A tutorial overview on theory and design of offset-free MPC algorithms

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Introduction to offset-free MPC

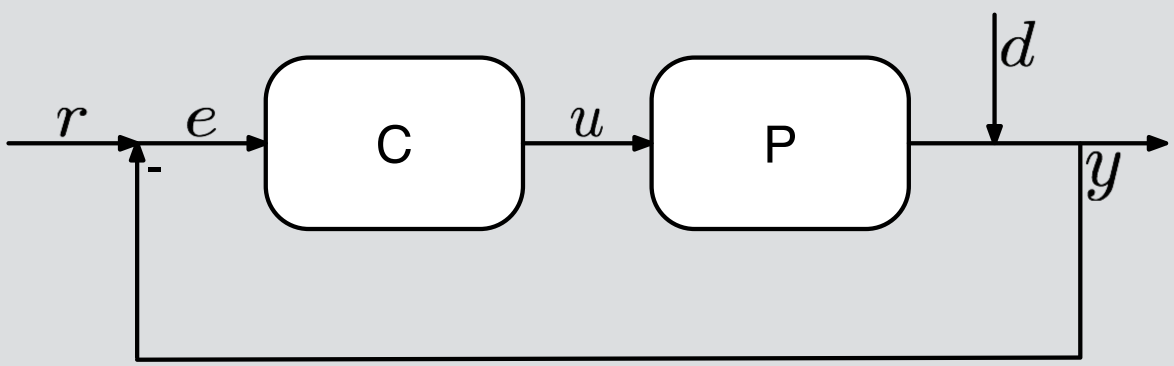
The role of feedback

- Feedback is central in control systems:
  - ✓ In PID the tracking error $e = r - y$ alone determines the control action
  - ✓ In MPC an OCP is resolved at each time because a new state measurement is available
- Necessary to cope with disturbances and modeling errors
- How to achieve offset-free tracking with persistent errors?
  … a matter of personal preference among different methods

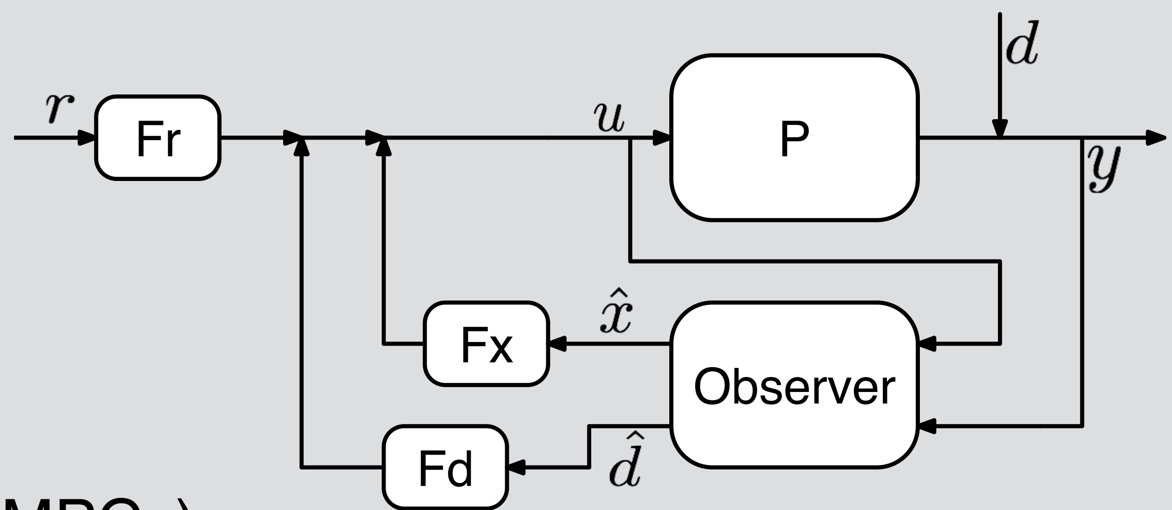
Introduction to offset-free MPC

Different approaches to offset-free control

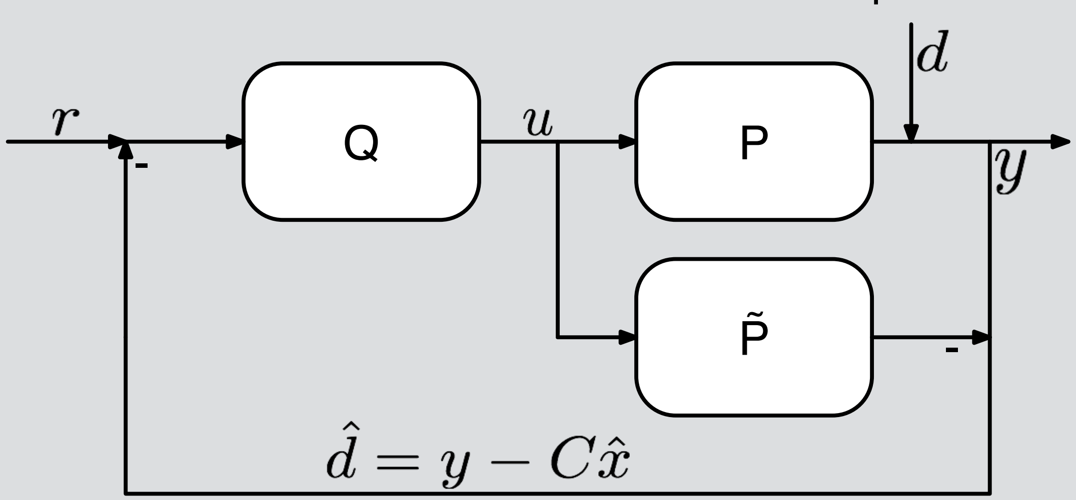
PID control:
tracking error is integrated



LQG (extended):
state/disturbance observer and compensator



IMC (but also early MPCs)
disturbance evaluation and compensation



Introduction to offset-free MPC

Objectives

- Present an **updated** and **comprehensive** description of offset-free MPC algorithms for nonlinear (and linear) discrete-time systems
- Clarify the **main concepts**
- Show **new results**
- Highlight **subtleties** by means of **challenging** applications

Outline of this talk

1. Offset-free **Nonlinear** MPC: principles, known and new results
2. Offset-free **Economic** MPC: idea, design, and results
3. Offset-free **Linear** MPC: new equivalence results
4. Application **examples** and conclusions

Part I: Offset-free Nonlinear MPC

Offset-free NMPC: assumptions

Plant

$$x_p^+ = f_p(x_p, u, w_p)$$
$$y = h_p(x_p, v_p)$$

Nominal model

$$x^+ = f(x, u)$$
$$y = h(x)$$

Disturbances

$$w := f_p(x_p, u, w_p) - f(x, u), \qquad v := h_p(x_p, v_p) - h(x)$$

General assumptions and requirements

- Output y is measur
- Disturbances (
- Input and output
  $u \in \mathbb{U}, \;\; y \in \mathbb{Y}$

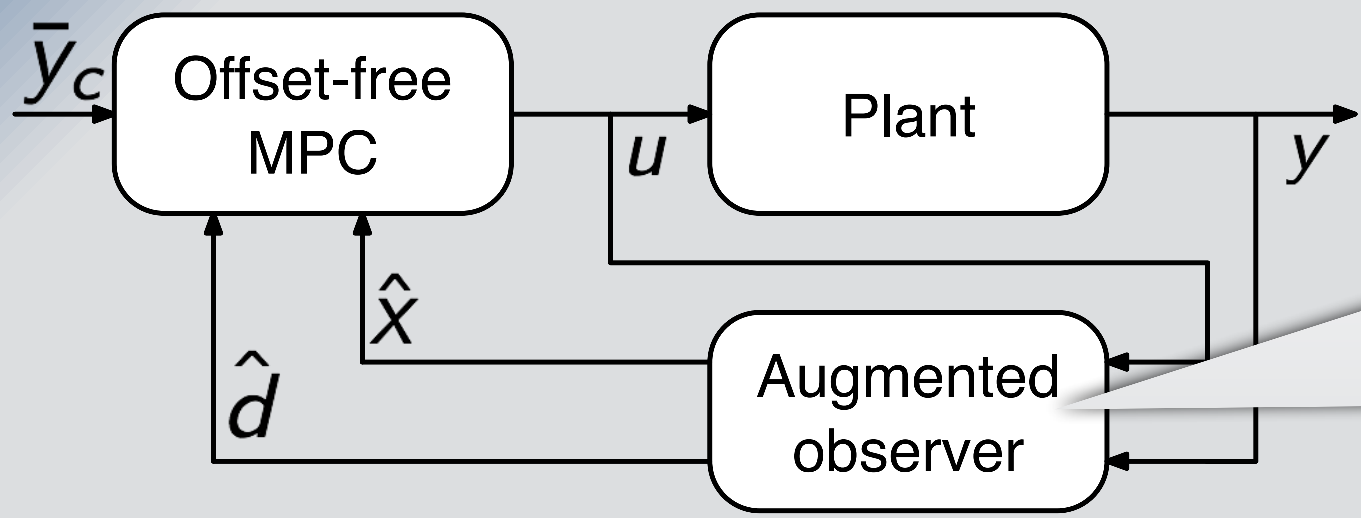
Assume that
disturbances (w,v) are
asymptotically constant

Offset-free tracking problem

- Controlled variable: $\quad y_c = r(y)$
- Design an output feedback MPC law $\quad u = \kappa(y)$ such that

$$\lim_{k\to\infty} y_c(k) = \bar{y}_c$$

Offset-free NMPC: Augmented model and observability



Augmented model

$$x^+ = F(x, d, u)$$

$$d^+ = d$$

$$y = H(x, d)$$

Requirements and remarks

- Consistent with nominal dynamics

$$F(x, 0, u) = f(x, u), \quad H(x, 0) = h(x)$$

- Detectable (observable)
  - ✓Possible iff the nominal system is detectable (observable)
  - ✓Dimension of ***d*** is limited $\quad n_d \leq n_y$
- The disturbance state ***d*** follows an integral dynamics

Offset-free NMPC: Observer design

Steady-state Kalman filter like framework

Predictor

$$\hat{x}_{k|k-1} = F(\hat{x}_{k-1|k-1}, \hat{d}_{k-1|k-1}, u_{k-1})$$

$$\hat{d}_{k|k-1} = \hat{d}_{k-1|k-1}$$

$$\hat{y}_{k|k-1} = H(\hat{x}_{k|k-1}, \hat{d}_{k|k-1})$$

Prediction error

$$e_k = y_k - \hat{y}_{k|k-1}$$

Filtering

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \kappa_x(e_k)$$

$$\hat{d}_{k|k} = \hat{d}_{k|k-1} + \kappa_d(e_k)$$

This generally implies $n_d = p$

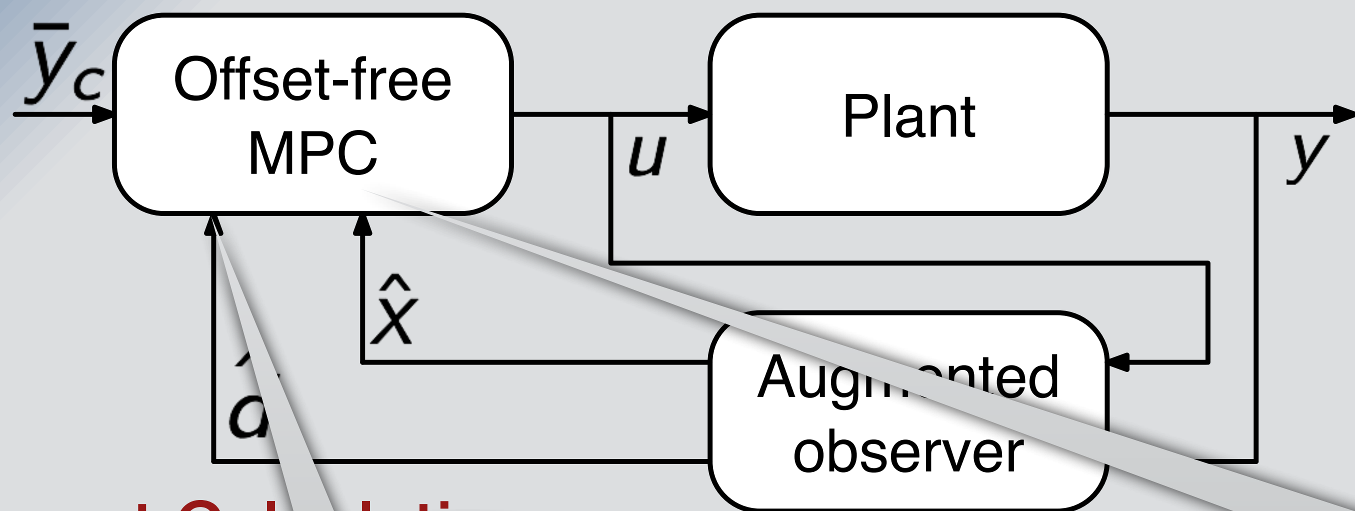
Assumption (Nominal c

- Functions (κ_x, κ_d) are continuous and satisfy

$$\kappa_x(0) = 0, \qquad \kappa_d(e) = 0 \Leftrightarrow e = 0$$

- The observer is nominally stable for the augmented model

Offset-free NMPC: Target and optimal control problems



Target Calculation

$$\min_{x,u,y} \ell_s(y, u)$$

subject to:

$$x = F(x, \hat{d}_{k|k}, u)$$

$$y = H(x, \hat{d}_{k|k})$$

$$r(y) = \bar{y}_c$$

$$y \in \mathbb{Y}, \quad u \in \mathbb{U}$$

Deviation variables

$$\tilde{x}_i := x_i - x_s(k)$$

$$\tilde{u}_i := u_i - u_s(k)$$

FHOCP

$$\min_{\mathbf{x},\mathbf{u}} \sum_{i=0}^{N-1} \ell(\tilde{x}_i, \tilde{u}_i) + V_f(\tilde{x}_N)$$

subject to:

$$x_0 = \hat{x}(k)$$

$$x_{i+1} = F(x_i, \hat{d}_{k|k}, u_i)$$

$$H(x_i, \hat{d}_{k|k}) \in \mathbb{Y}, \qquad u_i \in \mathbb{U}$$

$$\tilde{x}_N \in \mathbb{X}_f$$

Target and FHOCP can be merged together [LAAC08]

Offset-free NMPC: main results

Offset-free tracking property (MB02, PR03, MM12)

- Assume that the target and the FHOCP are feasible at all times, and the closed-loop system reaches an equilibrium (u_∞, y_∞)
- It follows that:

$$r(y_\infty) = \bar{y}_c$$

Sketch of proof.

At equilibrium

$$\hat{d}_\infty = \hat{d}_\infty + \kappa_d(y_\infty - \hat{y}_\infty)$$

$$\kappa_d(y_\infty - \hat{y}_\infty) = 0 \quad \Rightarrow \hat{y}_\infty = y_\infty$$

FHOCP solution (*l(x,u)* is PD)

$$(\hat{x}_\infty, u_\infty) = (x_{s,\infty}, u_{s,\infty})$$

From the target calculation

$$r(y_\infty) = r(\hat{y}_\infty) = r(y_{s,\infty}) = \bar{y}_c$$

Offset-free NMPC: main results

Asymptotic stability and offset-free control in state feedback

- If $(y = x)$ choose the following augmented system $(B_d K_d = I)$

$$F(x,u,d) := f(x,u) + B_d d, \; H(x,d) := x, \; \kappa_x(e) = e, \; \kappa_d(e) = K_d e$$

- It follows that:

$$\lim_{k\to\infty} x_k - \hat{x}_{k|k-1} = 0, \; \lim_{k\to\infty} \hat{d}_{k|k-1} = B_d^{-1}\bar{w}$$

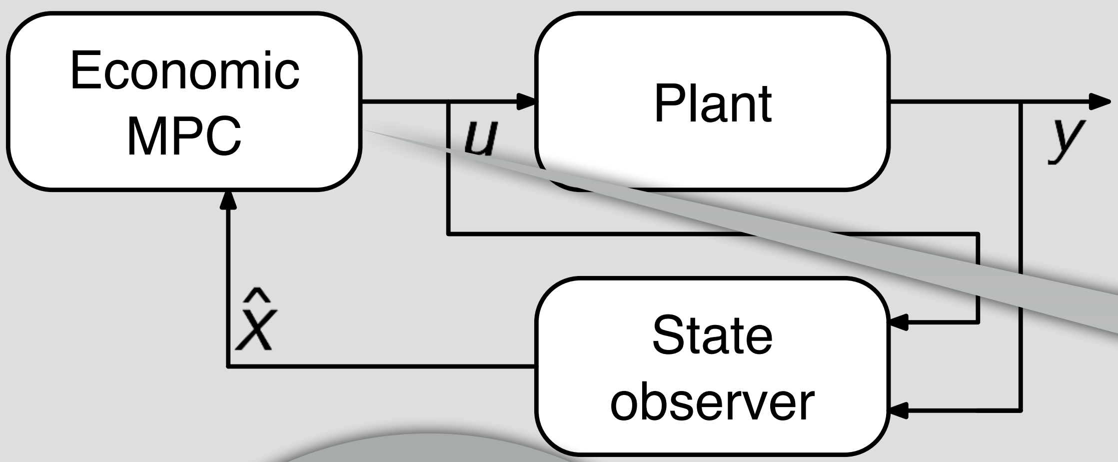
Remarks

1. The closed-loop system is assumed to reach a stable equilibrium
2. Establish when this occurs is very hard (e.g., see [BFS13])
3. How to design a nonlinear observer augmented system is a tricky task
4. When the state is measurable offset-free design and analysis is simpler

Part II: Offset-free Economic MPC

Economic MPC: a brief introduction

Output feedback framework



$$\min_{\mathbf{x},\mathbf{u}} \sum_{i=0}^{N-1} \ell_e(h(x_i), u_i)$$

subject to:

$$x_0 = \hat{x}(k)$$

$$x_{i+1} = f(x_i, u_i)$$

$$u_i \in \mathbb{U}, \;\; h(x_i) \in \mathbb{Y}$$

$$x_N = x_s$$

$$\min_{x,u} \ell_e(h(x), u)$$

subject to:

$$x = f(x, u)$$

$$u \in \mathbb{U}, \;\; h(x) \in \mathbb{Y}$$

(x_s, u_s) is an asymptotically stable equilibrium in which the economic cost is minimized [DAR11, AAR12]

Offset-Free Economic MPC: what do you mean?

Plant

$$x^+ = f(x,u) + w$$
$$y = h(x) + v$$

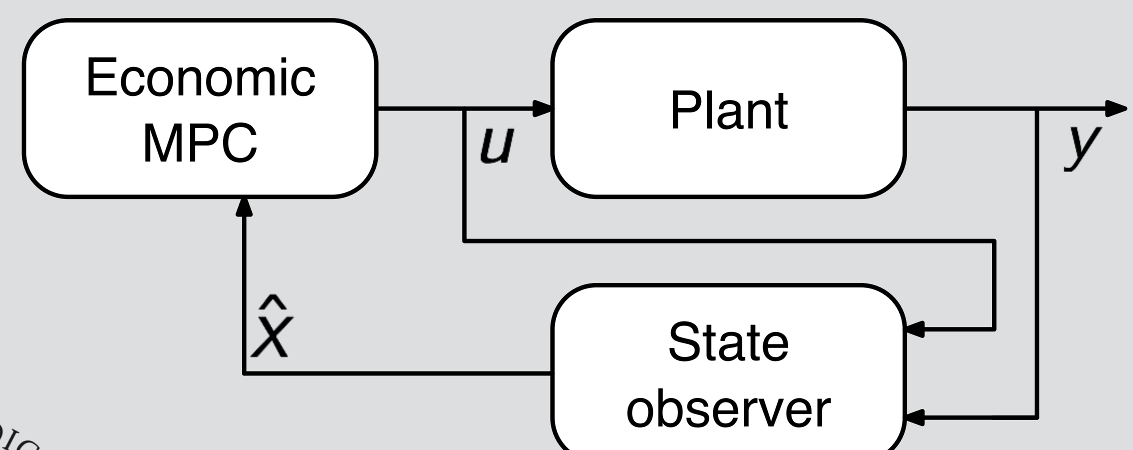
Best equilibrium

$$\min_{\bar{x}_p, \bar{u}, \bar{y}} \ell_e(\bar{y}, \bar{u})$$

subject to:

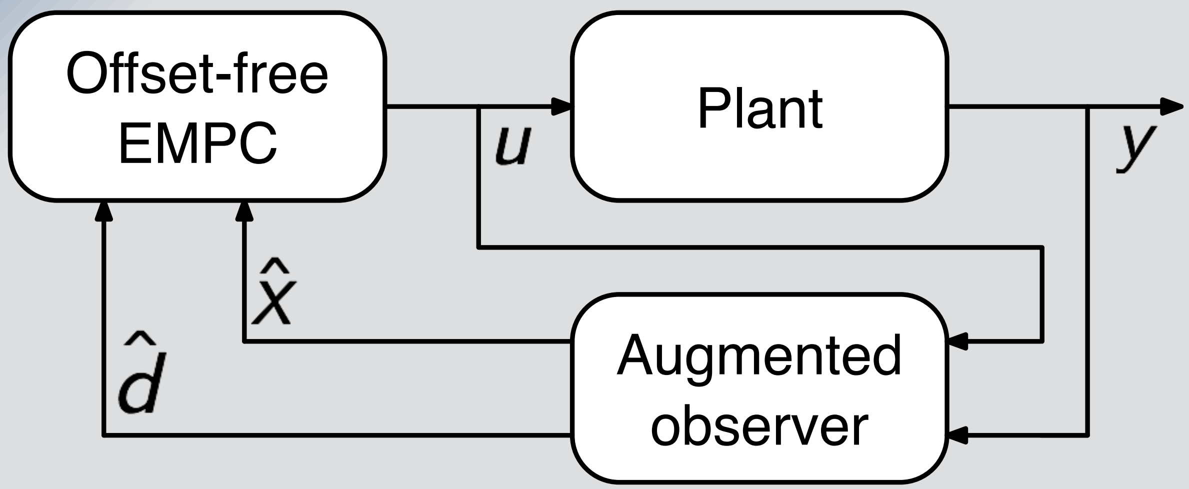
$$\bar{x}_p = f(\bar{x}_p, \bar{u}) + \bar{w}$$
$$\bar{y} = h(\bar{x}_p) + \bar{v}$$
$$\bar{u} \in \mathbb{U}, \qquad \bar{y} \in \mathbb{Y}$$

Nominal EMPC



- Where do we go if we use nominal EMPC?
  - ✓Unclear…
- Can we reach the best equilibrium?
  - ✓No with nominal EMPC…

Offset-free Economic MPC: design



Step 1. Augmented observer

Evaluate $(\hat{x}_{k|k}, \hat{d}_{k|k})$

Step 2. Current best equilibrium

$$(x_s(k), u_s(k), y_s(k)) := \arg\min_{x,u,y} \ell_e(y, u)$$

subject to:

$$x = F(x, \hat{d}_{k|k}, u), \qquad y = H(x, \hat{d}_{k|k})$$

$$u \in \mathbb{U} \leq 0, \qquad y \in \mathbb{Y}$$

Step 3. FHOCP

$$\min_{\mathbf{x},\mathbf{u}} \sum_{i=0}^{N-1} \ell_e(H(x_i, \hat{d}_{k|k}), u_i)$$

subject to:

$$x_0 = \hat{x}_{k|k}$$

$$x_{i+1} = F(x_i, \hat{d}_{k|k}, u_i)$$

$$u_i \in \mathbb{U}, \qquad H(x_i, \hat{d}_{k|k}) \in \mathbb{Y}$$

$$x_N = x_s(k)$$

Offset-free Economic MPC: main result

Convergence to the optimal equilibrium [PGA15]

- Assume that:
  - ✓ Equilibrium problem (Step 2.) and the economic MPC problem (Step 3.) are feasible at all times
  - ✓ The closed-loop system reaches an equilibrium with input and output (u_∞, y_∞)
- The achieved ultimate cost $\ell(y_\infty, u_\infty)$ is the best achievable for the actual plant.

The statement/proof in [PGA15] need a revision :(

Part III: Offset-free Linear MPC

Offset-free LMPC: Disturbance model

Nominal model

$$f(x,u) := Ax + Bu$$

$$h(x) := Cx, \; r(y) := Dy$$

Requirements

(A,B) stabilizable, (C,A) detectable

$$\text{rank} \begin{bmatrix} A - I & B \\ DC & 0 \end{bmatrix} = n_x + n_r$$

Disturbance model and observer

- The augmented system takes the form:

$$F(x,u,d) := Ax + Bu + B_d d, \; H(x,d) := Cx + C_d d$$

- The linear observer is: $\kappa_x(e) := K_x e, \; \kappa_d(e) := K_d e,$

Main results (design guidelines) [MB02,PR03]

- The augmented system is **observable** iff $\text{rank} \begin{bmatrix} A - I & B_d \\ C & C_d \end{bmatrix} = n_x + n_d$
- There exist **valid** (B_d, C_d) iff $n_d \leq n_y$
- If $n_d = n_y$ then K_d is **invertible**

A known alternative for LMPC: The velocity model

A known "alternative" [PR01, W04, GAM08, BFS12-13]

Extended model (state/control increments)

$$\delta x^+ = A\delta x + B\delta u$$

$$y^+ = y + C\delta x^+ = y + CA\delta x + CB\delta u$$

Requirements

(A,B) stabilizable, (C,A) detectable

$$\text{rank}\begin{bmatrix} A-I & B \\ C & 0 \end{bmatrix} = n_x + n_y$$

Remarks on the observer

- An observer is required to estimate (at least) δx
- Since y is measured, we can use $(K_y = I)$, i.e. a deadbeat observer

Main results (design guidelines)

- The velocity model is detectable if (A,C) is detectable
- The velocity model is stabilizable if (A,B) is stabilizable and the rank condition holds true: $n_y \leq n_u$
- A deadbeat observer $(K_y = I)$ is stable iff $(A - K_{\delta x} CA)$ is stable

The velocity model is not an alternative method

Main result [P15]

Consider the velocity form model, with a stable output deadbeat observer gain:

$$K_\delta = \begin{bmatrix} K_{\delta x} \\ I \end{bmatrix}$$

This is equivalent to using the following disturbance model and observer gains:

$$B_d = K_{\delta x}, \quad C_d = I - CK_{\delta x}, \quad K_x = K_{\delta x}, \quad K_d = I$$

...some thoughts

The velocity model is just a *particular case* of disturbance model. So, the question about which method is *better* makes no longer sense...

Offset-free LMPC: Further equivalence results in state feedback case

Some considerations

- When $y = x$, we can use a full deadbeat observer
  - ✓ Disturbance model:

$$\mathrm{rank}(B_d) = n, \quad C_d = 0, \quad K_x = I, \quad K_d = B_d^{-1}$$

  - ✓ Velocity model: $K_{\delta x} = I, \; K_y = I$

Equivalence result

Full
state disturbance model
with deadbeat observer

Consider the velocity form model with full deadbea

This is equivalent to using the following disturbance m
observer gains:

$$B_d = I, \quad C_d = 0, \quad K_x = I, \quad K_d = I$$

Another "alternative" method

State disturbance observer [T14]

- Introduced by Tatjewski [T14] to "avoid" state augmentation
- Choose an observer gain K such that $(A - KCA)$ is Hurwitz
- Define and compute a state disturbance and output bias

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K(y_k - C\hat{x}_{k|k-1})$$
$$d_x(k) := \hat{x}_{k|k} - \hat{x}_{k|k-1} = K(y_k - C\hat{x}_{k|k-1})$$
$$d_y(k) := y_k - C\hat{x}_{k|k}$$

- Use the following prediction model in the FHOCP:

$$x^+ = Ax + Bu + d_x(k)$$
$$y = Cx + d_y(k)$$

It is the
following disturbance model [P15]:
$B_d = K, \quad C_d = I - CK, \quad K_x = K, \quad K_d = I$

Yet another "alternative" method

A different velocity algorithm [GAM08]

- It uses the following augmented system with velocity input:

$$\begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix} + \begin{bmatrix} B \\ I \end{bmatrix} \delta u_k$$

$$y_k = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix}$$

- Both state components are estimated (at time *k*):

$$\hat{x}_{k|k} = \hat{x}_{k-1|k} + K_x e_k$$
$$\hat{u}_{k-1|k} = \hat{u}_{k-1|k-1} + K_u e_k$$

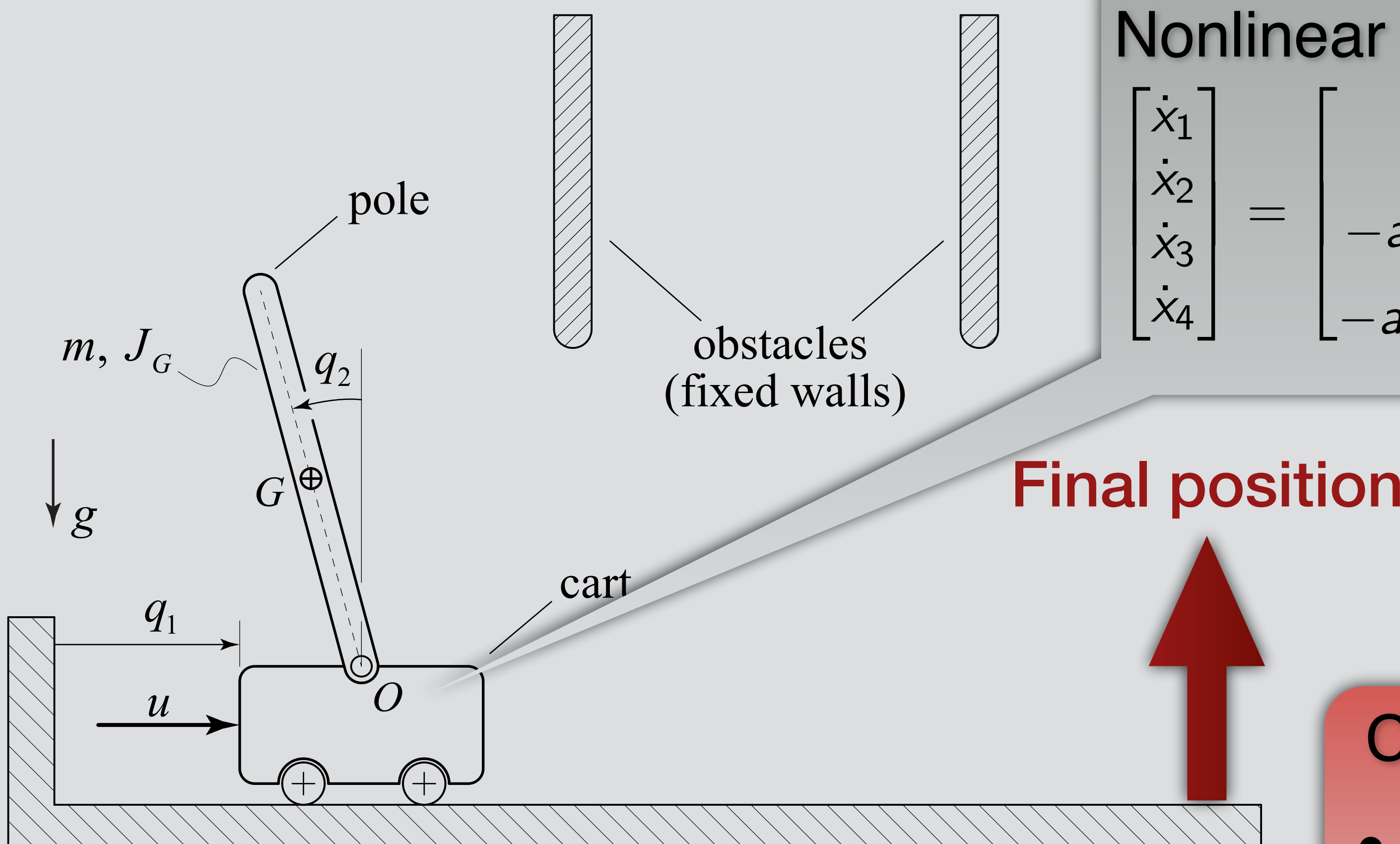
It is the **input disturbance model**:

$$B_d = B, \quad C_d = 0, \quad K_x = K_x, \quad K_d = K_u$$

Part IV: Application examples and conclusions

Application of offset-free NMPC

A cart-pole system with obstacles



Nonlinear dynamics

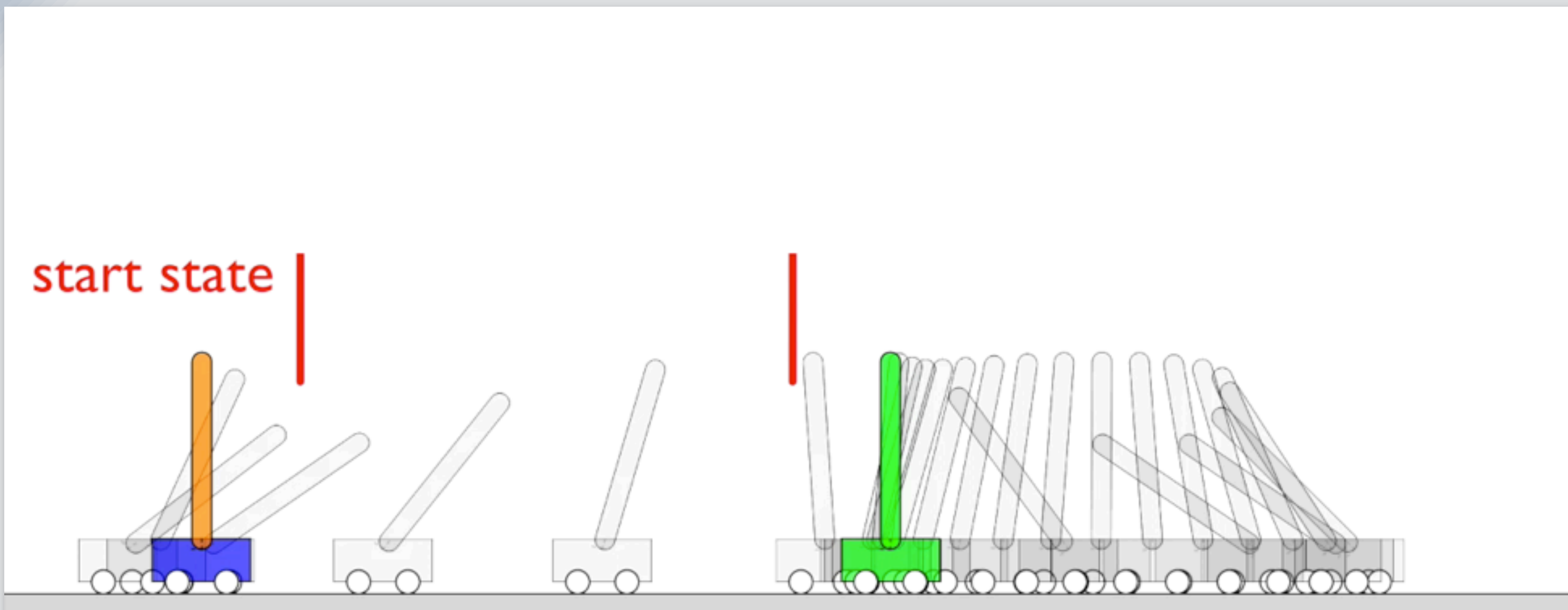
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ -a(x_2)mls_2\left(J_O x_4^2 - mglc_2\right) \\ -a(x_2)mls_2\left(mlx_4^2 c_2 - M_t g\right) \end{bmatrix} + a(x_2) \begin{bmatrix} 0 \\ 0 \\ J_O \\ mlc_2 \end{bmatrix} u$$

Control task specifications

- Control force is constrained
- Reach final position stably and avoiding obstacles
- Unknown disturbance forces occur

Application of offset-free NMPC

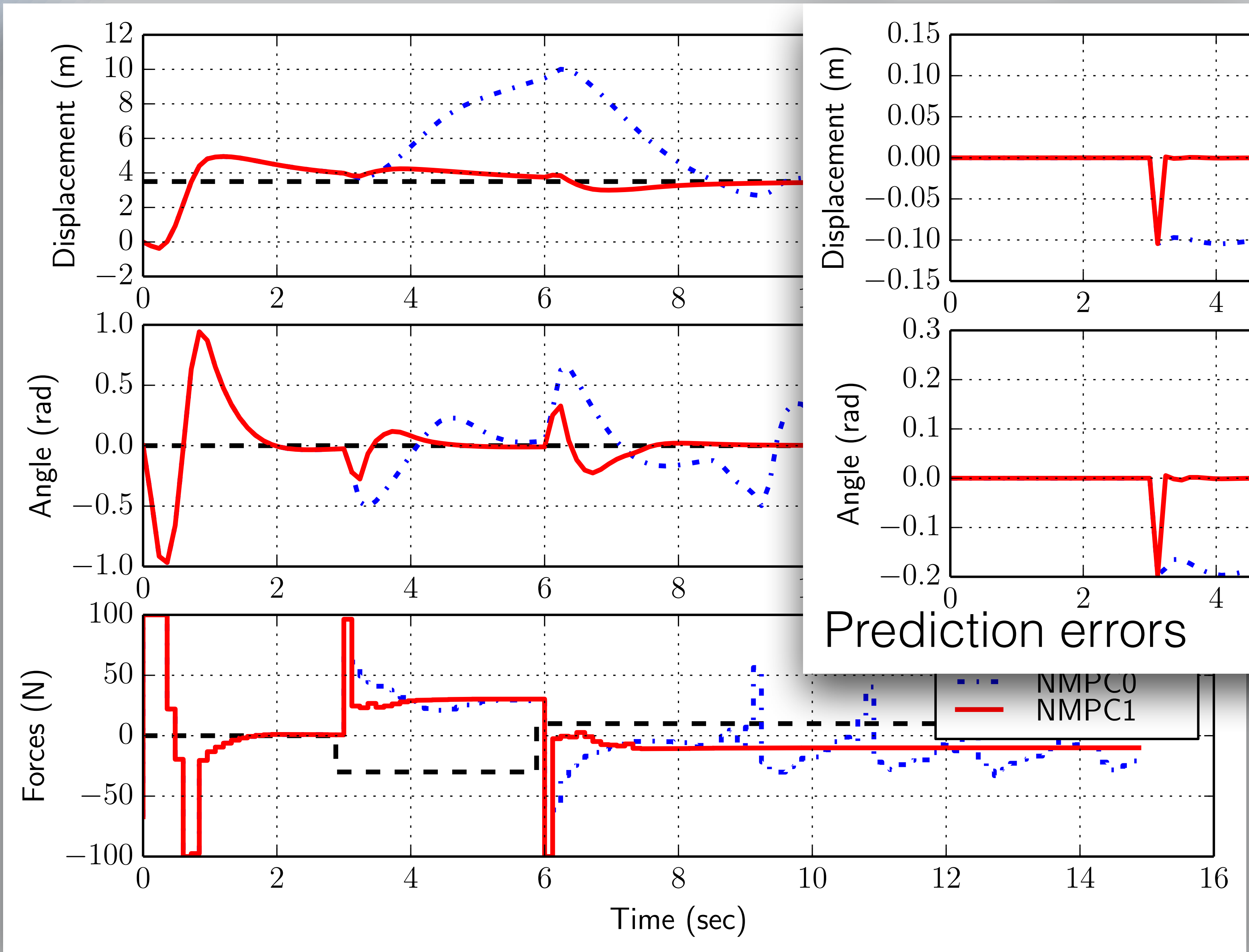
Showing the desirable behavior

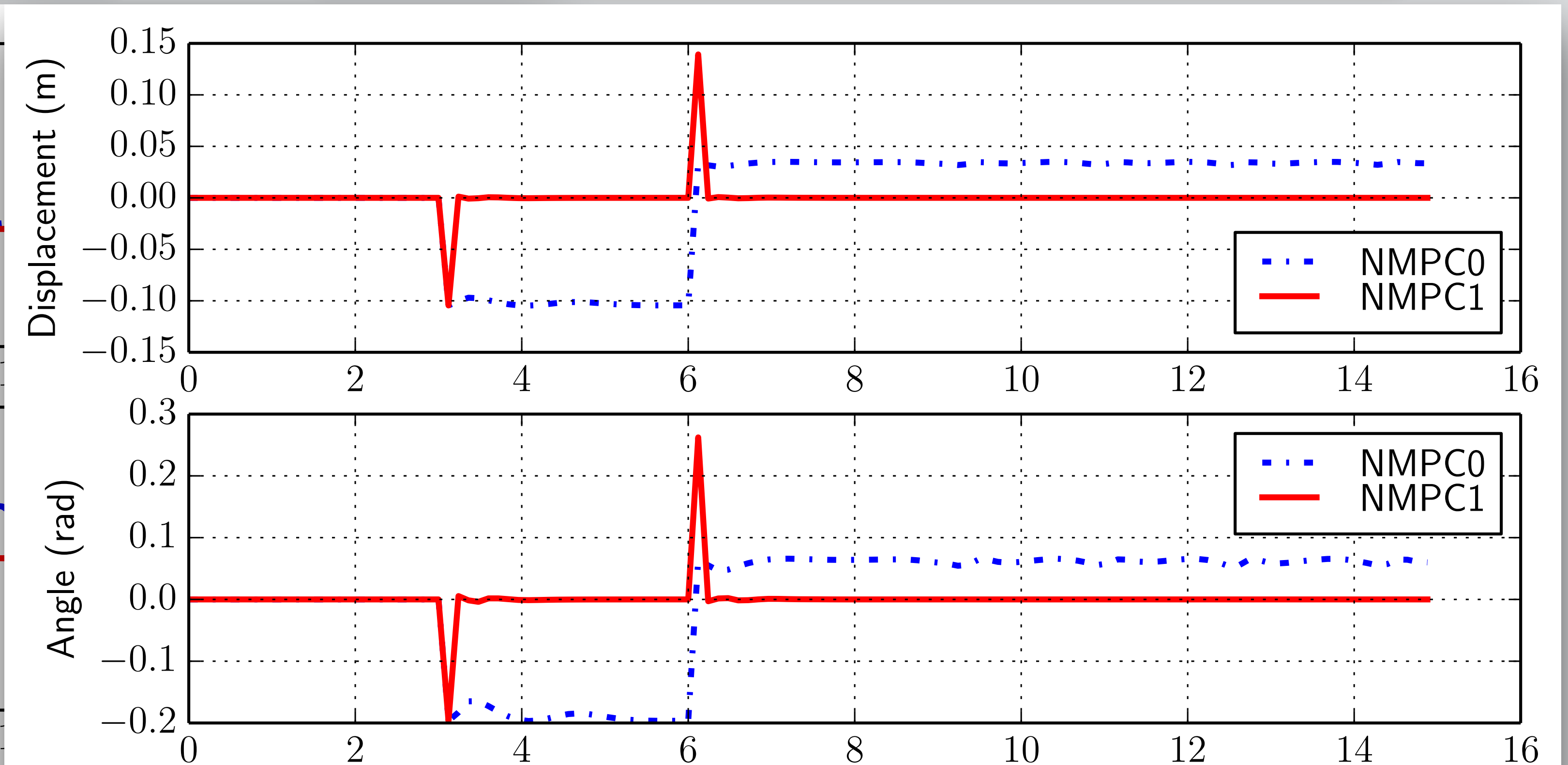


Application of offset-free NMPC

Cart-pole with obstacles: full state feedback

NMPC0 uses no disturbance model vs NMPC1 uses state disturbance model

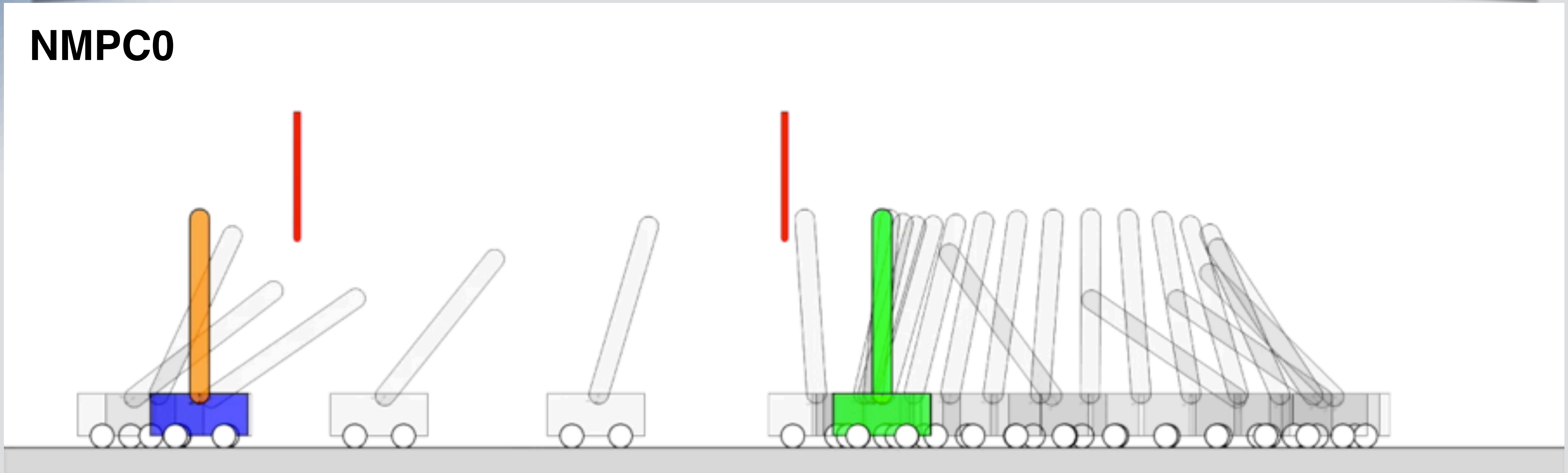


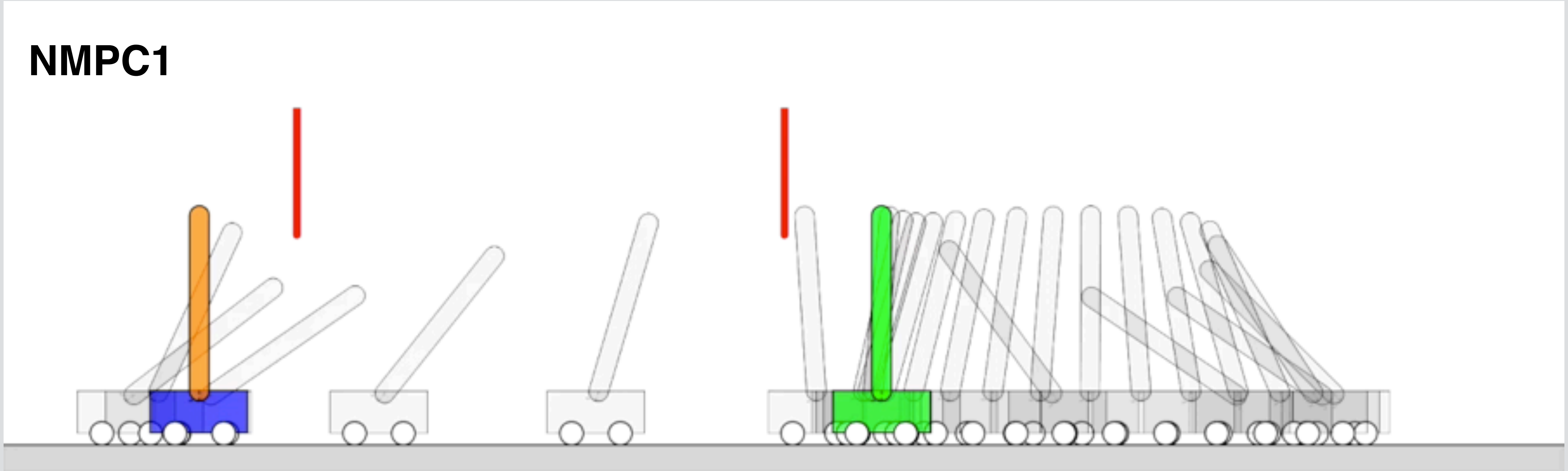


Prediction errors

Application of offset-free NMPC

Cart-pole with obstacles: full state feedback

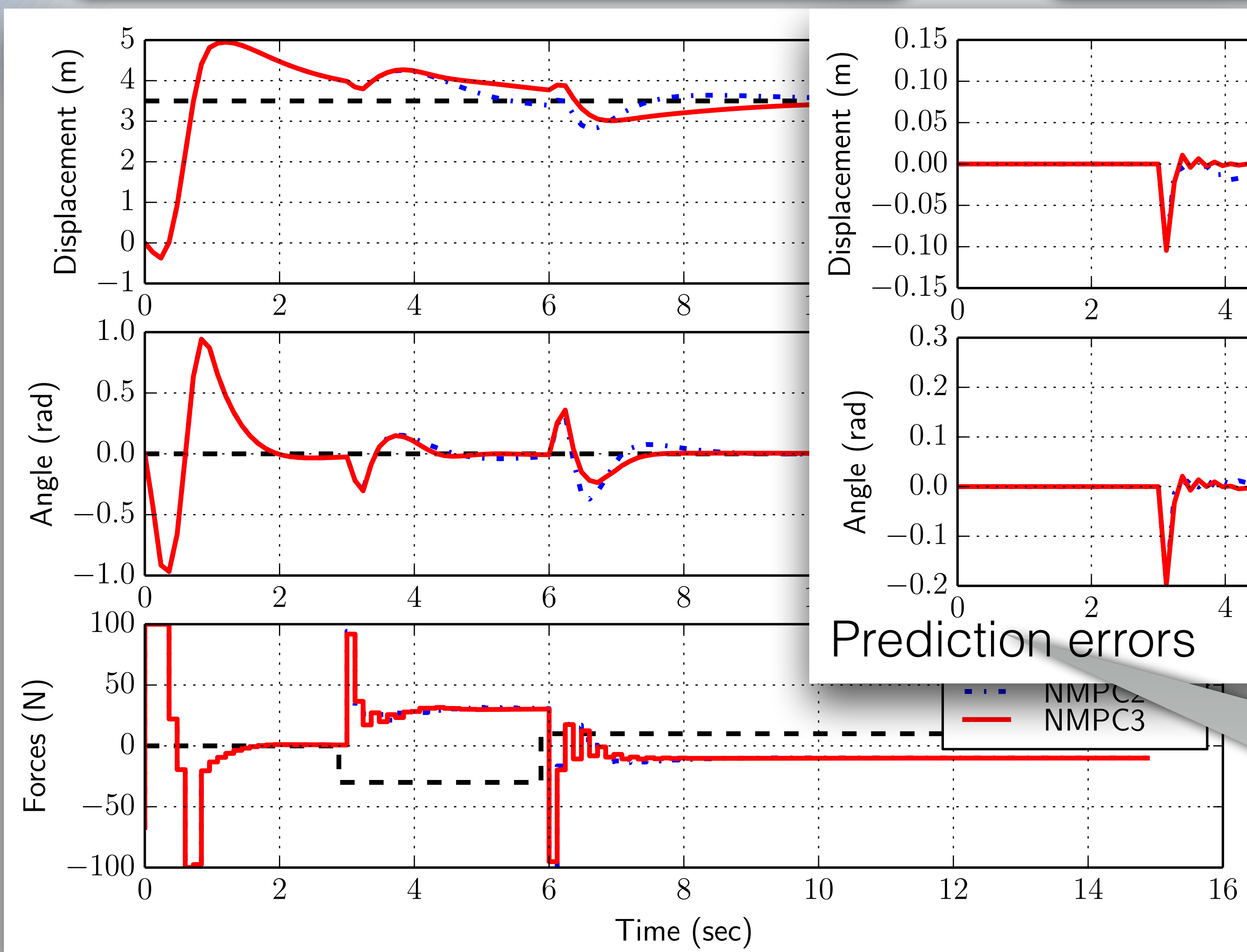


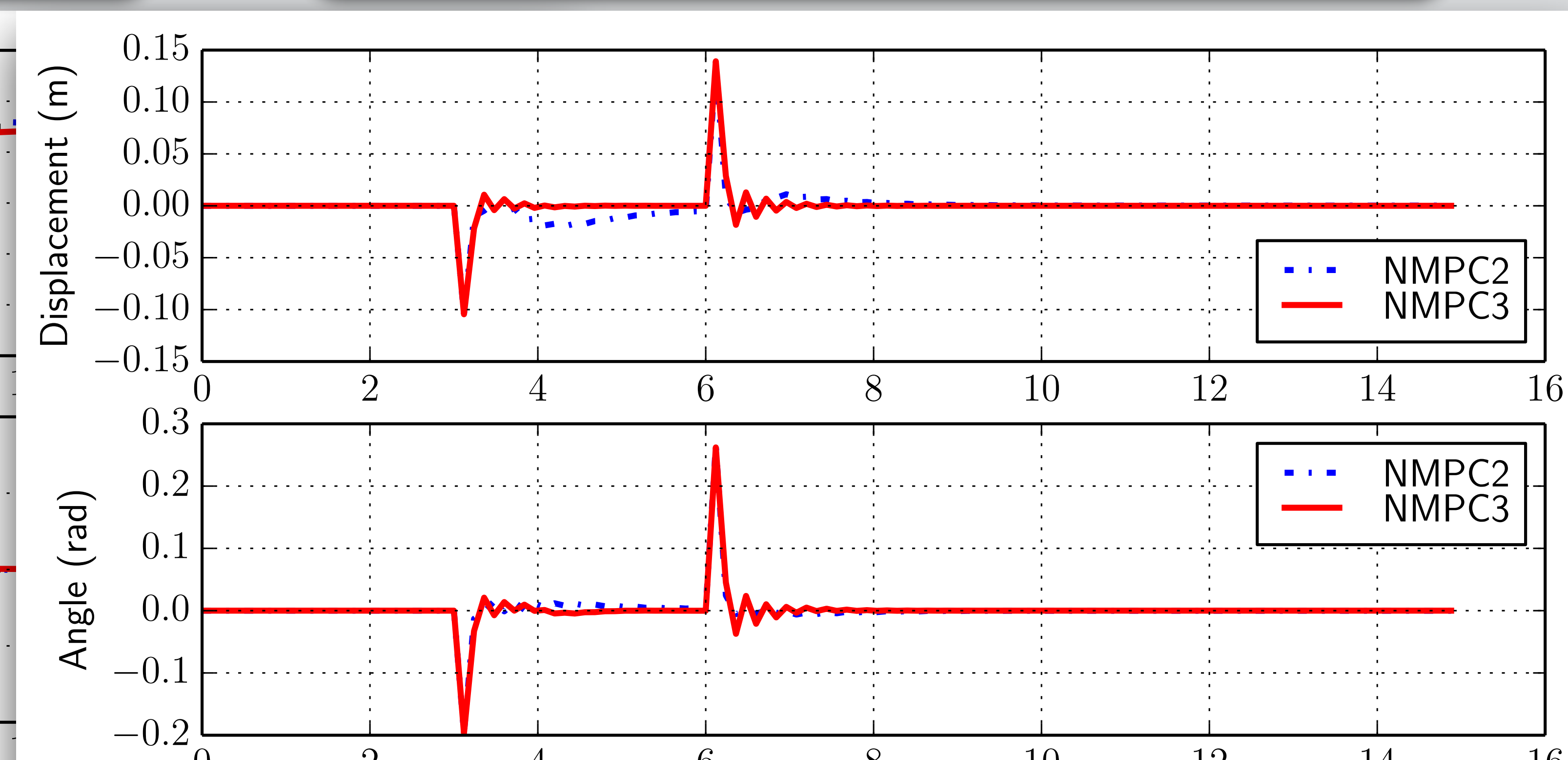


Application of offset-free NMPC

Cart-pole with obstacles: output feedback $n_y = 2$

NMPC2 uses a linear input disturbance model ($n_d = 1$) vs NMPC3 uses a linear input/output disturbance model ($n_d = 2$)



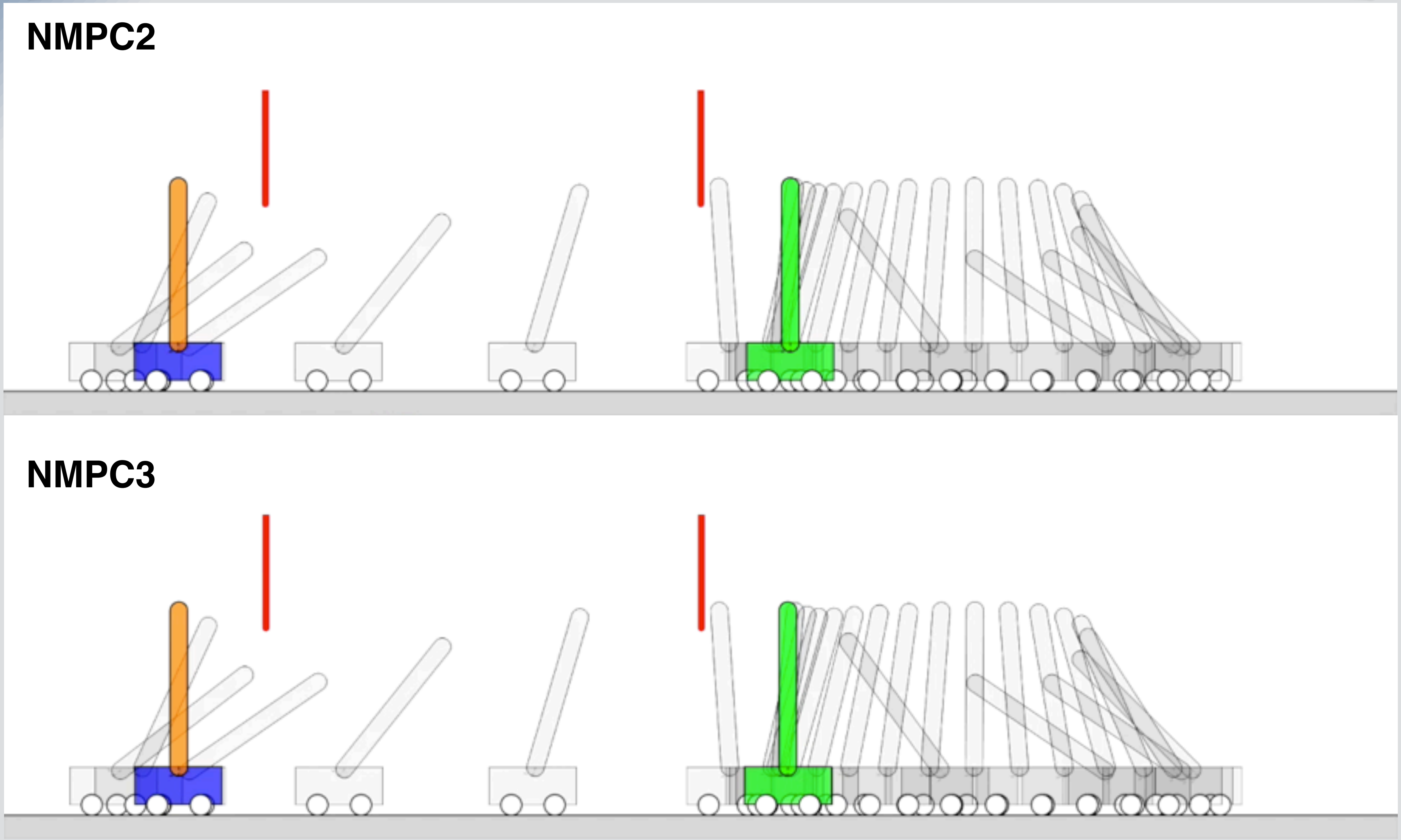


Prediction errors

Take Home Message

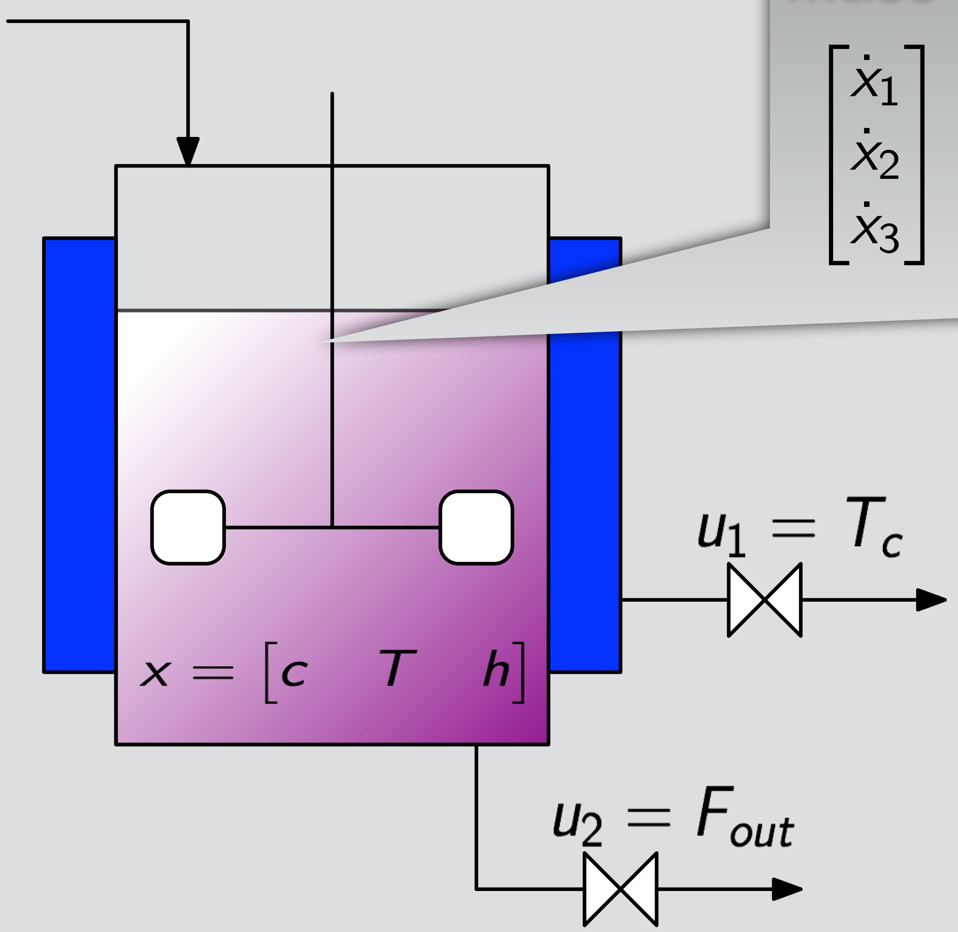
Can use $n_d < n_y$ in lucky cases
So, use always $n_d = n_y$

Application of offset-free NMPC



Application of offset-free EMPC

Continuous Stirred Tank Reactor with exothermal reaction



Mass and energy balances

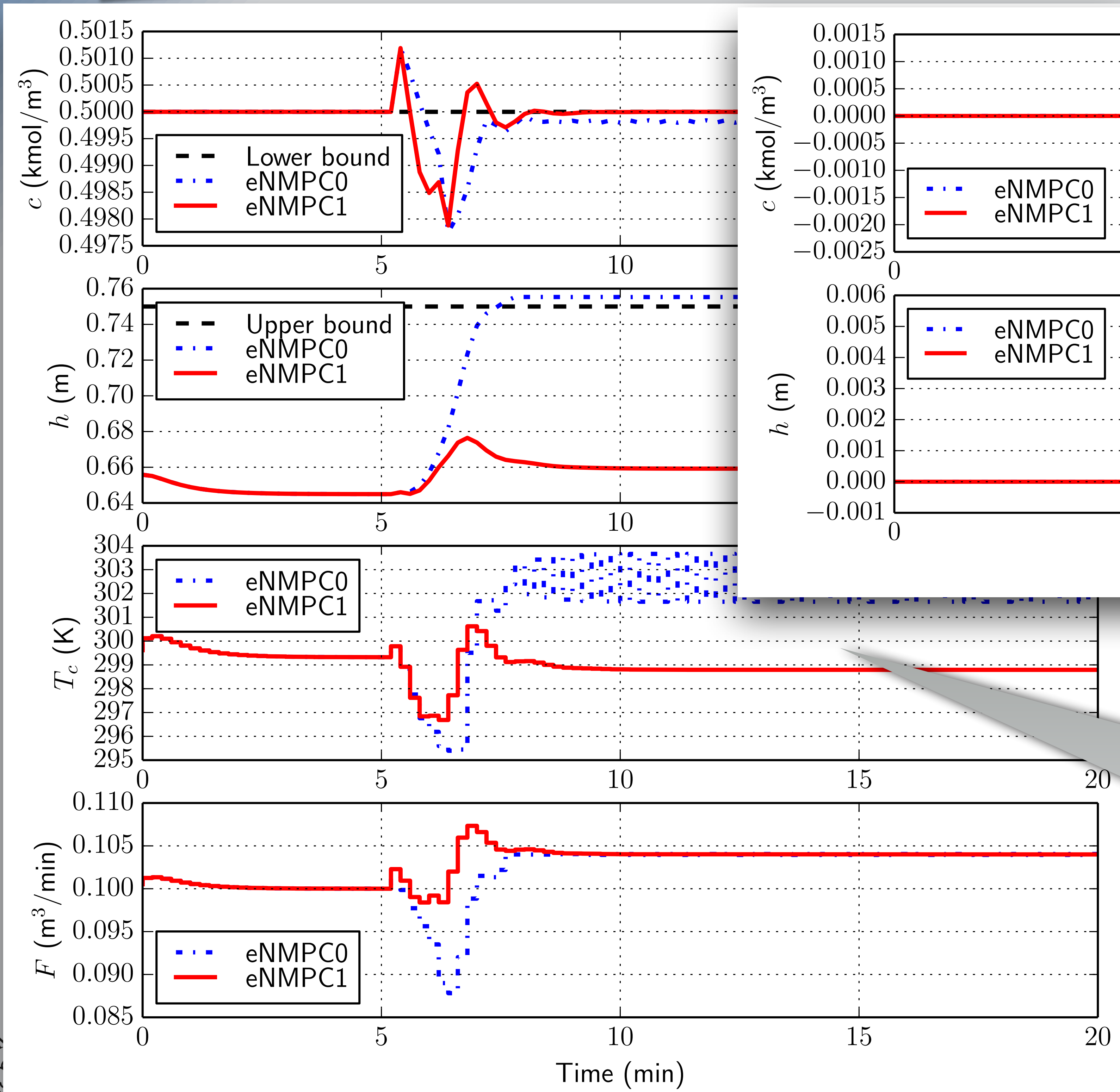
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \frac{F_0 c_0 - u_2 x_1}{\pi r^2 x_3} - k_0 \exp(-\frac{E}{R x_2}) x_1 \\ \frac{F_0 (T_0 - x_2)}{\pi r^2) x_3} - \frac{\Delta H}{\rho C_p} k_0 \exp(-\frac{E}{R x_2}) x_1 + \frac{2 U_0}{r \rho C_p} (u_1 - x_2) \\ \frac{F_0 - u_2}{\pi r^2} \end{bmatrix}$$

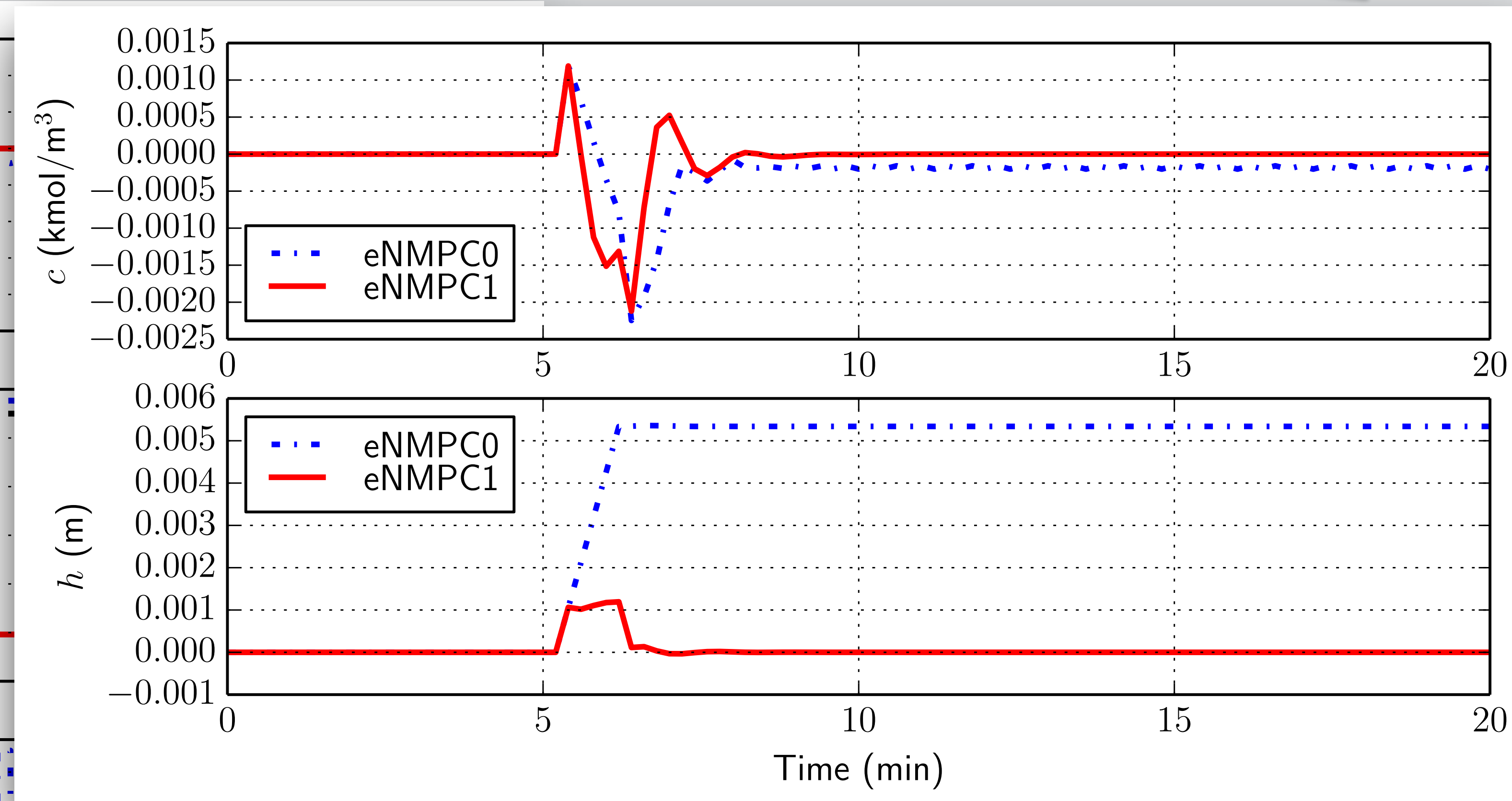
Control task specifications

- Measure (c, h)
- Enforce input and output constraints
- Economic cost $\ell_e(x, u) = x_1 = c$

Application of offset-free EMPC

CSTR: output feedback

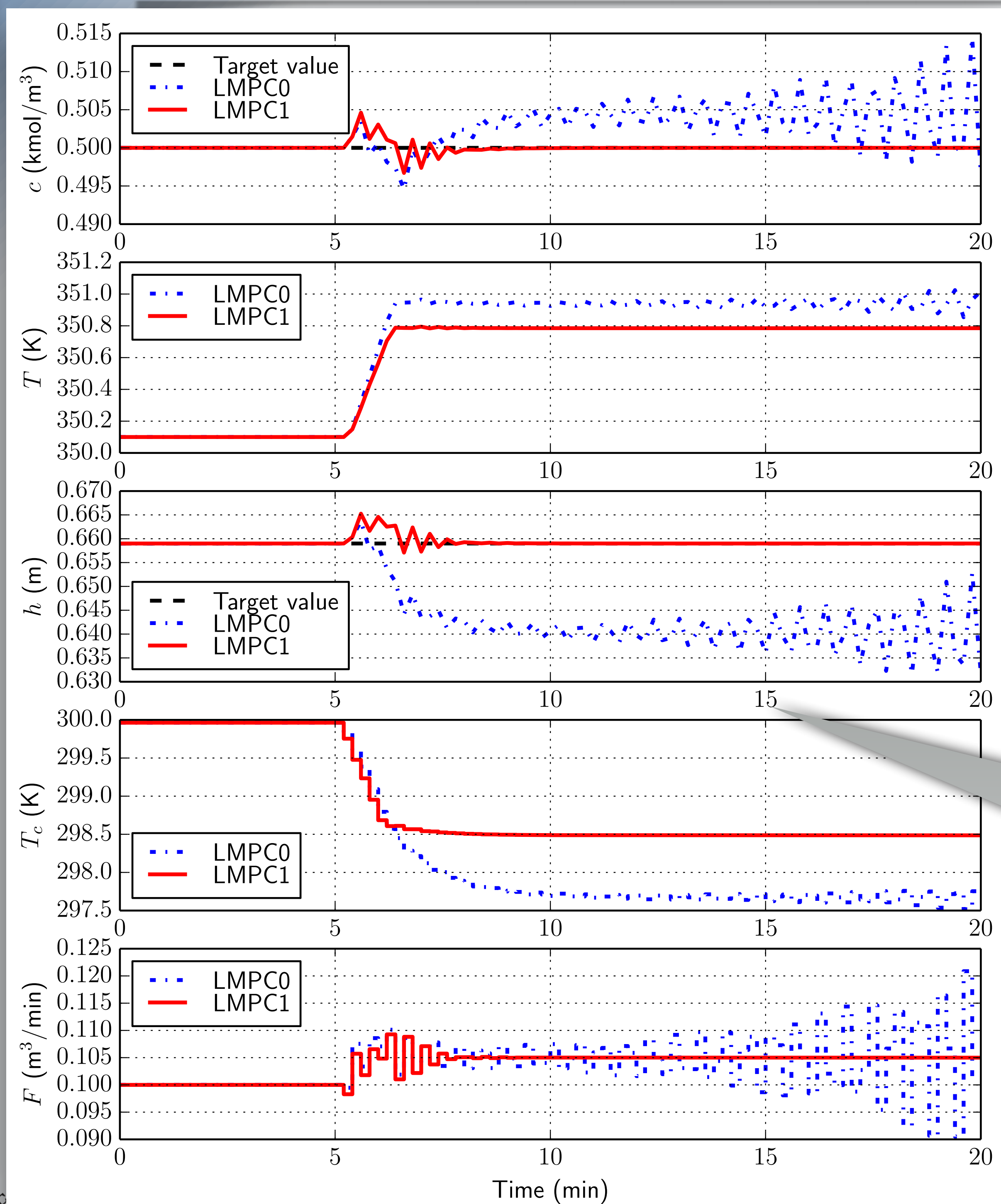




Take Home Message
eMPC pushes towards constraints
Offset-free output prediction is necessary to avoid instability

Application of offset-free LMPC

CSTR: full state feedback, tracking cost



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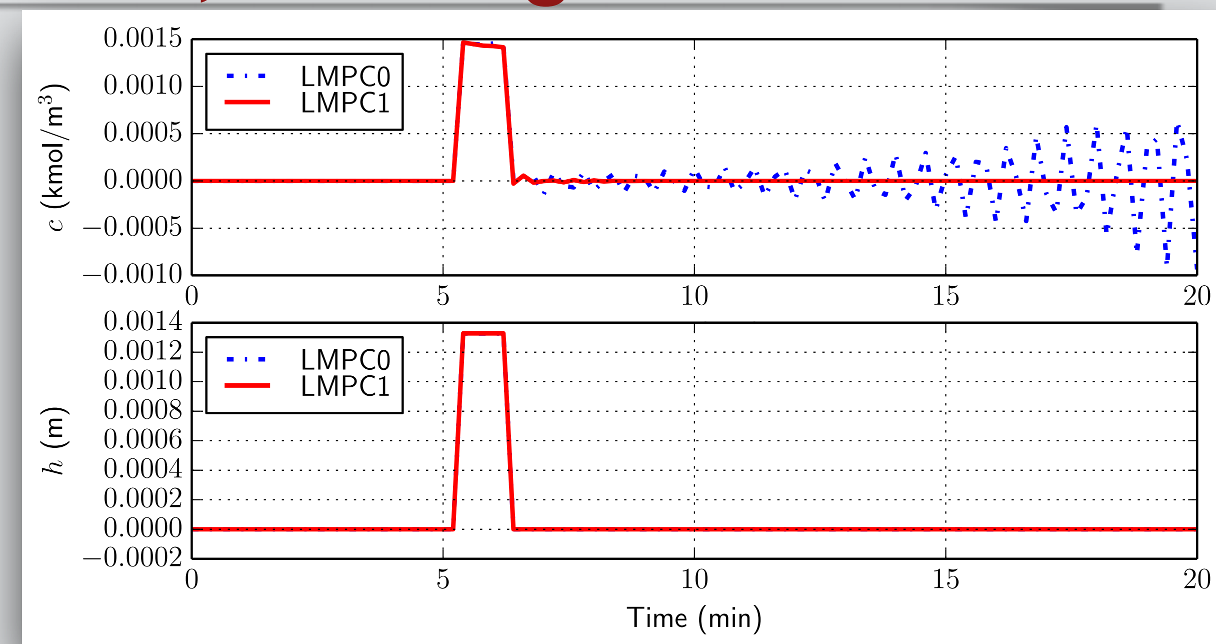

Take Home Message

Offset-free LMPC can be effective on nonlinear systems if correctly designed

Again the choice $n_d < n_y$ is not safe!

Conclusions

- Described the latest advances in the design of offset-free MPC algorithms
- A self-contained summary of the available results for nonlinear MPC:
    - ✓ use of disturbance models and observers
    - ✓ extended the existing asymptotic convergence results
- Extended the concept of offset-free estimation to the design of economic MPC for systems with persistent errors/disturbances.
- Linear MPC:
    - ✓ a commonly known method based on the velocity form model is a particular case of disturbance model, and not an alternative route to offset-free tracking
- Challenging examples of nonlinear processes (controlled by NMPC, LMPC, and EMPC)
    - ✓ highlight the significance of the presented results
    - ✓ emphasize specific subtleties related to the number of used disturbances and to the process dynamics, which may result in an incorrect design

Some open problems...

- Effective nonlinear disturbance modeling coupled with nonlinear (MHE) observers
- More general results for economic MPC
- Robust stability questions
- Time varying systems (tracking and economic MPC)

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