CONT: $\quad y(1), \quad y(2), \quad \dots$

$\boxed{\text{i.i.d.}}$ $\quad$ $\boxed{\text{independent identically distributed}}$

$$p_{y(1),y(2)}(y(1),y(2)) = p_{y(1)}(y(1)) \cdot p_{y(2)}(y(2))$$

$$p_{y(1)}(y) = p_{y(2)}(y) \quad \forall y$$

$$G(y(1),y(2)) = 0$$

TYPICAL ASS. FOR MEASUREMENTS / NOISE

SAMPLE VARIANCE REVISITED:

$$\hat{\theta}_N(y_N) = \frac{1}{N}\sum_{k=1}^{N}\left(y(k) - \frac{1}{N}\sum_{i=1}^{N} y(i)\right)^2$$

$y(i)$ i.i.d. WITH MEAN μ AND VARIANCE σ^2

$$\mathbb{E}\{\hat{\theta}_N\} = \, ?$$

$$= \mathbb{E}\left\{\left(y(1) - \frac{1}{N}\sum_{i=1}^{N} y(i)\right)^2\right\}$$

$$= \mathbb{E}\{(\varepsilon(1) - \frac{1}{N}\sum \varepsilon(i))^2$$

$y(i) = \mu + \varepsilon(i)$

MEAN OF ε ZERO,

VARIANCE OF ε σ^2

$$\mathbb{E}\left\{ \left(\varepsilon(1) - \frac{1}{N}\sum_{i=1}^{N} \varepsilon(i)\right)^2 \right\}$$

$$\mathrm{cov}(X,Y) = \mathbb{E}(X\cdot Y) \cancel{-\mu_x \cdot \mu_y}$$

$$= \mathbb{E}\left\{ \varepsilon(1)^2 - 2\,\frac{\varepsilon(1)}{N}\sum_{i=1}^{N} \varepsilon(i) + \left(\frac{1}{N}\sum_{i=1}^{N} \varepsilon(i)\right)^2 \right\}$$

$$= \sigma^2 - \frac{2}{N}\sigma^2 + \frac{1}{N^2}\,\mathbb{E}\left\{\sum_{i=1}^{N} \varepsilon(i) \sum_{j=1}^{N} \varepsilon(j)\right\}$$

$$= \sigma^2 - \frac{2}{N}\sigma^2 + \frac{1}{N^2}\underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N} \mathbb{E}\{\varepsilon(i)\,\varepsilon(j)\}}_{= N\cdot\sigma^2}$$

$$= \sigma^2 - \frac{2}{N}\sigma^2 + \frac{N}{N^2}\sigma^2$$

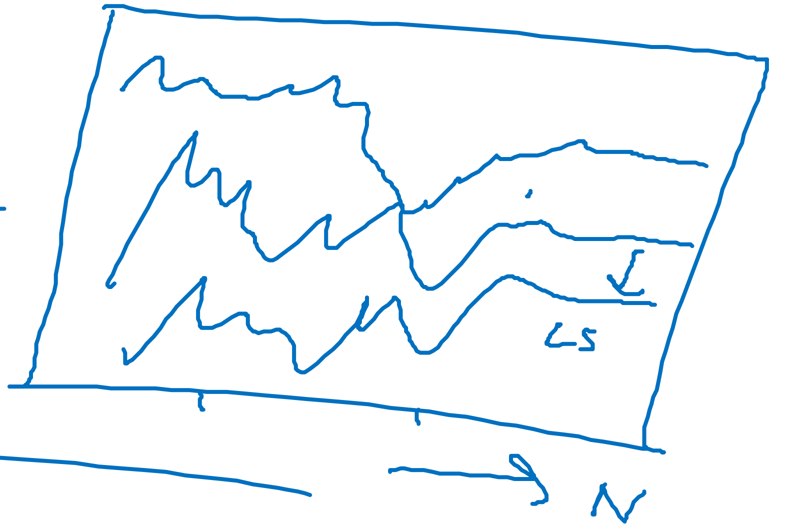
$$= \frac{N\sigma^2 - \sigma^2}{N} = \frac{N-1}{N}\sigma^2 \quad \square$$

2.5 ANALYSIS OF RESISTANCE ESTIMATION

3 $$\hat{R}_{SA}(N) = \frac{1}{N}\sum_{k=1}^{N}\frac{u(k)}{i(k)}$$

1 $$\hat{R}_{EV}(N) = \frac{\frac{1}{N}\sum u(k)}{\frac{1}{N}\sum i(k)}$$

2 $$\hat{R}_{LS}(N) = \frac{\frac{1}{N}\sum u(k)\cdot i(k)}{\frac{1}{N}\sum i(k)\cdot i(k)}$$



ASSUME : $u(k) = u_0 + n_u(k)$

$i(k) = i_0 + n_i(k)$

u_0, i_0 : TRUE

n_i, n_u : NOISE, i.i.d., ZERO MEAN, VARIANCES σ_i^2, σ_u^2

(SYMMETRIC)

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} y(k) = E\{Y\}$$

$$\lim_{N\to\infty}\frac{1}{N}\sum u(k) = u_0$$

$$\lim_{N\to\infty}\frac{1}{N}\sum n_i(k)\, n_u(k) = 0$$

$$\lim_{N\to\infty}\frac{1}{N}\sum n_i(k)^2 = \sigma_i^2$$

2.5.1 EV-EST

$$\lim_{N\to\infty} \hat{R}_{EV}(N) = \frac{\lim_{N\to\infty} \frac{1}{N}\sum u(k)}{\lim_{N\to\infty} \frac{1}{N}\sum i(k)} = \frac{u_0}{i_0} = R_0 \; !$$

CORRECT ESTIMATOR !

2.5.2 LS-EST.

$$\lim_{N\to\infty} \hat{R}_{LS}(N) = \frac{\lim_{N\to\infty} \frac{1}{N}\sum u(k)\cdot i(k)}{\lim \frac{1}{N}\sum i(k)^2} = \frac{\lim \frac{1}{N}\sum (u_0+n_u(k))(i_0+n_i(k))}{\lim \frac{1}{N}\sum (i_0+n_i(k))^2}$$

NUMERATOR

$$= \lim \frac{1}{N}\sum \left(u_0\, i_0 + \underbrace{u_0\, n_i(k)}_{\to 0} + \underbrace{n_u(k)\cdot i_0}_{\to 0} + \underbrace{n_u(k)\cdot n_i(k)}_{\to 0}\right)$$

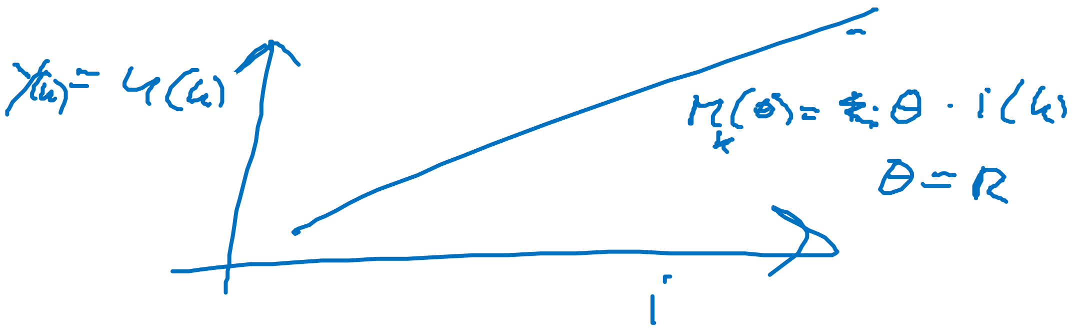
$$= u_0 \cdot i_0$$

DENOMINATOR $= \lim \frac{1}{N}\sum \left(i_0^2 + \underbrace{2\, i_0\, n_i(k)}_{\to 0} + n_i(k)^2\right) = i_0^2 + \sigma_i^2$

$$\lim_{N\to} \hat{R}_{LS}(N) = \frac{u_0\, i_0}{i_0^2+\sigma_i^2} = \frac{u_0}{i_0}\cdot\frac{1}{\left(1+\frac{\sigma_i^2}{i_0^2}\right)} < R_0$$

CH 3 UNCONSTRAINED OPTIMIZATION IN A NUTSHELL

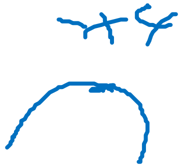
$$f(\theta) = \|y - M(\theta)\|_2^2 = \sum_{k=1}^{N} \left(y(k) - M_k(\theta)\right)^2$$



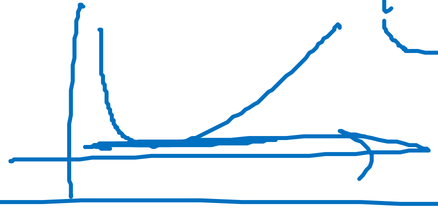
$\boxed{f(x)}$

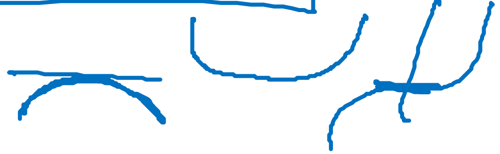
$x \in \mathbb{R}^n$ $f : \mathbb{R}^n \to \mathbb{R}$

AIM: $\boxed{\min_{x \in \mathbb{R}^n} f(x)}$



ALSO: $f(x) = \frac{1}{x} + x$ FOR $x \in \mathbb{R}_{++}$





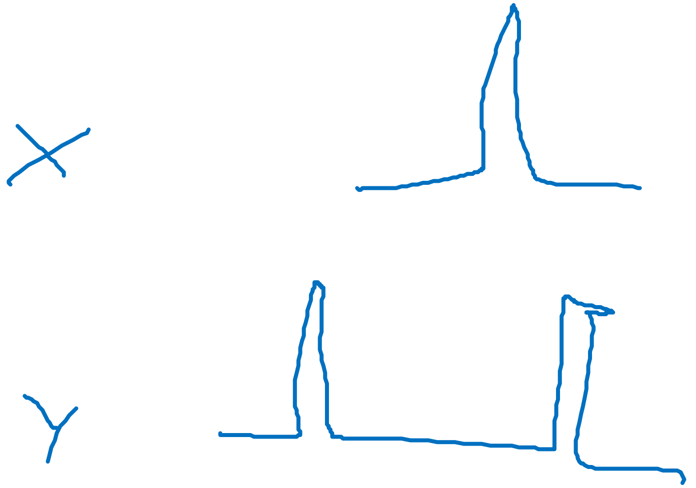
3.1 OPTIMALITY CONDITIONS

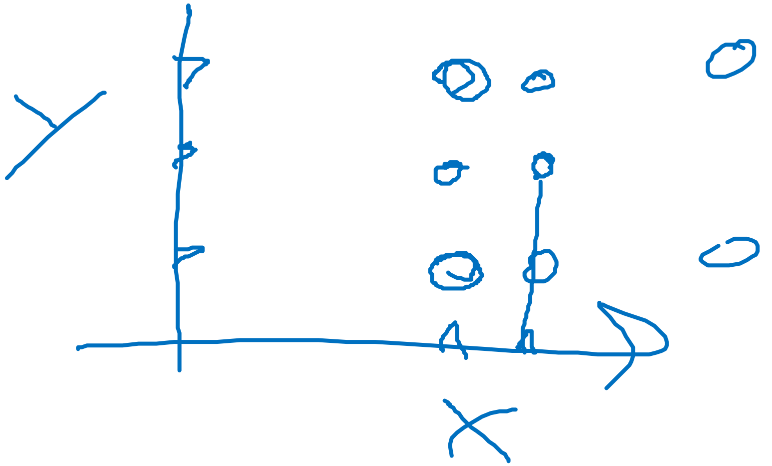
THM 1 (FONC) x^* IS LOCAL MINIMIZER $\Longrightarrow \nabla f(x^*) = 0$

THM 2 (SONC) " $\Longrightarrow \underline{\nabla^2 f(x^*) \succeq 0}$ (ALL EVs NON-NEGATIVE)

NECESSARY BUT NOT SUFFICIENT

ANSWER TO QUESTION ON EX-SHEET:





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$$f(x_1, x_2) = x_1 \cdot x_2^2 \qquad \nabla_x f(x) = \begin{bmatrix} x_2^2 \\ 2x_2 \cdot x_1 \end{bmatrix} \qquad \nabla^2 f = \begin{bmatrix} 0 & 2x_2 \\ 2x_2 & 2x_1 \end{bmatrix}$$

HESSIAN

THM 3 (SOSC & STABILITY UNDER PERTURBATION)

IF x^* IS STATIONARY (i.e. $\nabla f(x^*) = 0$) AND $\nabla^2 f(x^*) \succ 0$ (POS. DEF.) THEN x^* IS LOCAL MINIMIZER

AND

x^* IS STABLE AGAINST PERTURBATIONS, i.E. EXISTS C SO THAT

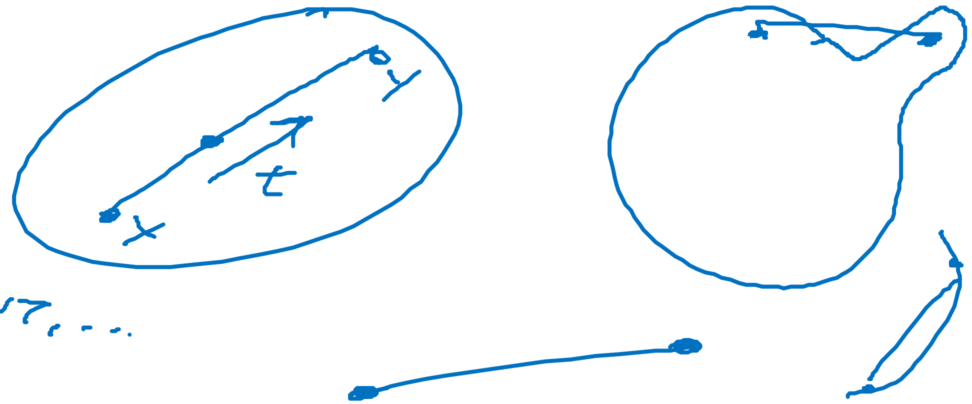
$$\boxed{\left\| x^* - \arg\min_{x \in \mathbb{R}^n} \left(f(x) + p^T x \right) \right\| \leq C \cdot \|p\|}$$

IN ESTIMATION, WE ONLY LIKE MINIMIZERS WHICH SATISFY (SOSC). OTHERWISE, WE SPEAK OF "ILL-POSED" ESTIMATION PROBLEMS.

3.2 CONVEX OPTIMIZATION

DEF: SET $\Omega \in \mathbb{R}^n$ IS CONVEX IFF

$$\forall x,y \in \Omega \quad t \in [0,1] \quad x + t(y-x) \in \Omega$$



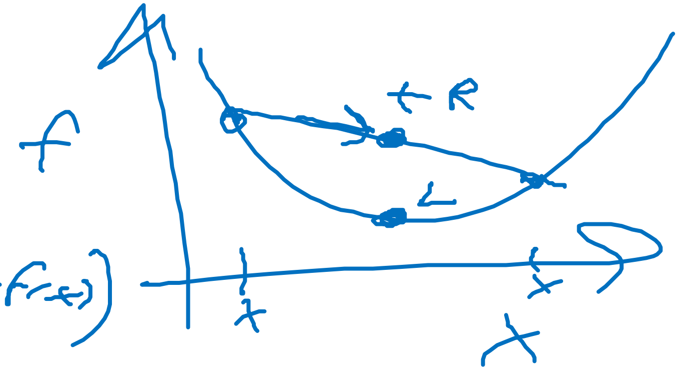
EX: BALL, SQUARE, $\mathbb{R}^n$, POINT, LINE SEGMENT, ...

DEF: FCN $f : \Omega \to \mathbb{R}$ IS CONVEX IFF

Ω IS CONVEX &

$\forall x,y \in \Omega,\ t \in [0,1]$:

$$\underbrace{f(x + t(y-x))}_{L} \le \underbrace{f(x) + t\big(f(y) - f(x)\big)}_{R}$$



THM 4: FOR A CONVEX OPT. PROBLEM ($\min_{x \in \Omega} f(x)$}, WITH f CONVEX) EVERY LOCAL IS ALSO A GLOBAL MINIMIZER

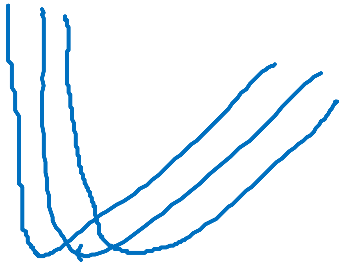
THM 5: ASSUME f IS CONVEX AND DIFF.ABLE. IF $\nabla f(x^*) = 0$ THEN x^* IS <u>GLOBAL</u> MINIMIZER

WHEN IS FCN CONVEX ?

THM 6: $f : \Omega \to \mathbb{R}$ TWICE DIFF.ABLE, Ω CONVEX,

THEN (f CONVEX) $\Longleftrightarrow$ $\nabla^2 f(x) \succeq 0 \quad \forall x \in \Omega$

EX: $f(x) = \frac{1}{x} + x$ $\qquad \Omega = (0, \infty)$



$\nabla f(x) = -\frac{1}{x^2} + 1$

$\nabla^2 f(x) = \frac{1}{x^3} > 0 \implies$ CONVEX

$\nabla f(x^*) = 0$

$\Longleftrightarrow 1 = \frac{1}{x^2}$ $\quad \boxed{x^* = 1}$ GLOB. MINIMIZ.

$\nabla^2 f(x^*) = 1 > 0$ (SOSC)

PERTURBATION

min $\underbrace{\frac{1}{x} + x}_{f} + p \cdot x$ $\qquad \nabla f = -\frac{1}{x^2} + 1 + p$ $\Longleftrightarrow 1 + p = \frac{1}{x^2}$

$x^*_{(p)} = \frac{1}{\sqrt{1+p}}$