# Constrained Optimization

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*(some slide material was provided by W. Bangerth, K. Mombaur)*

# Nonlinear Programming (Problem Class 3)

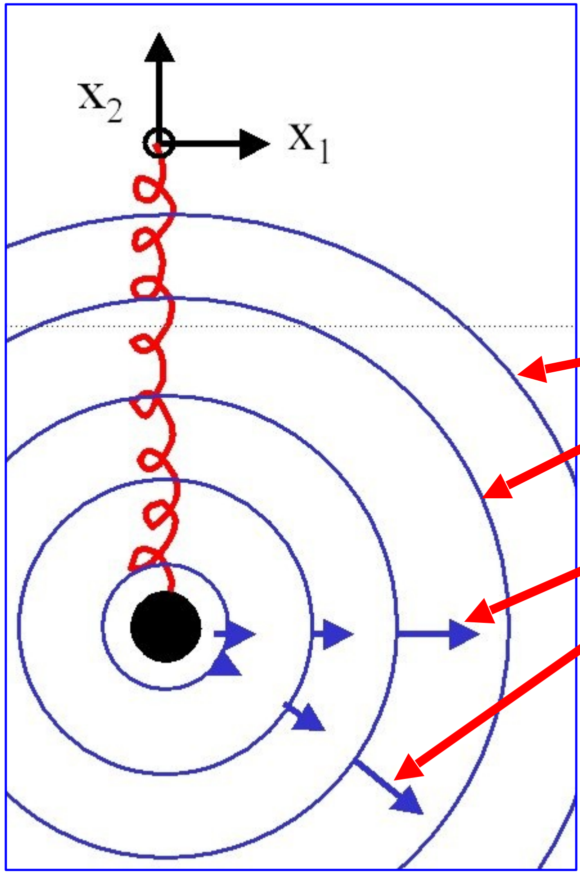
- General problem formulation:

$$
\boxed{
\begin{array}{rcllc}
\min f(x) & & & f: & D \subset R^n \rightarrow R \\
s.t.\, g(x) & = & 0 & g: & D \subset R^n \rightarrow R^l \\
h(x) & \geq & 0 & h: & D \subset R^n \rightarrow R^k
\end{array}
}
$$

*f* objective function / cost function
*g* equality constraints
*h* inequality constraints

f,g,h shall be smooth (twice differentiable) functions

# Recall: ball on a spring without constraints




$$\boxed{\min_{x \in R^2} x_1^2 + x_2^2 + m x_2}$$

contour lines of *f(x)*

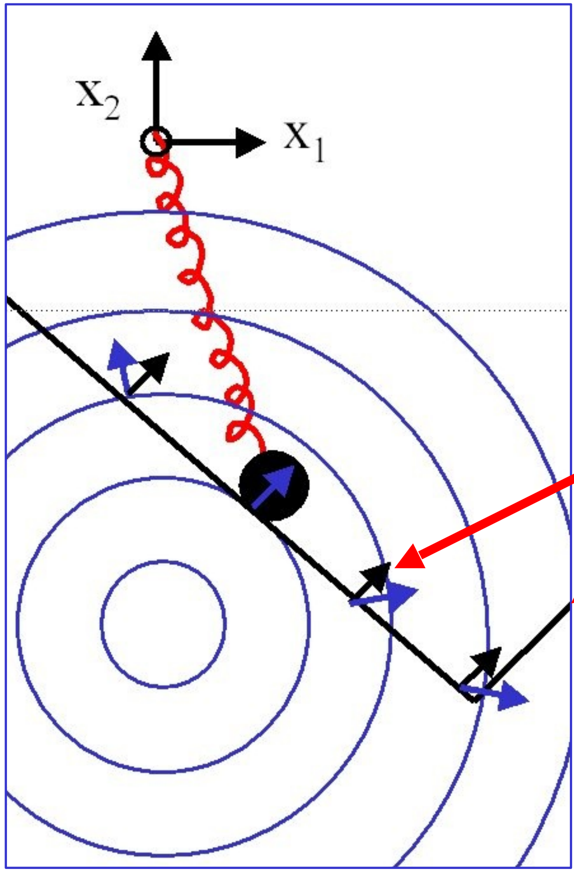
gradient vector

$$\nabla f(x) = (2x_1, 2x_2 + m)$$

unconstrained minimum:

$$0 = \nabla f(x^*) \Leftrightarrow (x_1^*, x_2^*) = (0, -\frac{m}{2})$$

# Now: ball on a spring with constraints




$$\min f(x)$$

$$h_1(x) := 1 + x_1 + x_2 \quad \geq \quad 0$$

$$h_2(x) := 3 - x_1 + x_2 \quad \geq \quad 0$$

gradient $\nabla h_1$ of active constraint

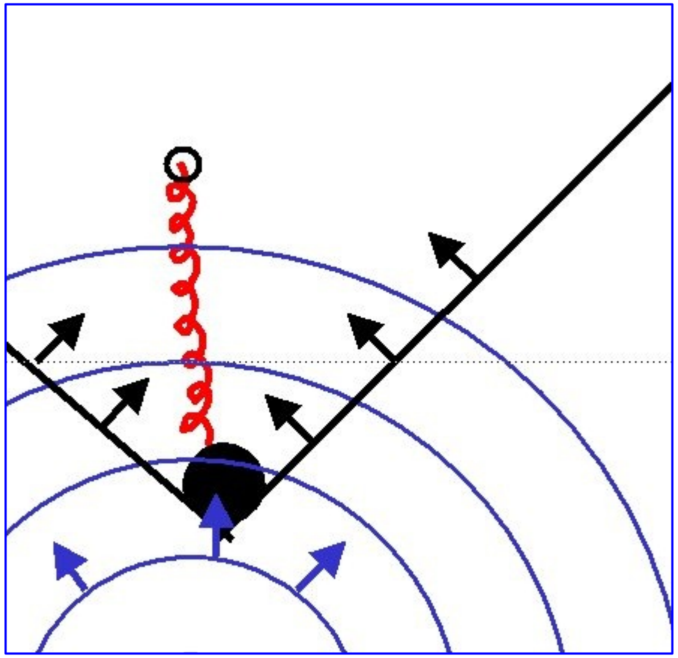
inactive constraint $h_2$

constrained minimum:

$$\boxed{\nabla f(x^*) = \mu_1 \nabla h_1(x^*)}$$

*Lagrange multiplier*

# Ball on a spring with two active constraints



$$\min f(x)$$

$$h_1(x) := 1 + x_1 + x_2 \quad \geq \quad 0$$

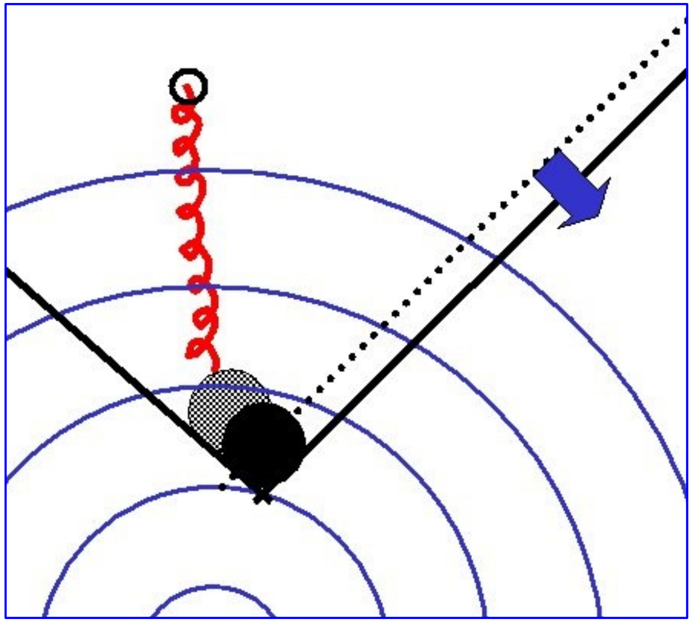
$$h_2(x) := 3 - x_1 + x_2 \quad \geq \quad 0$$

„equilibrium of forces“

$$\nabla f(x^*) = \mu_1 \nabla h_1(x^*) + \mu_2 \nabla h_2(x^*) \qquad \mu_1, \mu_2 \geq 0$$

*„constraint forces“*

# Multipliers as „shadow prices“



What happens if we relax a constraint?

Feasible set becomes bigger,

so new minimum $f(x_\varepsilon^*)$ becomes smaller.

How much would we gain?

$$\boxed{f(x_\varepsilon^*) \approx f(x^*) - \mu\varepsilon}$$

old constraint: $h(x) \geq 0$

new constraint: $h(x) + \varepsilon \geq 0$

*Multipliers show the hidden cost of constraints.*

# The Lagrangian Function

For constrained problems, introduce modification of objective function:

$$\boxed{L(x,\lambda,\mu) := f(x^*) - \sum \lambda_i g_i(x) - \sum \mu_i h_i(x)}$$

- equality multipliers $\lambda_i$ may have both signs in a solution
- inequality multipliers $\mu_i$ cannot be negative (cf. shadow prices)
- for inactive constraints, multipliers $\mu_i$ are zero

# Optimality conditions (constrained)

*Karush-Kuhn-Tucker necessary conditions (KKT-conditions):*

- $x^*$ feasible
- there exist $\lambda^*$, $\mu^*$ such that

$$\boxed{\nabla_x L(x^*, \lambda^*, \mu^*) = 0}$$

$$(\Leftrightarrow \text{"Equilibrium"}\ \nabla f = \sum \lambda_i \nabla g_i + \sum \mu_i \nabla h_i)$$

- $\mu^* \geq 0$ holds
- and it holds the complementarity condition

$$\boxed{\mu^{*T} h(x^*) = 0}$$

i.e. $\mu_i^* = 0$ or $h_i(x^*) = 0$ for each $i$

# Sequential Quadratic Programming (SQP)

Constrained problem:

$$\boxed{\begin{aligned} &\min f(x) \\ g(x) &= 0 \\ h(x) &\geq 0 \end{aligned}}$$

*SQP Idea:* Consider successively quadratic approximations of the problem:

$$\boxed{\begin{aligned} &\min_{\Delta x} \; (\nabla f^k)^T \Delta x + \frac{1}{2} \Delta x^T H^k \Delta x \\ &g(x^k) + \nabla g(x^k)^T \Delta x = 0 \\ &h(x^k) + \nabla h(x^k)^T \Delta x \geq 0 \end{aligned}}$$

# SQP method

- if we use the exact hessian of the Lagrangian

$$H = \nabla^2 L(x, \lambda, \mu)$$

  this leads to a newton-method for the optimality conditions and feasibility.
- with update-formulas for $H^k$, we obtain quasi-Newton SQP-methods.
- if we use appropriate update-formulas, we can have superlinear convergence.
- global convergence can be achieved by using a stepsize strategy.

# SQP algorithm

0. Start with $k$=0, start value $x^0$ and $H^0=I$
1. Compute $f(x^k)$, $g(x^k)$, $h(x^k)$, $\nabla f(x^k)$, $\nabla g(x^k)$, $\nabla h(x^k)$
2. If $x^k$ feasible and

$$\left\|\nabla L(x,\lambda,\mu)\right\| < \varepsilon$$

   then *stop* ➨ convergence achieved
3. Solve quadratic problem and get $\Delta x^k$
4. Perform line search and get stepsize $t^k$
5. Iterate

$$x^{k+1} = x^k + t^k \Delta x^k$$

6. Update hessian
7. $k$=$k$+1, goto step 1

# Summary

- Lagrangian function plays important role in constrained optimization
- Lagrange multipliers of inequalities have positive sign
- KKT conditions are necessary optimality conditions
- Look at SQP again in the following presentation…