Real-Time Dilemma and Real Time Iteration (RTI) Scheme

SQP for NMPC in a nutshell

NMPC at time i

$$\min_{u,s} \sum_{k=0}^{N} \|s_k - x_{ref}\|_Q^2 + \sum_{k=0}^{N-1} \|u_k - u_{ref}\|_R^2$$

$$\text{s.t.} \quad s_{k+1} = f(s_k, u_k)$$

$$h(s_k, u_k) \geq 0,$$

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Quadratic Problem Approximation

QP (for a given s, u)

$$\min_{\Delta u, \Delta s} \quad \frac{1}{2} \begin{bmatrix} \Delta s & \Delta u \end{bmatrix} \quad \begin{bmatrix} \Delta s \\ \Delta u \end{bmatrix} + \quad \begin{bmatrix} \Delta s \\ \Delta u \end{bmatrix}$$

$$\text{s.t.} \quad \Delta s_{k+1} = \quad + \quad \Delta s_k + \quad \Delta u_k,$$

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Iterative procedure (at each time i):

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Real Time Iterations

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Preparation Phase
Without knowing $\hat{x}_i$

- **Linearize**
- (Gauss-Newton $\Rightarrow B \succ 0$)
- Prepare the QP

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Feedback Phase:

- Solve QP once $\hat{x}_i$ available
  $\rightarrow$ same latency as linear MPC

CasADi

www.casadi.org

Key Properties of Casadi

- Open Source (LGPL)
- Automatic Differentiation
- Python interface

What is it good for?

- Easy and powerful way of formulating NLPs/OCPs
- Linked to NLP solvers
- Linked to Sundials integrators

ACADO Toolkit

www.acadotoolkit.org

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Fast implementations for real-time execution
$\longrightarrow$ ACADO Code Generation tool:

Export tailored solver in plain C code