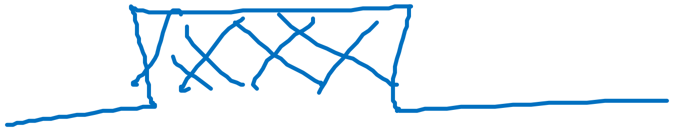
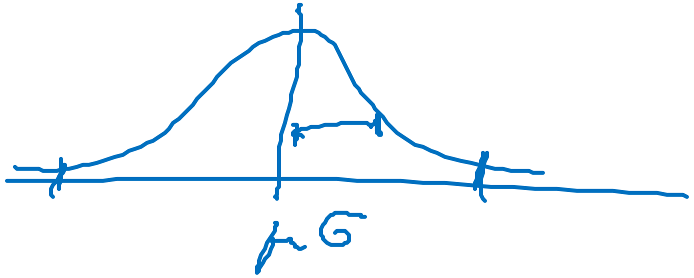
PDF





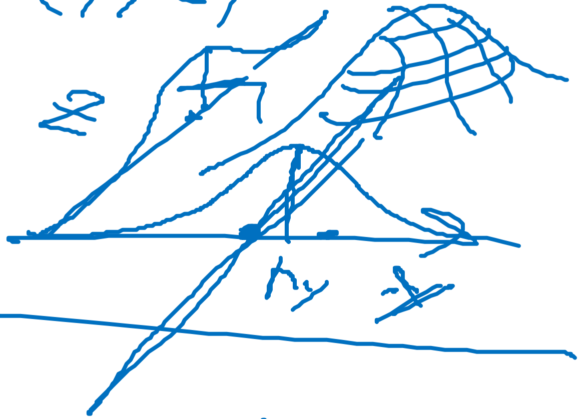
$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

2.2.4 COVARIANCE & CORRELATION

" $p_{Y,Z}(y,z)$

$Y, Z \in \mathbb{R}$

JOINT PDF $p(y,z)$



MARGINAL DISTRIBUTION

$$p_Y(y) = \int_{-\infty}^{\infty} p(y,z)\, dz$$

$\mu_Y, \mu_Z, \sigma_Y^2, \sigma_Z^2$

COVARIANCE

$$\sigma(Y,Z) := \mathbb{E}\{(Y-\mu_Y)(Z-\mu_Z)\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y-\mu_Y)\cdot(z-\mu_Z)\, p(y,z)\, dz\, dy$$

TEASER: IF Y & Z <u>INDEP</u>, WHAT IS $\sigma(Y,Z)$?

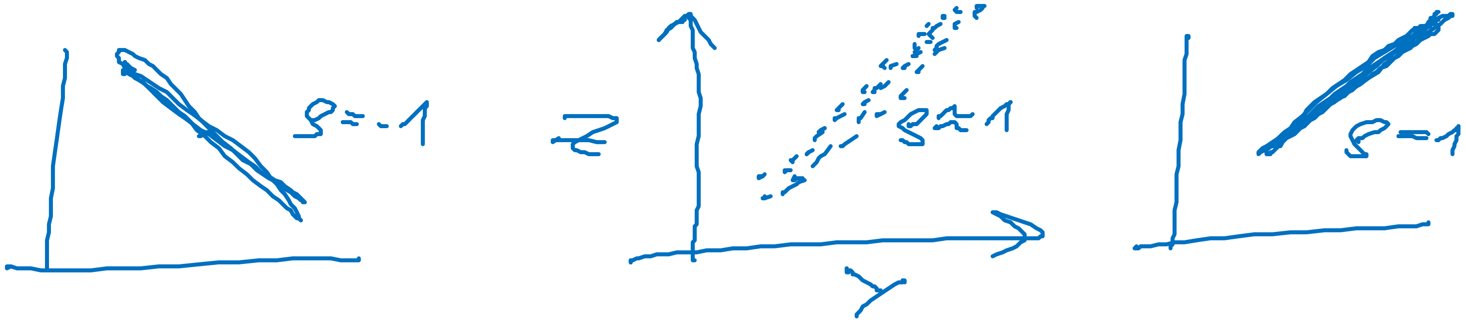
$$p(y,z) = p_y(y) \cdot p_z(z)$$

$$\sigma(Y,Z) = \underbrace{\int_{-\infty}^{\infty} \underbrace{(y-\mu_y)\, p_y(y)\, dy}}_{=0} \cdot \underbrace{\int_{-\infty}^{\infty} \underbrace{(z-\mu_z)\, p_z(z)\, dz}}_{=0}$$

$$= 0$$

$$\frac{\sigma(Y,Z)}{\sigma_Y \cdot \sigma_Z} =: \rho(Y,Z)$$

<u>CORRELATION</u> $\in [-1, 1]$



2.3 MULTIDIM. RANDOM VARS.

$X \in \mathbb{R}^n$ $\qquad f: \mathbb{R}^n \to \mathbb{R}^m$

$$\boxed{\mathbb{E}\{f(X)\} := \int_{\mathbb{R}^n} f(x) \cdot \underset{\uparrow \text{PDF}}{p_X(x)} \cdot d^n x}$$

$\int p_X(x)\, d^n x = 1$

[CONTENTS MISSING (TECHNICAL PROBLEMS)]

$$\Sigma_X = \mathbb{E}\{(X-\mu_X)(X-\mu_X)^T\} =: \mathrm{cov}(X)$$

Σ_X is POSITIVE SEMI-DEFINITE MATRIX

DIAGONAL: VARIANCES

OFF-DIAGONALS: COVARIANCES

$$\mathrm{cov}(X) = \mathbb{E}\{XX^T\} - \mu_X\mu_X^T$$

CORRELATION MATRIX

$$\begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{12} & \ddots & & \vdots \\ & & \ddots & \\ \rho_{1n} & \cdots & & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sigma_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sigma_n} \end{pmatrix} \mathrm{cov}(X) \begin{pmatrix} \frac{1}{\sigma_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sigma_n} \end{pmatrix}$$

2.3.2 MULTIDIM. NORMAL DISTRIBUTION (GAUSSIAN)

$x \in \mathbb{R}^n$

PDF $$\boxed{p(x) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)}$$

$E\{X\} = \mu \in \mathbb{R}^n$

$\mathrm{cov}(X) = \Sigma \in \mathbb{R}^{n \times n}$

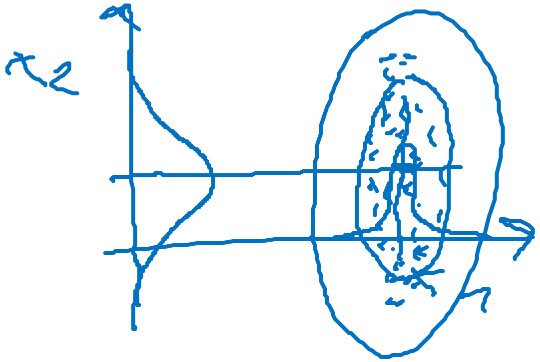
EX: X_1, X_2 INDEP. $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$

σ_1^2, σ_2^2

$$\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$$

$$p(x) = \frac{1}{\sqrt{\det\left(2\pi \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}\right)}} \exp\left(-\frac{1}{2} \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}^T \begin{pmatrix} \sigma_1^{-2} & 0 \\ 0 & \sigma_2^{-2} \end{pmatrix} \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}\right)$$

NO CORRELATION



$$= \quad '' \quad \exp\left(-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma_1^2}\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{1}{2}\frac{(x_1-\mu_1)^2}{\sigma_1^2}\right) \exp\left(-\frac{1}{2}\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)$$

$$= p_{x_1}(x_1) \cdot p_{x_2}(x_2)$$

2.4 STATISTICAL ESTIMATORS

UNKNOWN PARAMETER : $\Theta \in \mathbb{R}^n$, (TRUE VALUE Θ_0)

GIVEN: MEASUREMENTS $y(1), y(2), \ldots, y(N)$

$$y_N := \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix} \qquad Y_N = \begin{bmatrix} Y(1) \\ \vdots \\ Y(N) \end{bmatrix}$$

SMALL y_N : REAL VALUES

BIG Y_N : RANDOM VECTOR

ESTIMATOR: MAP FROM Y_N $\Rightarrow$ VALUES OF Θ (ESTIMATES)

$$\hat{\Theta}_N(Y_N) \qquad \hat{\Theta}_N : \mathbb{R}^N \to \mathbb{R}^n$$

EX: ESTIMATING MEAN : $\boxed{\hat{\Theta}_N(Y_N) := \frac{1}{N} \sum_{k=1}^{N} Y(k)}$ "SAMPLE MEAN"

DEF: (BIASED & UNBIASEDNESS) ESTIM. $\hat{\Theta}$ IS

UNBIASED IFF (IF & ONLY IF)

$$\boxed{\mathbb{E}\{\hat{\Theta}_N(Y_N)\} = \Theta_0}$$

WHERE Θ_0 IS THE (UNKNOWN) TRUE VALUE

OTHERWISE IT IS BIASED

EX: $\hat{\Theta}_N(Y_N) = \frac{1}{N}\sum_{k=1}^{N} y(k)$ IS UNBIASED ESTIMATOR FOR THE TRUE MEAN OF $y(k)$ (IF ALL $y(k)$ FOLLOW SAME PDF WITH MEAN μ_y)

$\Theta_0 = \mu_y$

$$\mathbb{E}\{\hat{\Theta}_N(Y_N)\} = \frac{1}{N}\sum_{k=1}^{N} \mathbb{E}\{Y(k)\} = \frac{1}{N}\sum_{k=1}^{N} \mu_y = \mu_y$$

OTHER EX:

$$\hat{\Theta}_N(Y_N) = \frac{y(N)+y(1)}{2}$$

UNBIASED :)

$\mathrm{Cov}(\hat{\Theta}_N(Y_N))$ IS A PERFORMANCE MEASURE FOR UNBIASED ESTIMATORS

BIGGER COV $\hat{=}$ WORSE PERFORMANCE

$\mathrm{COV}(\hat{\Theta}_A) \succeq \mathrm{COV}(\hat{\Theta}_B)$ (~~MATRIX~~ MATRIX INEQUALITY)

$\Updownarrow$

$\hat{\Theta}_B$ HAS BETTER OR EQUAL PERFORM. COMP. TO $\hat{\Theta}_A$

A, B SYMM. MATRICES)

$\boxed{A \succeq B} \iff A - B \succeq 0 \iff (A-B)$ IS POS. SEMI DEFINITE

$\iff (A-B)$ HAS NON-NEGATIVE EIGENVALUES

$\boxed{\succeq} \neq \boxed{\geq}$

MATRIX INEQUALITY (FOR SYMM. MATRICES)

RECALL: IF $A = A^T$, $A \in \mathbb{R}^{n \times n}$

$\exists$ $R \in \mathbb{R}^{n \times n}$ WITH $R R^T = I$,

$$\boxed{A = R \cdot D\, R^T}$$

WITH $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$

$d_n \in \mathbb{R}$

(FROM LIN. ALG.)

DEF 2: ASYMPTOTIC UNBIASEDNESS:

$$\boxed{\lim_{N\to\infty} E\{\hat{\theta}_N(Y_N)\} = \theta_0}$$

EX: ESTIMATE <u>VARIANCE</u> OF $Y(k)$ (WITH SAME PDF, μ_Y, σ_Y^2)

$$M(Y_N) := \frac{1}{N}\sum_{k=1}^{N} Y(k)$$

$$\hat{\theta}_N(Y_N) = \frac{1}{N}\sum_{k=1}^{N}\left(Y(k) - M(Y_N)\right)^2$$

$$E\{\hat{\theta}_N\} = \frac{1}{N}E\left\{\sum_{k=1}^{N}\left(Y(k) - \frac{1}{N}\sum_{i=1}^{N} Y(i)\right)^2\right\}$$

$$= \frac{1}{N}\left(\sum_{k=1}^{N}\left(Y(k)^2 - 2\,Y(k)\cdot\frac{1}{N}\sum_{i=1}^{N} Y(i) + \frac{1}{N^2}\left(\sum_{i=1}^{N} Y(i)\right)^2\right)\right)$$

$$= \dots \quad \dots = \frac{N-1}{N}\sigma_Y^2 \longrightarrow \sigma_Y^2$$ FOR $N\to\infty$

ALTERNATIVE:

$$\frac{1}{N-1}\sum\left(Y(k) - M(Y_N)\right)^2$$

UNBIASED

$=: S^2$

"SAMPLE VARIANCE"

DEF 3: CONSISTENCY: ESTIMATOR $\hat{\theta}_N$ IS CONSISTENT IF FOR $\varepsilon > 0$ THE PROBABILITY

$$P\left(\hat{\theta}_N(Y_N) \in [-\varepsilon + \theta_0, \theta_0 + \varepsilon]\right) \longrightarrow 1$$

FOR $N \rightarrow \infty$

(a) asymptotically unbiased

(b) Cov $\rightarrow 0$ WITH $N \rightarrow \infty$

$\Downarrow$

CONSISTENCY