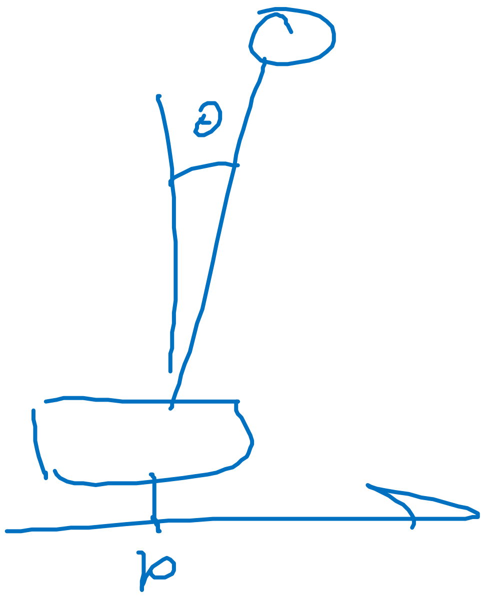
EX 7



$$x = \begin{bmatrix} p \\ v \\ \theta \\ \omega \end{bmatrix} =$$

$v = \dot{p}$

$\omega = \dot{\theta}$

$$\dot{x} = \begin{bmatrix} \dot{p} \\ \dot{v} \\ \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} v \\ \cdot \\ \omega \\ \cdot \end{bmatrix}$$

$v = \dot{p}$

$\dot{\omega} = \ddot{\theta}$

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2 (a)

$$\| M_N(\theta) - Y_N \|^2_{\Sigma^{-1}} = \| \Sigma^{-\frac{1}{2}} (M(\theta) - Y_N) \|_2^2 = \| \Sigma^{-\frac{1}{2}} M_N(\theta) - \Sigma^{-\frac{1}{2}} Y_N \|_2^2$$

$$= \| \tilde{M}(\theta) - \tilde{Y} \|_2^2$$

$$V^* = \min_\theta \| \tilde{M}(\theta) - \tilde{Y} \|_2^2 \approx N - d$$

$$\boxed{\tilde{\sigma}^2 = \frac{V^*}{N-d}}$$

$\theta \in \mathbb{R}^d$

$\tilde{Y} \in \mathbb{R}^N$

FOR LLS, $\tilde{M}(\theta) = \Phi_N$

$$\Sigma_{\hat{\theta}} = \tilde{\sigma}^2 \cdot (\Phi_N^T \Phi_N)^{-1}$$

FOR NLS

$$\Phi_N \approx \frac{\partial \tilde{M}}{\partial \theta}(\hat{\theta})$$

JACOBIAN

SUMMARY LAST LECTURE

$\boxed{\text{DFT (FFT)}}$

$U = \text{fft}(u)$ $\quad u \in \mathbb{R}^N$

$\Updownarrow$ $\boxed{O(N \cdot \log N)}$ $\quad U \in \mathbb{C}^N$

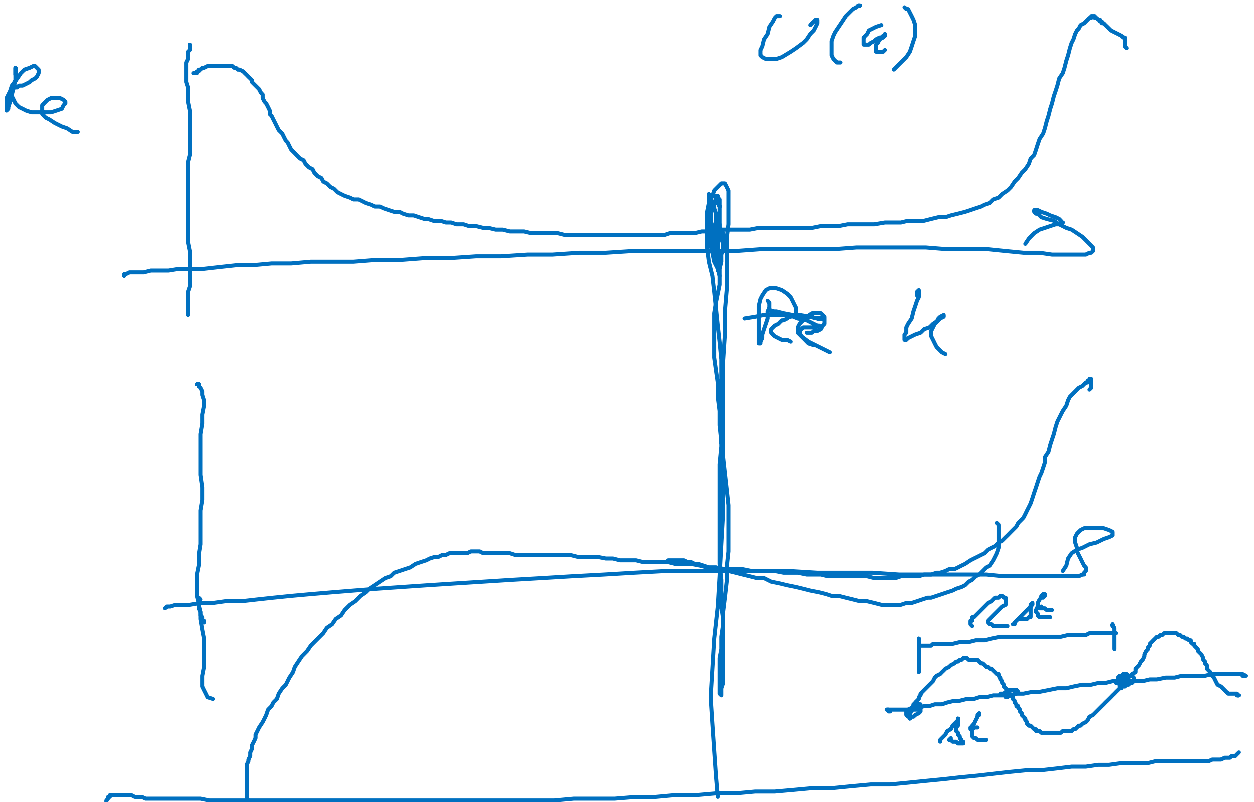
$U = M \cdot u$ $\quad \boxed{O(N^2)}$

$M : \mathbb{C}^{N \times N}$, UNITARY $M^* \cdot M = I$

$u_{new} = \text{ifft}(U) = M^* \cdot U$

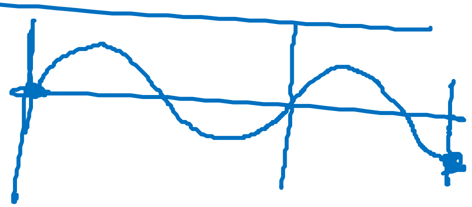
$\boxed{\|u\|_2^2 = \|U\|_2^2}$ PARSEVAL-IDENTITY

PROOF: $\|U\|_2^2 = \|Mu\|_2^2 = u^T M^* \cdot M u = u^T I u = \|u\|_2^2$

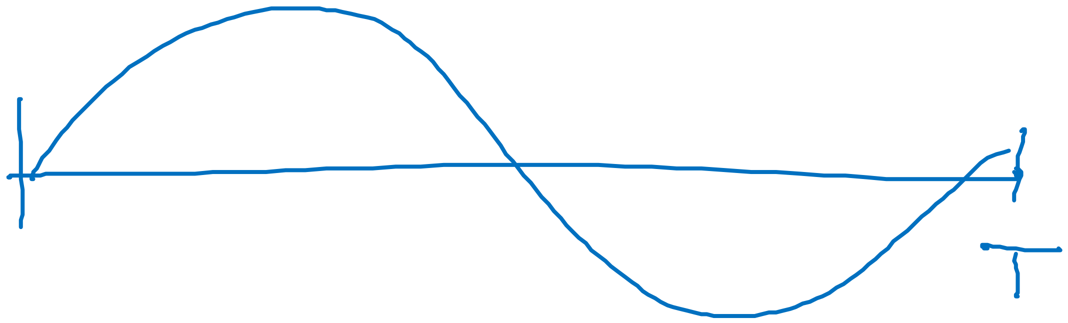


ALIASING: SAMPLING TOO SLOW FOR FREQU.
(NYQUIST $\frac{1}{2\Delta t}$)

LEAKAGE: "WINDOWING EFFECT": PERIODS DO NOT FIT EXACTLY INTO WINDOW



BASE FREQUENCY



$$\omega_{BASE} = \frac{2\pi}{T}$$

8.3 MULTISINE EXCITATION SIGNAL

AIM:
CREATE PERIODIC EXC. SIGNAL WITH ALL DESIRED FREQUENCIES AT ONCE.

ADVANTAGE: WAIT ONLY ONCE UNTIL TRANSIENTS DIE OUT. (ASSUMING STABLE SYSTEMS)

WINDOW LENGTH T

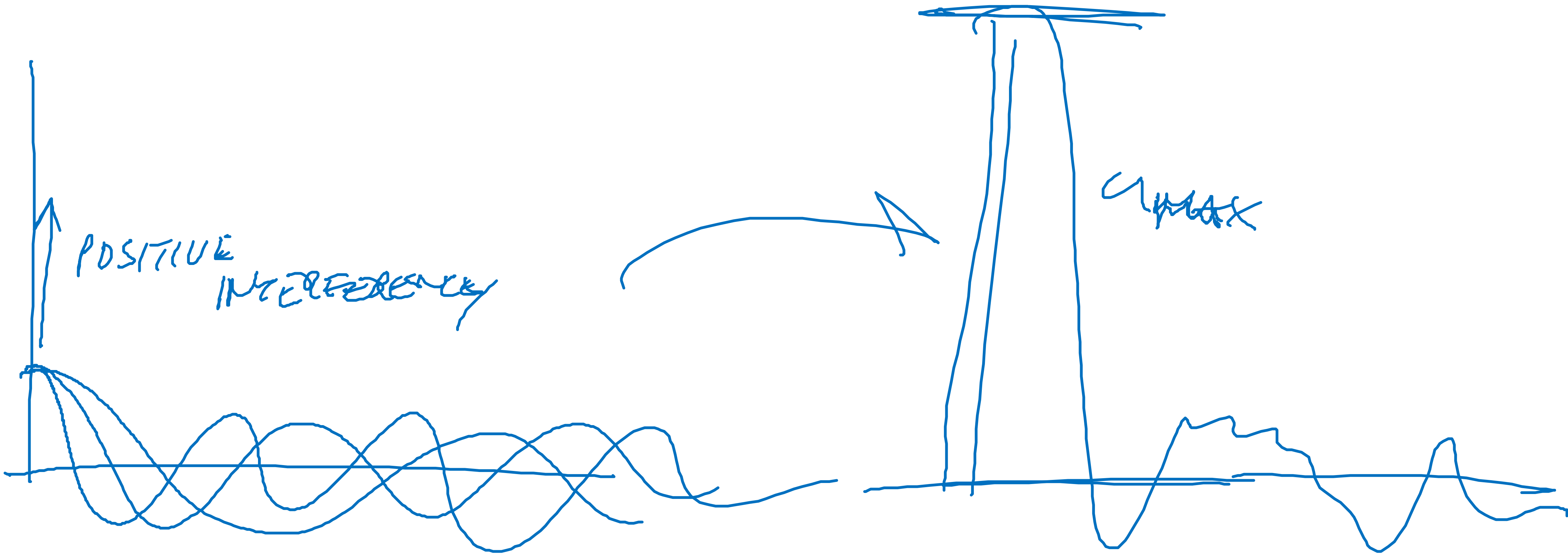
SAMPLING TIME Δt

1. $T = N \cdot \Delta t$ WITH N INTEGER (OFTEN $N = 2^n$, GOOD FOR FFT) E.G. 4096
2. ONLY ADD FREQUENCIES THAT ARE INT. MULT. OF ω_{BASE} $2\omega_{BASE}$, $3\omega_{BASE}$, ...
3. DO NOT ADD ANY FREQUENCY HIGHER OR EQUAL TO $\omega_{NYQUIST} = \frac{2\pi}{2\Delta t} = \frac{\pi}{\Delta t}$ (STOP EARLIER, EG. AT $\frac{\omega_{NYQUIST}}{2}$)

$\omega_{BASE} = \frac{2\pi}{T}$

$\frac{\omega_{NYQ}}{2} = \frac{2\pi}{4\Delta t}$

$\frac{\omega_{NYQ}/2}{\omega_{BASE}} = \frac{T}{4\Delta t} = \frac{N}{4}$ IS HIGHEST INT. MULT. OF BASE FREQ.



$$C_{CR} := \frac{U_{MAX}}{U_{RMS}} \quad \text{(FOR ANY SIGNAL)}$$

$$U_{MAX} = \max_{t \in [0,T]} |u(t)|$$

$$U_{RMS} := \sqrt{\frac{1}{T} \int_0^T |u(t)|^2 \, dt} = \sqrt{\frac{1}{N} \Sigma |u(k)|^2} = \sqrt{\frac{1}{N} \Sigma |U(k)|^2}$$

BAD / HIGH CREST FACTOR C_{CR}

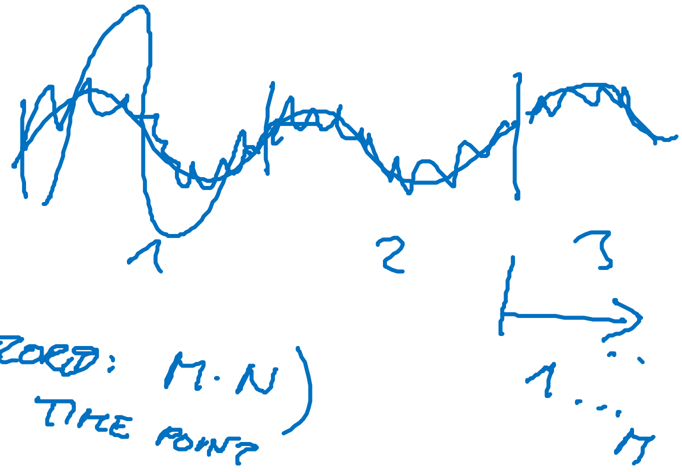
IDEALLY, HAVE SMALL CREST FACTOR

8.3.3 MULTISINE ID. PROCED.

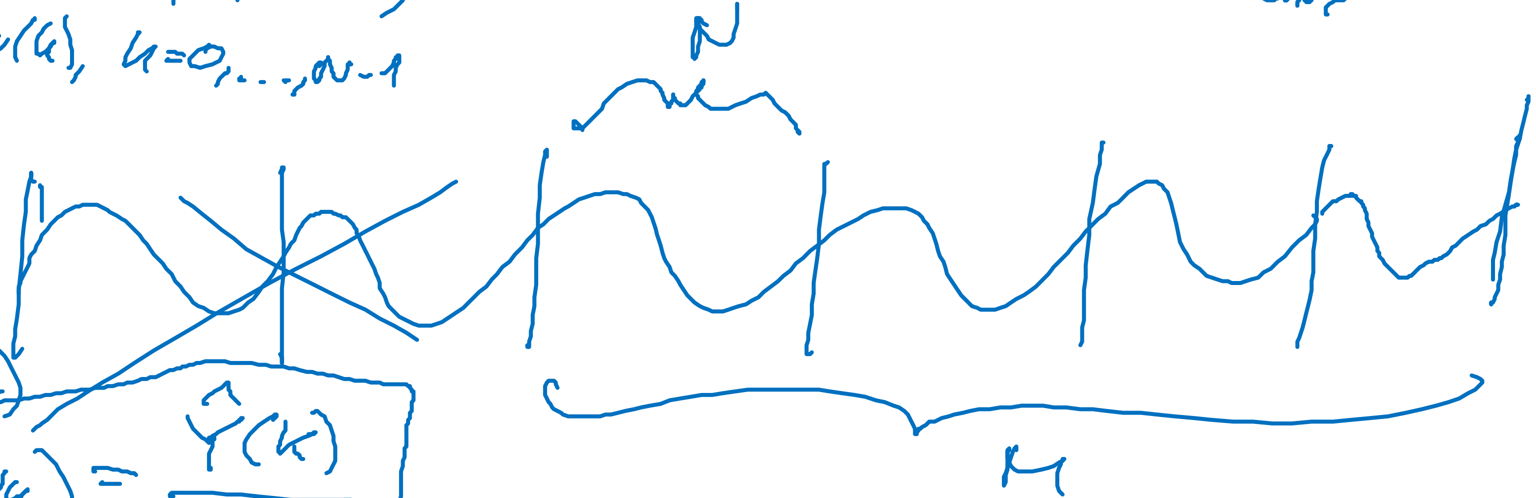
STEP 1: INPUT DESIGN $\left(\omega_k := \omega_{BASE} \cdot k\right)$

STEP 2: EXP. & ANALYSIS

- APPLY MULTISINE INPUT MANY PERIODS TO THE SYSTEM
  WAIT UNTIL TRANSIENTS DIED OUT,



- USE LATEST $\underline{\underline{M}} \in \mathbb{N}$ PERIODS FOR IDENTIFICATION (RECORD: $M \cdot N$ TIME POINTS)
- AVERAGE THEM $\left(u(k), y(k)\right)$
  GET $\hat{u}(k)$, $\hat{y}(k)$, $k = 0, \dots, N-1$
- APPLY DFT TO BOTH, GET $\hat{Y}(k)$, $\hat{U}(k)$
  (FOR k IN THE SIGNAL)



- ESTIMATE $\boxed{\hat{G}(j\omega_k) = \dfrac{\hat{Y}(k)}{\hat{U}(k)}}$

8.3.4 M.S. ERROR ANALYSIS

TWO WAYS TO LOOK AT MS PROC.

1. AVERAGE IN TIME DOMAIN 2. DO DFT 3. BUILD QUOTIENT $\hat{G} = \frac{\hat{Y}}{\hat{U}}$	$\Longleftrightarrow$	1. DO DFT OF EACH WINDOW 2. AVERAGE IN FR. DOMAIN 3. QUOTIENT
↑ YOU DO THIS		↑ ANALYSE THIS

WHY THE SAME? $\bar{U}_{[j]} = M_{DFT} \cdot u_{[j]}$

$$\sum_{j=1}^{M} \bar{U}_{[j]} = \sum M_{DFT} u_{[j]} = M_{DFT} \cdot \sum_{j=1}^{M} u_{[j]}$$

BECAUSE FFT IS LINEAR OP, SO INTERCHANGES WITH AVERAGING

ASS:
1. NOISE IN M WINDOWS UNCORRELATED, ZERO MEAN
2. NOISE ON U & Y IS UNCORR.
3. AFTER DFT OF EACH WINDOW, NOISE ON $U(k)$ & $Y(k)$ IS "CIRCULAR COMPLEX NORMALLY DISTR." (CCND) WITH NOISE LEVELS $G_U^2(k)$ & $G_Y^2(k)$ [k IS FREQUENCY]

4. DUE AVERAGING

$$\boxed{\begin{aligned} \hat{U}(k) &= \bar{U}_0(k) + N_{\hat{U}}(k) \\ \hat{Y}(k) &= \bar{Y}_0(k) + N_{\hat{Y}}(k) \end{aligned}}$$

NOISE $N_{\hat{U}}$ & $N_{\hat{Y}}$ WILL BE CCND WITH VARIANCES

$$G_{\hat{U}}^2(k) = \frac{G_U^2(k)}{M}, \quad G_{\hat{Y}}^2(k) = \frac{G_Y^2(k)}{M}$$

ANAL. QUOTIENT:

$$\hat{G}(j\omega_k) = \frac{\hat{Y}(k)}{\hat{U}(k)} = \frac{Y_0(k) + N_{\hat{Y}}(k)}{U_0(k) + N_{\hat{U}}(k)} = G_0(j\omega_k) \cdot \frac{\left(1 + \frac{N_{\hat{Y}}(k)}{Y_0(k)}\right)}{\left(1 + \frac{N_{\hat{U}}(k)}{U_0(k)}\right)}$$

$$\approx G_0(j\omega_k) \cdot \left(1 + \frac{N_{\hat{Y}}(k)}{Y_0(k)} - \frac{N_{\hat{U}}(k)}{U_0(k)} + \cancel{\ldots}\right)$$

$$\mathbb{E}\{\hat{G}(j\omega_k)\} = G_0(j\omega_k)$$ UNBIASED ✓

$$\boxed{G_0(j\omega_k) = \frac{Y_0}{U_0} \qquad \text{SNR}: \frac{|U_0(k)|}{\sigma_U(k)}}$$

$$\sigma^2_{\hat{G}}(k) = |G_0(j\omega_k)| \cdot \left(\frac{\sigma^2_{\hat{Y}}(k)}{|Y_0(k)|^2} + \frac{\sigma^2_{\hat{U}}(k)}{|U_0(k)|^2}\right)$$

VARIANCE

$$= |\quad " \quad| \cdot \frac{1}{M} \cdot \left(\frac{\sigma^2_Y(k)}{|Y_0(k)|^2} + \frac{\sigma^2_U(k)}{|U_0(k)|^2}\right)$$

SHRINKS WITH M

SHRINK WITH INVERSE OF SIGNAL TO NOISE RATIO