

**Exercise 3: Linear Least Squares and Confidence Ellipsoids**  
(to be returned on Nov 24, 2015, 8:15 in HS 26, or before in building 102, 1st floor, 'Anbau')

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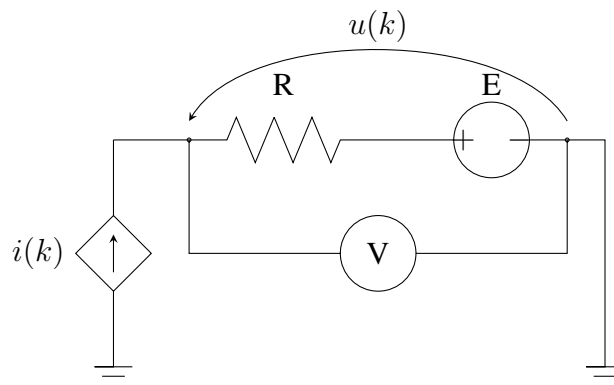
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The aim of this sheet is to strengthen your knowledge in least squares estimation and introduce to covariance ellipsoids.

Your MATLAB solution has to run from a main script called `main.m`, which can call other functions/scripts, but when running this script all the necessary results and plots should be clearly visible. In order to submit your code, include all the necessary files in a single folder using a folder format name `Ex03_Surname1` up to the four possible surnames of the group members. Compress the folder in a `.zip` file and send it to `fabiangirrbach@gmail.com`. Please state also your name and the names of your team members in the e-mail.

### Exercise Tasks

1. Given a sequence of i.i.d. scalar random variables  $x(1), \dots, x(N)$ , each with mean  $\mu$  and variance  $\sigma^2$ , what is the expected value and variance of their arithmetic mean, i.e. of the random variable  $y_N$  defined by  $y_N = \frac{1}{N} \sum_{k=1}^N x(k)$ . (1 point)
2. What is the minimizer  $x^*$  of the function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $x \mapsto f(x) = \|b + A^T x\|_2^2$  (with  $A^T$  of rank  $n$ )? (1 point)
3. Regard a random variable  $x \in \mathbb{R}^n$  with zero mean and covariance matrix  $P$ . What is the mean of  $z = x^T x$  as a function of  $P$ ? Prove your answer. (2 points)  
**Hint:** The trace is defined as  $\text{trace}(A) = a_{11} + a_{22} + \dots + a_{nn} = \sum_{i=1}^n a_{ii}$
4. Recall the following experimental set up to estimate the values of  $E$ ,  $R$  from the last exercise sheet. Note that we leave out the temperature dependent term  $k_3$  in this exercise.



The experiment consists of  $N$  measurements of the voltage  $u(k)$  for different values of  $i(k)$ . We can assume that the input variable  $i(k)$  is not affected by noise, but that the measurements  $u(k)$  are affected by i.i.d. additive noise and can be modeled by:

$$u(k) = E + Ri(k) + n_u(k).$$

Tasks:

- (a) Load the datasets provided on the website containing the measurements into MATLAB. Plot each dataset in a corresponding plot using the `subplot` command. (1 point)
- (b) Use a Least Squares estimator in matrix form to find the experimental values of  $R$ ,  $E$  for each dataset. Plot the linear fit through the measurement data. (1 points)
- (c) Plot a histogram of the residuals defined as  $r(k) = m(k) - u(k)$ ,  $m(k)$  being the voltage determined using the model, your LS-estimates and the values of  $i(k)$ ;  $u(k)$  are the measurements. What can be said about the noise in each data set? Give an educated guess of the type of noise. (2 points)
- (d) Estimate the variance of the measurement noise for each dataset. Then estimate the covariance matrix  $\Sigma$  for your earlier calculated LS-estimators for each data set. **Hint:** Check section 4.7 on the script. (3 points)
- (e) The following code extracts from the covariance  $\Sigma$  a confidence ellipsoid associated with  $E$  and  $R$ ; more specifically, the  $1\sigma$  confidence ellipsoid. Keep in mind that in the code example we assume that  $\hat{\theta} = [E, R]$ , if your estimate vector considers the parameters in different order your indexing will have to change. Make sure you understand how the confidence ellipsoids are calculated. Plot and compare both confidence ellipsoids on the same plot. Can you explain why the ellipsoids are different, despite using the same model for both measurements and similar values for the estimates? (2 points)

```
%Compute eigenvalues and eigenvectors for estimators
[V1,D1] = eig(sigmaEst1); % eigenvalues and eigenvectors
[V2,D2] = eig(sigmaEst2); % eigenvalues and eigenvectors

%Generate coordinates of 50 points on a unit circle
xy=[cos(linspace(0,2*pi,50)); sin(linspace(0,2*pi,50))];

%Generate the points of the confidence ellipse
xy_ellipse1 = [E_est1; R_est1]*ones(1,50) + V1*sqrt(D1)*xy;
xy_ellipse2 = [E_est2; R_est2]*ones(1,50) + V2*sqrt(D2)*xy;
```

*This sheet gives in total 13 points*