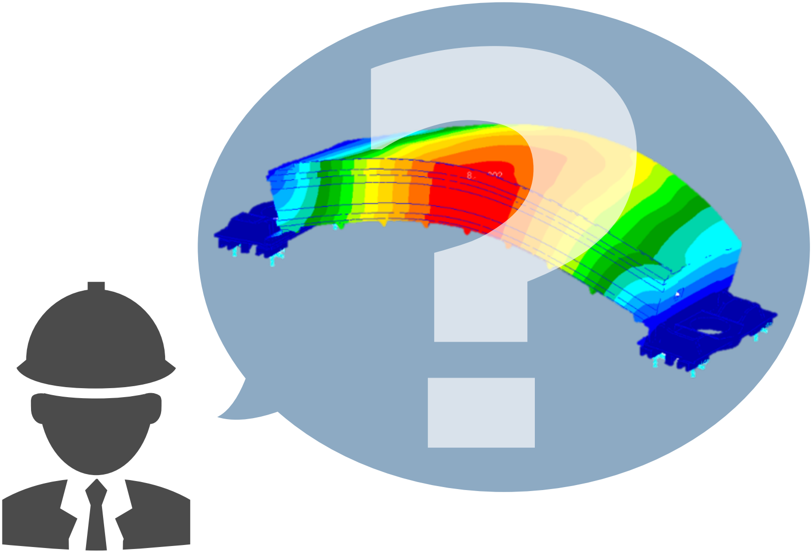

UNCERTAINTY
ROBUST OPTIMIZATION

Uncertainty in Finance

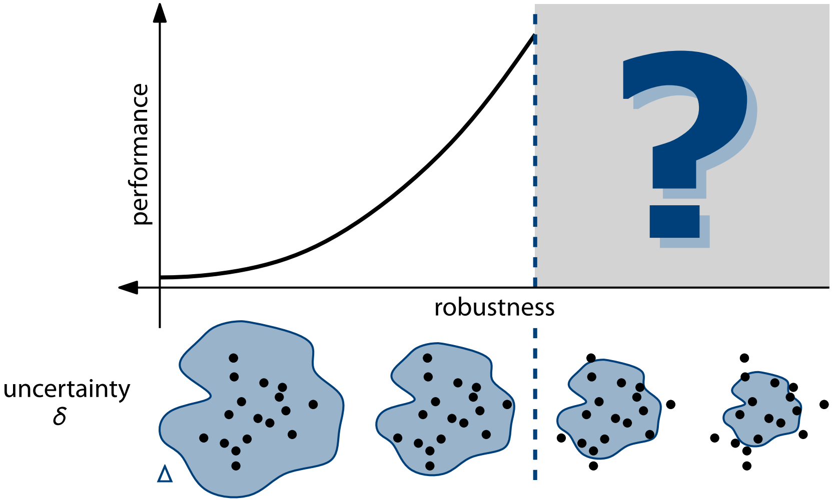
Uncertainty in Finance

Uncertainty in Engineering

Uncertainty in Engineering



The General Picture



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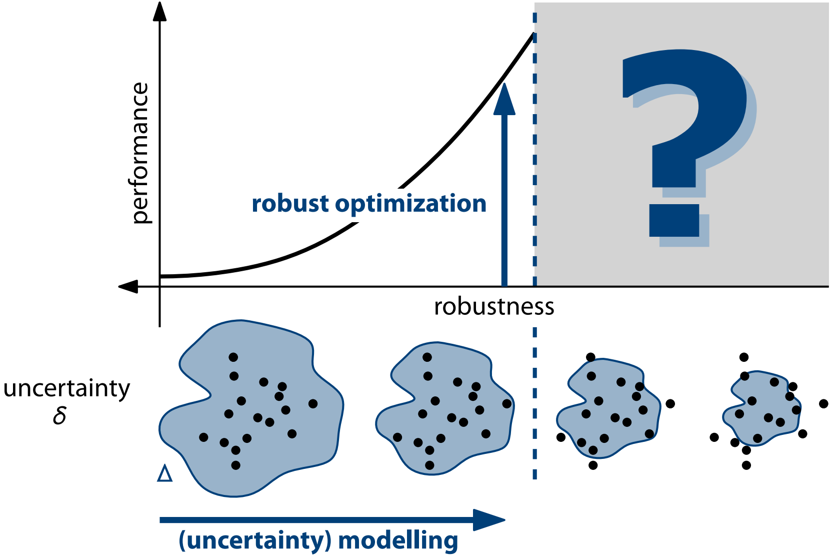

Robust Optimization

Find the solution (portfolio, control, design …),

that is adequate for all possible δ in Δ,

and guarantees the highest performance.

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$$
\begin{array}{ll}
\underset{x,\gamma}{\text{maximize}} & \gamma \\
\text{subject to} & f_0(x, \delta) \geq \gamma, \quad \text{for all } \delta \in \Delta \\
& f_i(x, \delta) \geq 0, \quad \text{for all } \delta \in \Delta, \\
& \qquad\qquad\qquad\;\; i = 1, \ldots, m
\end{array}
$$

1. *Beyond the original raison d'être*
2. *Solving robust optimization problems*
3. *Results in applications*

Motion Planning

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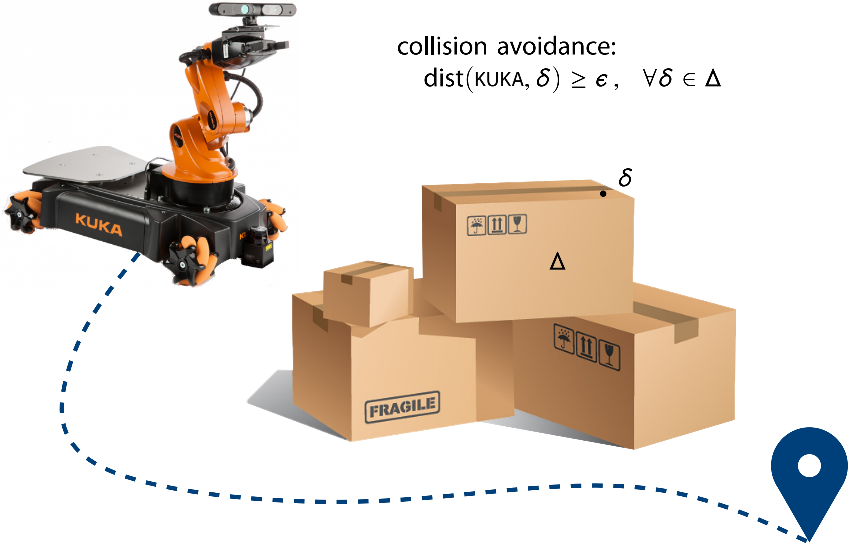
collision avoidance:

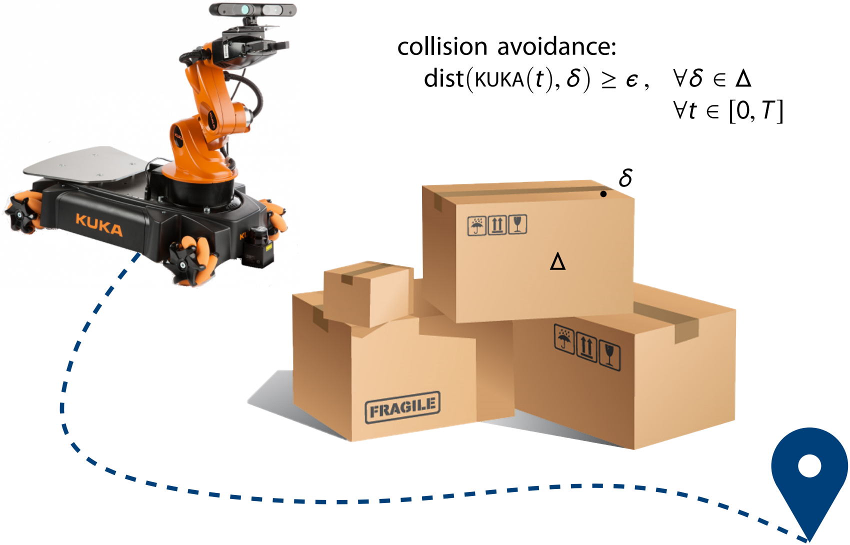
$$\mathrm{dist}(\text{KUKA}, \delta) \geq \epsilon\,, \quad \forall \delta \in \Delta$$



Motion Planning

collision avoidance:

$$\mathrm{dist}(\mathsf{KUKA}(t), \delta) \geq \epsilon\,, \quad \forall \delta \in \Delta$$
$$\forall t \in [0, T]$$



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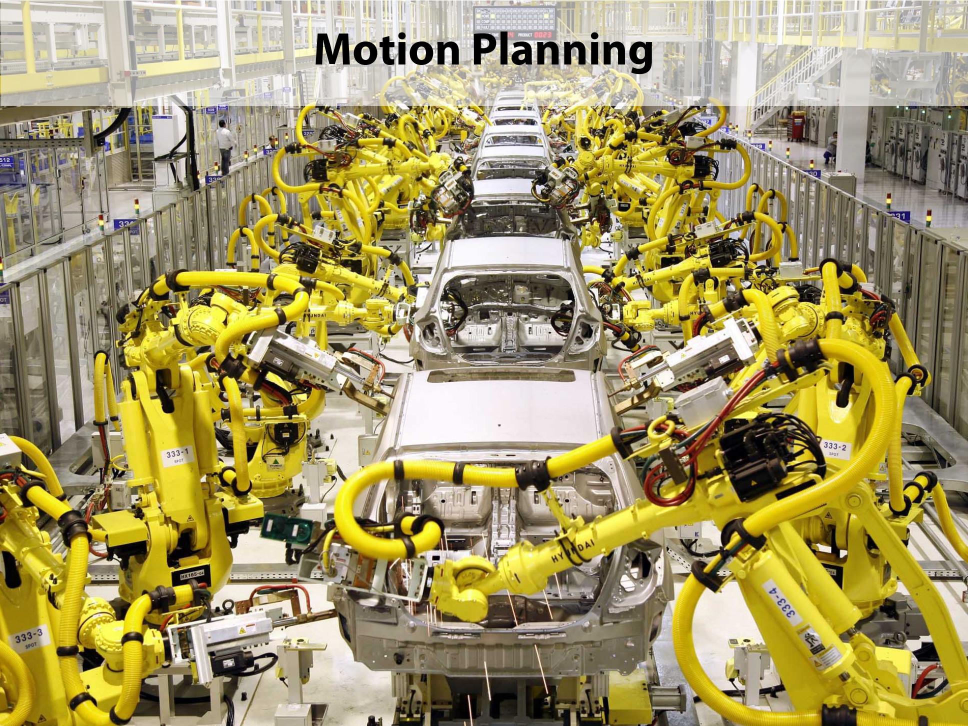

Interconnected Systems

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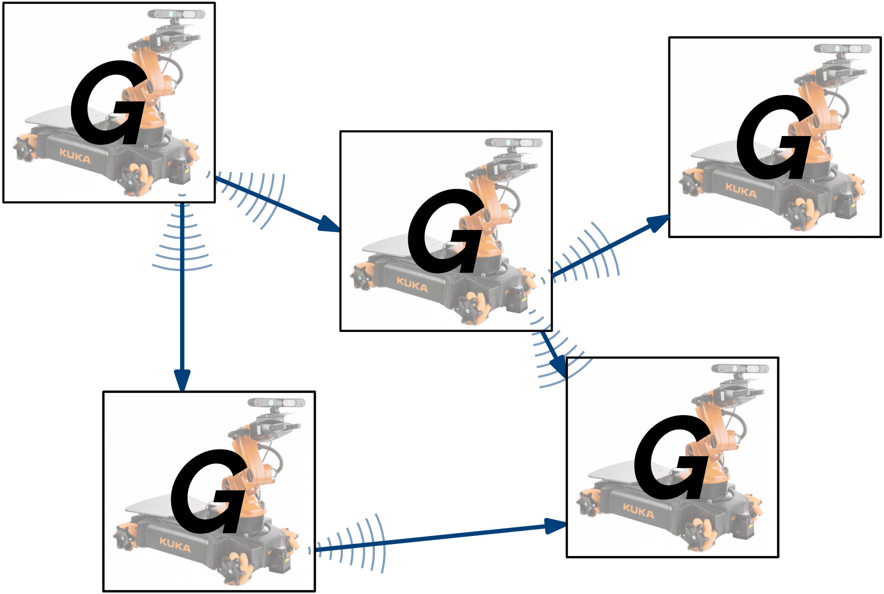

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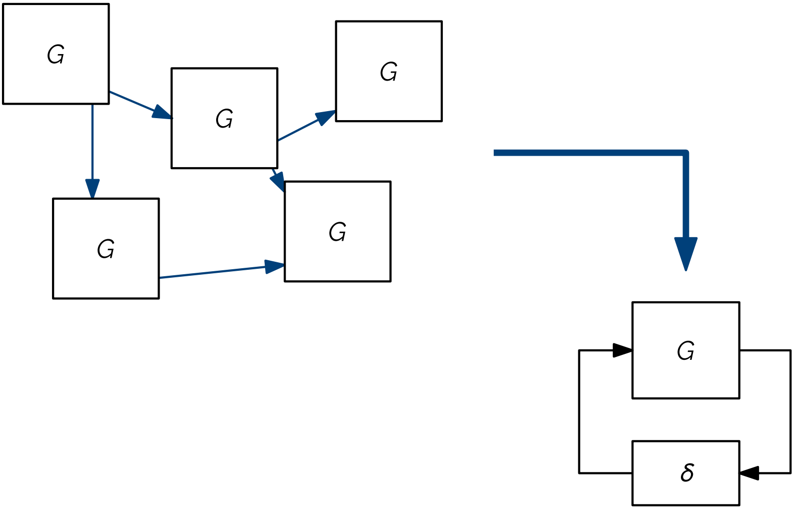

$\Delta \sim$ interconnection

Interconnected Systems

Parametric Programming

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & f_0(x, \delta) \\ \text{subject to} \quad & f_i(x, \delta) \geq 0\,, \quad i = 1, \ldots, m \end{aligned}$$

$\longrightarrow$

$x^{\mathrm{opt}}(\delta)$
for $\delta \in \Delta$

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$\longrightarrow$

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Trade-off analysis

$\hookrightarrow$ δ: weight on objectives

$$f_0(x, \delta) = \sum_k \delta_k \, f_{0,k}(x)$$

Parametric Programming

$$\begin{array}{lll} \underset{x}{\text{maximize}} & f_0(x,\delta) & \\ \text{subject to} & f_i(x,\delta) \geq 0\,, & i = 1,\ldots,m \end{array} \longrightarrow \begin{array}{c} x^{\text{opt}}(\delta) \\ \text{for } \delta \in \Delta \end{array}$$

Linear parameter varying control

$\hookrightarrow$ δ: measurable parameters affecting the system dynamics

Parametric Programming

$$\begin{array}{ll} \underset{x}{\text{maximize}} & f_0(x,\delta) \\ \text{subject to} & f_i(x,\delta) \geq 0\,, \quad i = 1,\ldots,m \end{array}$$

$$\longrightarrow$$

$x^{\text{opt}}(\delta)$
for $\delta \in \Delta$

Computing approximation $\hat{x}(\delta)$ of $x^{\text{opt}}(\delta)$

$$\hat{x}(\delta) = x_0 + x_1\,\delta + x_2\,\delta^2 \ldots$$

$\hookrightarrow$ solve robust optimization problem to compute coefficients

$$\begin{array}{ll} \underset{x_0,\,x_1,\,\ldots}{\text{maximize}} & \displaystyle\int_\Delta f_0(\hat{x}(\delta),\delta)\,d\delta \\ \text{subject to} & f_i(\hat{x}(\delta),\delta) \geq 0\,, \quad \text{for all } \delta \in \Delta\,, \\ & \qquad\qquad\qquad\qquad i = 1,\ldots,m \end{array}$$

1. ***Beyond the original raison d’être***

2. ***Solving robust optimization problems***

3. ***Results in applications***

The Main Challenge

$$f(x, \delta) \geq 0, \text{ for all } \delta \in \Delta$$

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$$f_x(\delta) \geq 0, \;\; \text{for all } \; \delta \in \Delta$$

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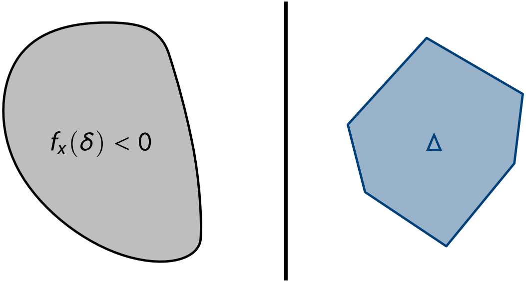
$$\{\delta \,|\, f_x(\delta) < 0\} \cap \Delta = \varnothing$$

Few Cases: Tractable Reformulation

Linear separation



Quadratic separation

- S-procedure, Kalman-Yakubovich-Popov lemma …

Otherwise: Relaxations

$$f(x,\delta) \geq 0\,, \quad \text{for all } \delta \in \Delta$$

$$\Uparrow$$

$$\mathcal{F}_{\Delta}(x) \geq 0$$ ***relaxation***

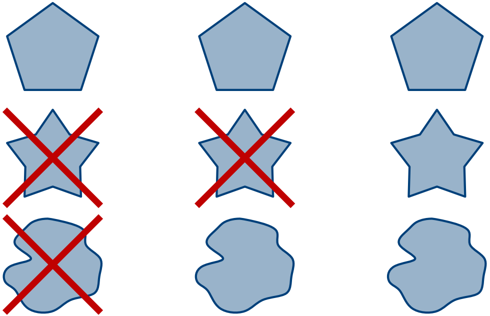
Otherwise: Relaxations

A novel scheme for constructing relaxations

- more general than existing schemes

	Pólya	sum-of-squares	**new**
$f_x(\delta)$	polynomial	polynomial	piecewise polynomial

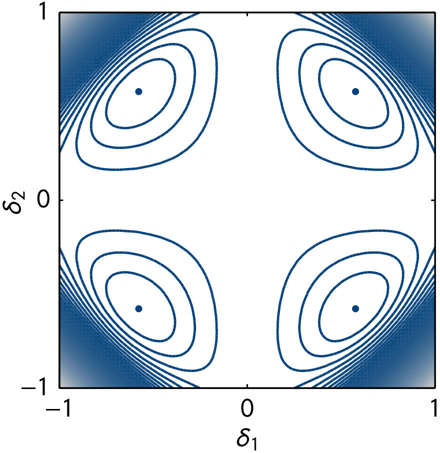
Δ



Otherwise: Relaxations

A novel scheme for constructing relaxations

- more general than existing schemes
- computationally more efficient than existing schemes



$$\begin{array}{ll} \underset{\gamma}{\text{maximize}} & \gamma \\ \text{subject to} & f(\delta) \geq \gamma\,, \ \text{for all} \ \delta \in \Delta \end{array}$$

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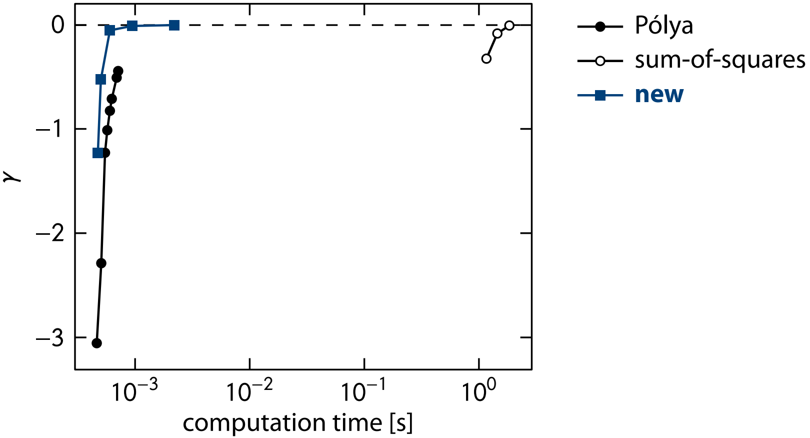


B-spline Relaxations

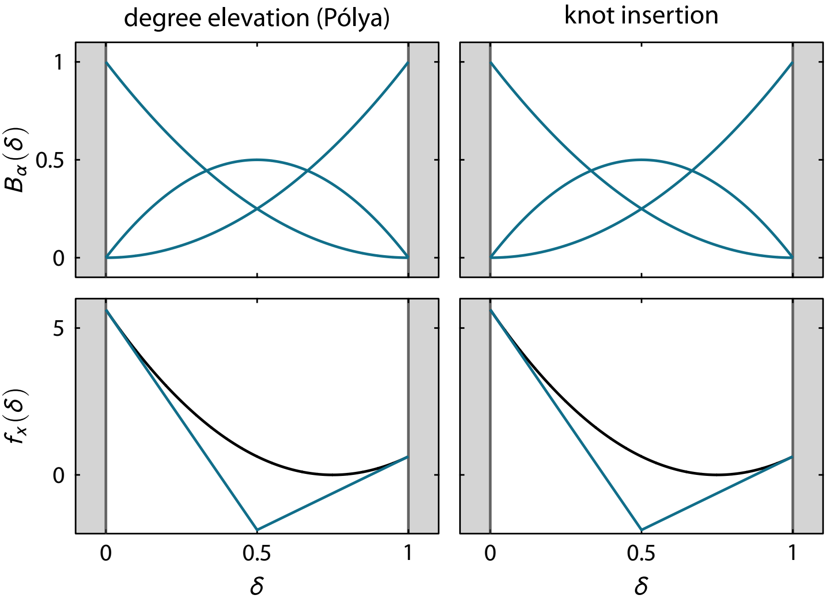
Overall idea

- B-splines: piecewise polynomial basis functions B_α
  - partition of unity on Δ: $\sum_{\alpha \in \boldsymbol{\alpha}} B_\alpha(\delta) = 1, \ \forall \delta \in \Delta$
  - positive on Δ: $B_\alpha(\delta) \geq 0, \ \forall \delta \in \Delta$

- relaxation: $$f(x, \delta) = \sum_{\alpha \in \boldsymbol{\alpha}} c_\alpha(x) B_\alpha(\delta) \geq 0, \ \ \forall \delta \in \Delta$$
$$\Uparrow$$
$$c_\alpha(x) \geq 0, \ \ \forall \alpha \in \boldsymbol{\alpha}$$

- refinement: higher-dimensional B-spline bases
  - piecewise polynomial of higher degree
  - piecewise polynomial on finer grid

B-spline Relaxations



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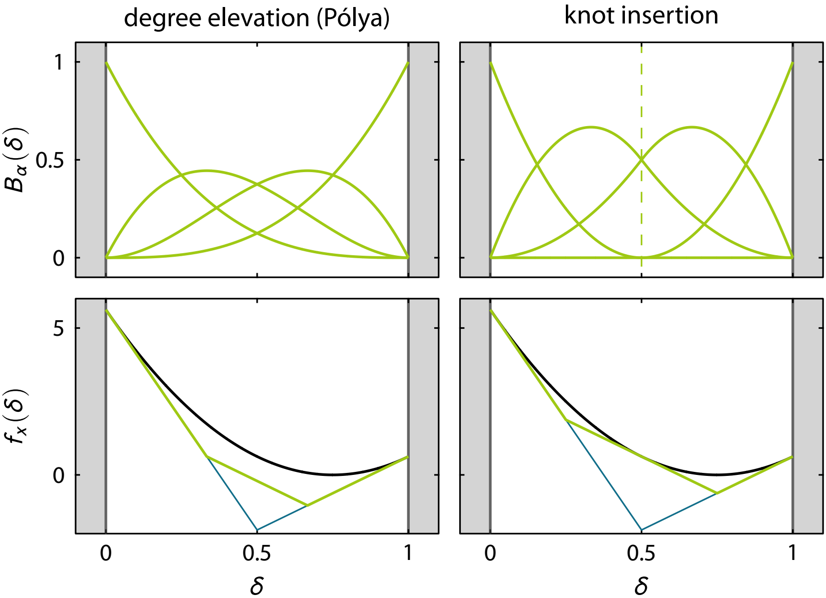

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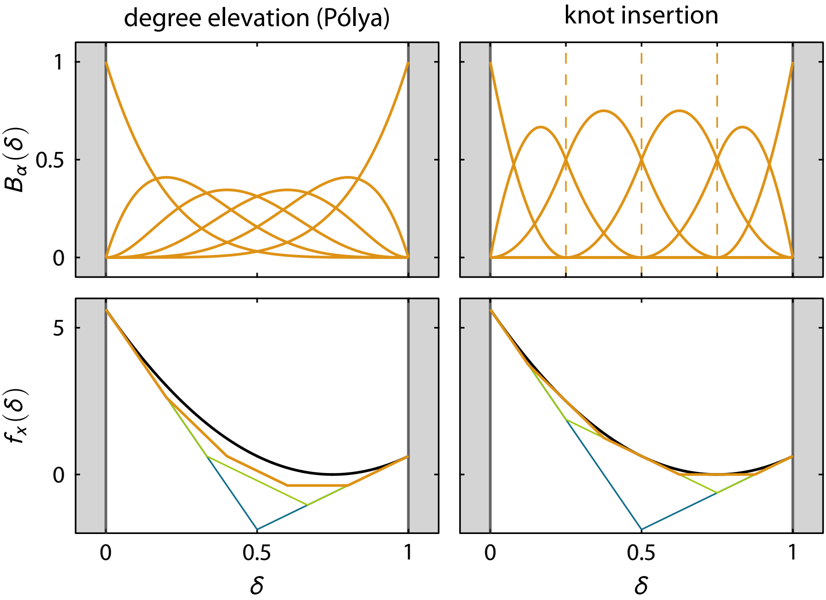

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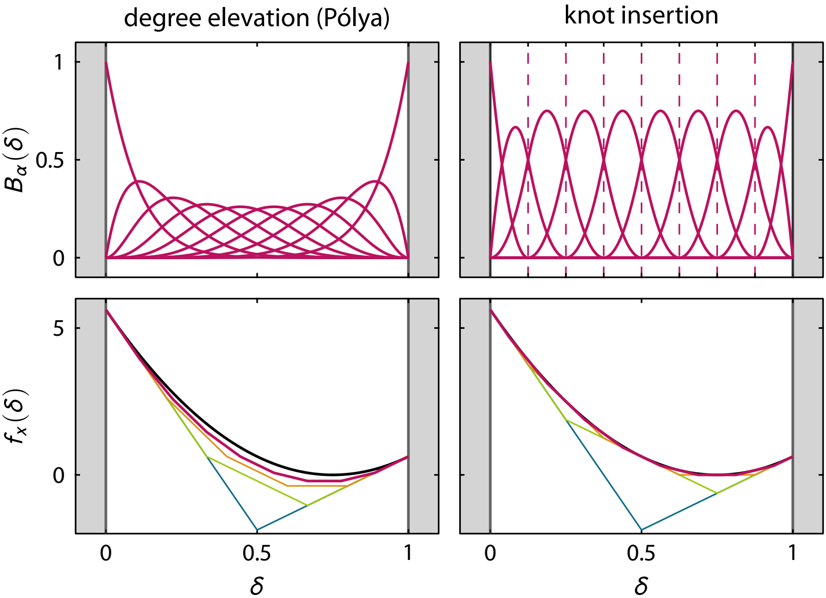
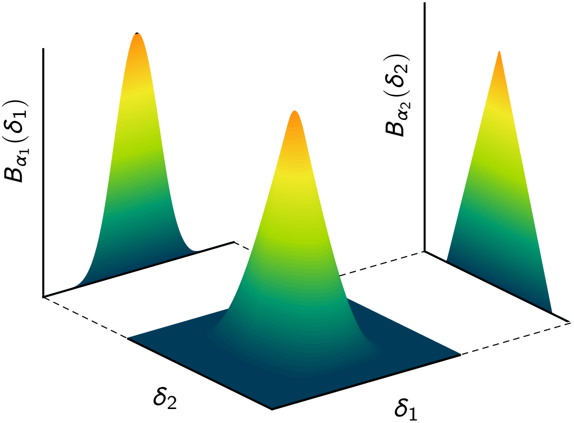


B-spline Relaxations

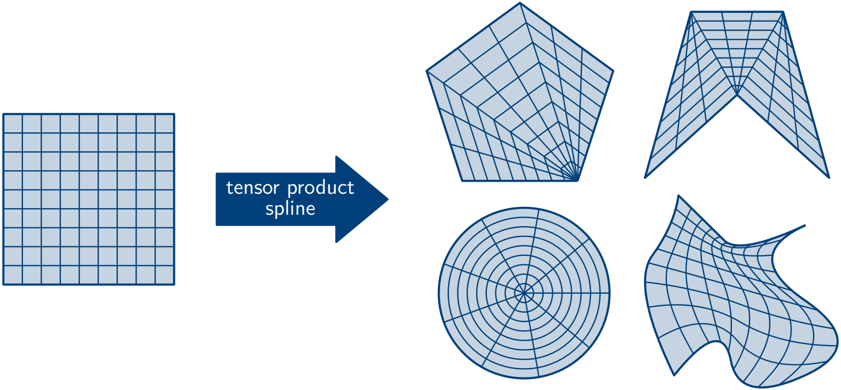
Tensor product B-splines

$$B_{\alpha}(\delta) = B_{\alpha_1}(\delta_1) B_{\alpha_2}(\delta_2)$$



B-spline Relaxations

Towards non-hyperrectangular domains



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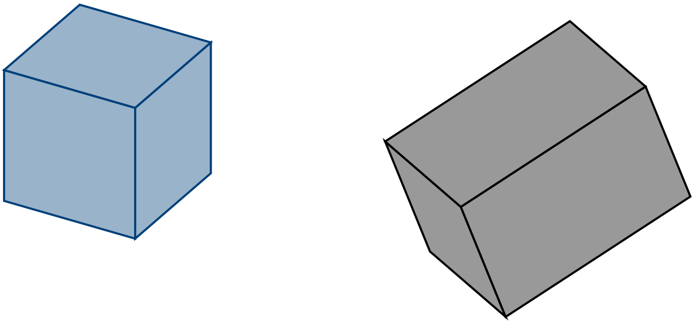
Motion Planning

Avoiding collision

- exact reformulation using linear separation



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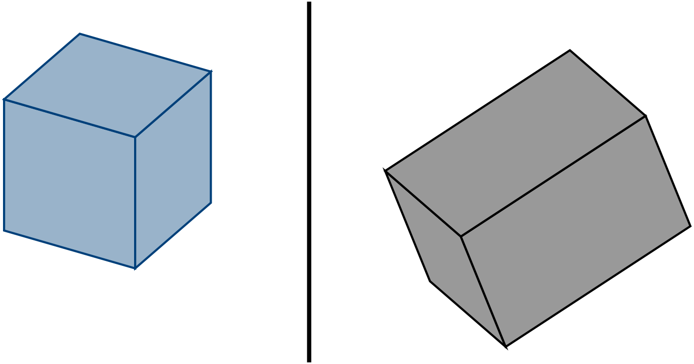

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Avoiding collision

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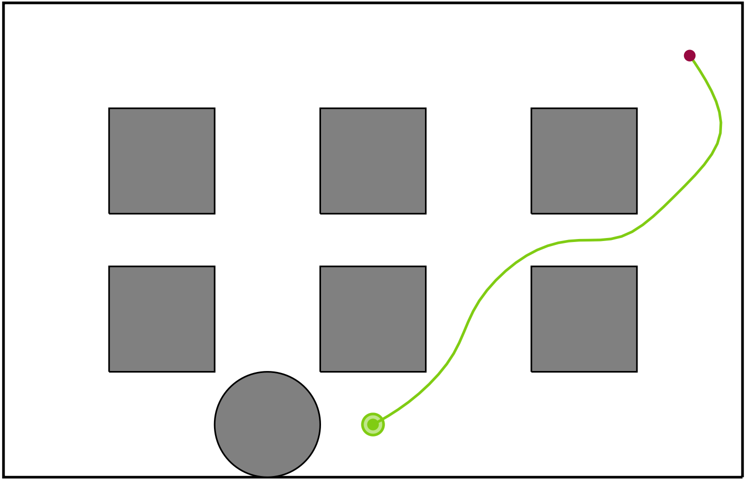
Enforcing constraints at all times

- B-spline parametrization of motion trajectory
- novel relaxation scheme for efficient constraint satisfaction

Motion Planning

20 ms for simple kinematic models



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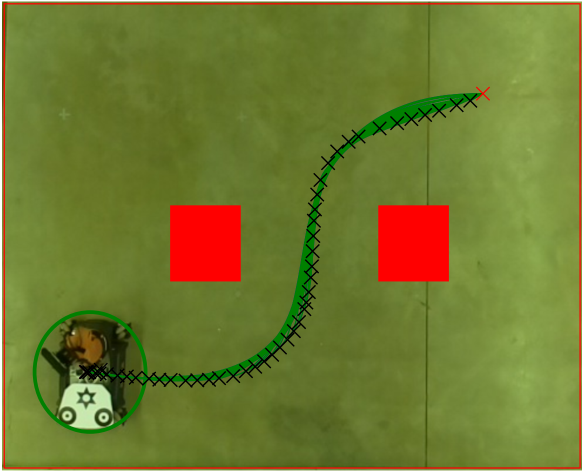
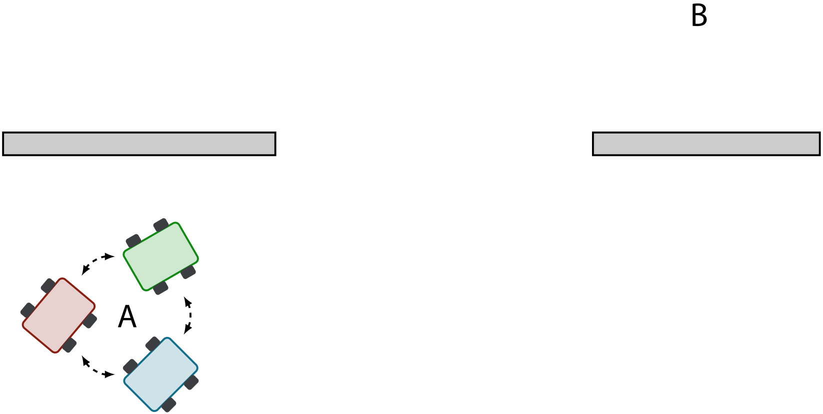


Motion Planning

Extension to multi-agent motion planning

- optimization distributed over agents using ADMM
- 1 ADMM iteration per update



Motion Planning

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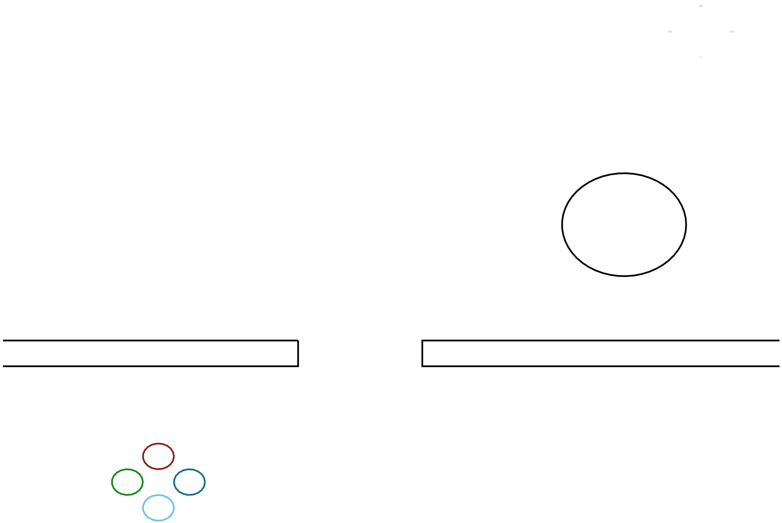

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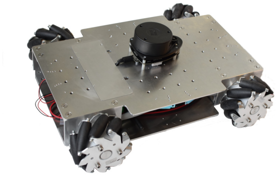
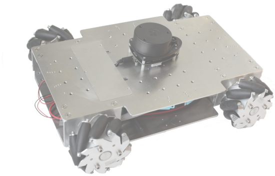
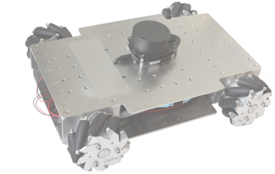


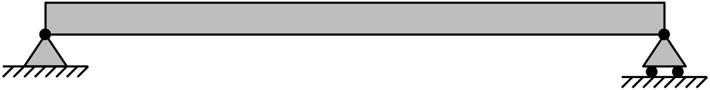




Interconnected Systems

Application in vibro-acoustics

Interconnected Systems



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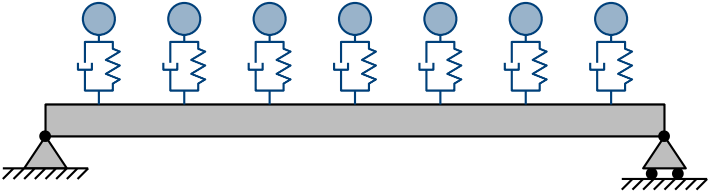

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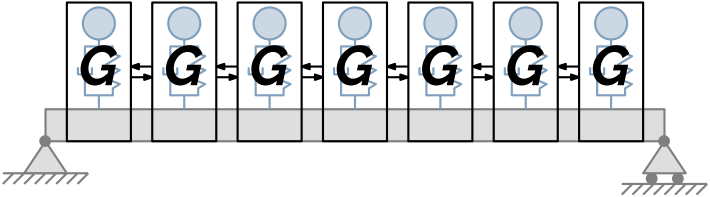
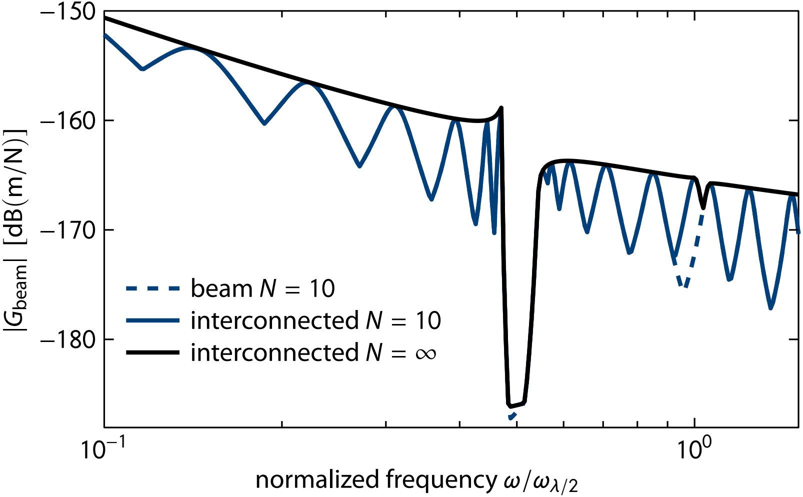


Interconnected Systems

Efficient analysis using quadratic separation



Parametric Programming

A novel B-spline based framework

- B-spline parameterized $\hat{x}(\delta)$ for high flexibility
- novel relaxation scheme for low conservatism

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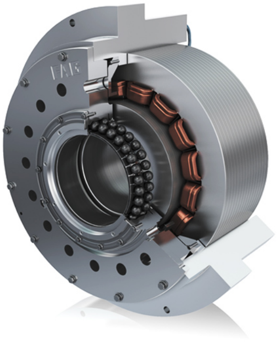
Trade-off analysis in active bearing control



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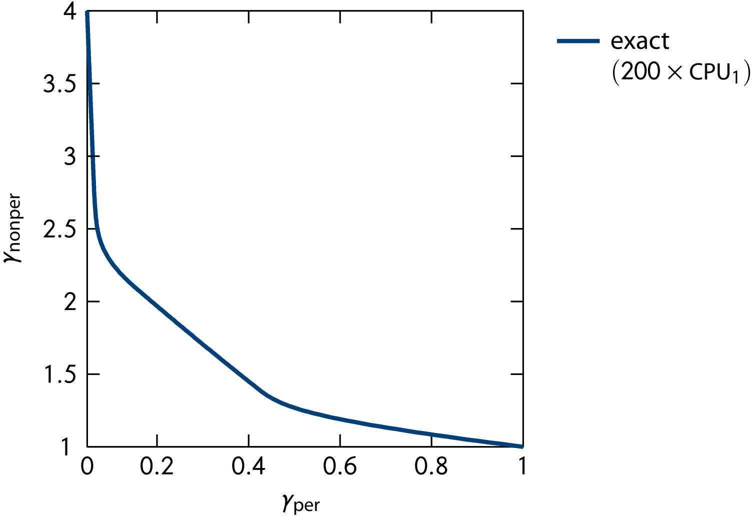
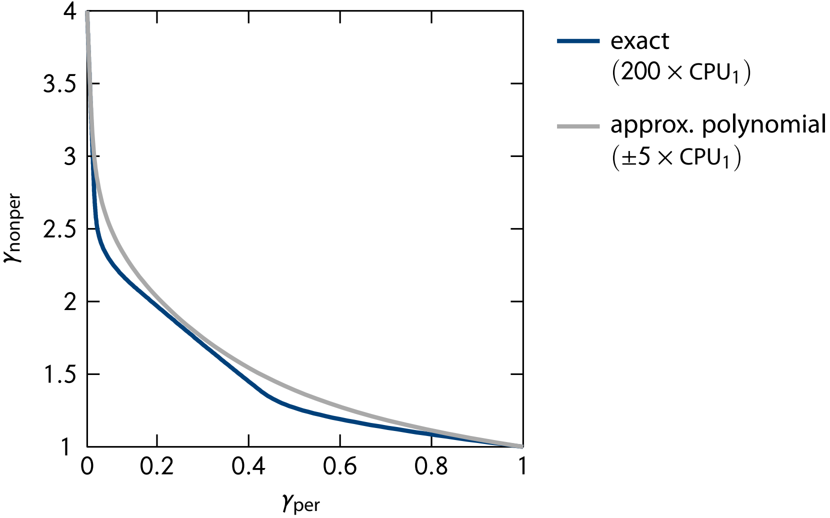
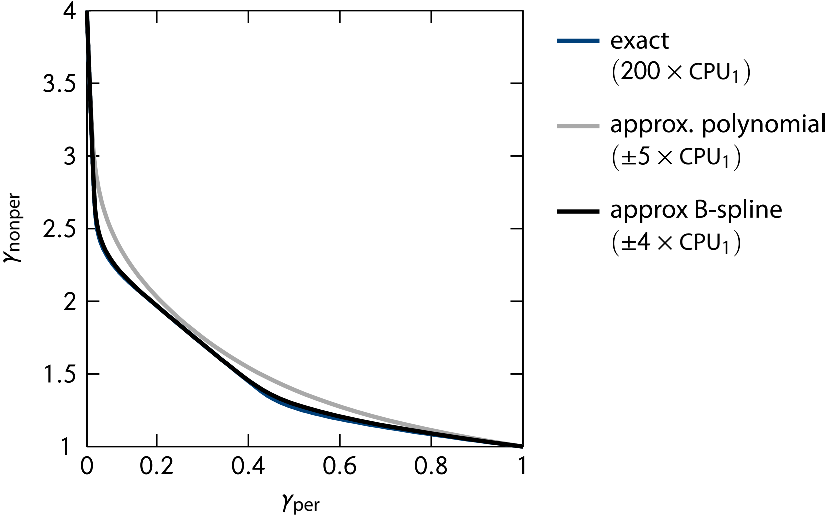


Parametric Programming



Parametric Programming



Conclusion

1. ***Robust optimization***
   has many applications in engineering

2. ***General and effective strategy***
   for solving robust optimization problems

3. ***Results in applications***

Thank You!

The MECO research group
Wannes Van Loock, Ruben Van Parys, Tim Mercy

Claus Claeys, Elke Deckers

fwo Research Foundation Flanders
Opening new horizons