Numerical Optimal Control with DAEs
Lecture 9: Indirect Optimal Control

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AWESCO PhD course

Objectives of the Lecture

- Basics of the Pontryagin Maximum Principle (PMP)
- Numerical difficulties of PMP & how to tackle them
- Some properties of PMP
- Singular OCPs & their impact on Direct Optimal Control

Outline

1. Introduction to the Pontryagin Maximum Principle (PMP)
2. Interpretation of $H_{\mathbf{u}}$
3. Input bounds in Indirect Optimal Control
4. Singular Optimal Control problems
5. General constraints in Indirect Optimal Control

Pontryagin Maximum Principle

Simple continuous problem:

$$\begin{aligned} \min_{\mathbf{x},\mathbf{u}} \quad & \phi(\mathbf{x}(t_\mathrm{f})) + \int_{t_0}^{t_\mathrm{f}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \mathrm{d}t \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right), \\ & \mathbf{x}\left(t_0\right) = \mathbf{x}_0 \end{aligned}$$

Define the Hamiltonian function

$$H\left(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}\right) = L\left(\mathbf{x}, \mathbf{u}\right) + \boldsymbol{\lambda}^\top \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right)$$

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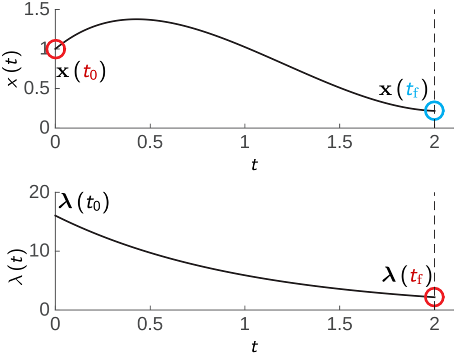

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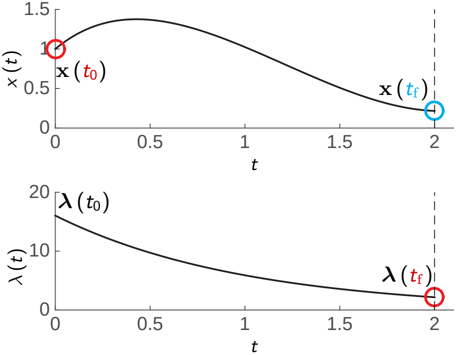

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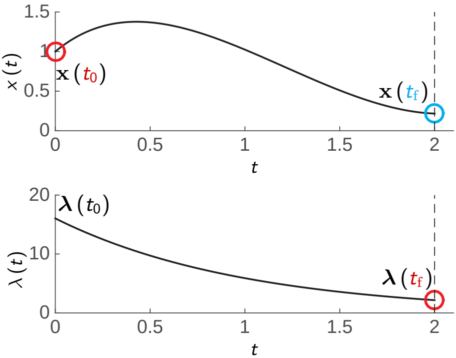

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Labelled a **Two Points Boundary Value Problem** (TPBVP)

Overview

	Continuous Equations	Discrete Equations
Global	Hamilton-Jacobi-Bellman (HJB)	Dynamic Programming (DP)
Local	Pontryagin (PMP)	Direct Optimal Control (DOC)

Continuous problem:

$$\begin{aligned} \min_{\mathbf{u}\in\mathcal{U}} \quad & \phi(\mathbf{x}(t_\mathrm{f})) + \int_{t_0}^{t_\mathrm{f}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \mathrm{d}t \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right), \\ & \mathbf{x}\left(t_0\right) = \mathbf{x}_0 \end{aligned}$$

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PMP: define the Hamiltonian function

$$H\left(\mathbf{x},\boldsymbol{\lambda},\mathbf{u}\right) = L\left(\mathbf{x},\mathbf{u}\right) + \boldsymbol{\lambda}^\top \mathbf{F}\left(\mathbf{x},\mathbf{u}\right)$$

Get the optimal input $\mathbf{u}\left(\mathbf{x},\boldsymbol{\lambda}\right) = \arg\min_{\mathbf{u}} H\left(\mathbf{x},\boldsymbol{\lambda},\mathbf{u}\right)$
Use it in the state-costate integration:

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- PMP equations provide an "∞"-dimensional input profile $\mathbf{u}\left(.\right)$
- State constraints hard to handle

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D.O.C. describes the solution as a finite set of variables $\mathbf{w}$ transform the problem into a discrete one

Solve the resulting Nonlinear Program (NLP):

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi\left(\mathbf{w}\right) \\ \text{s.t.} \quad & \mathbf{g}\left(\mathbf{w}\right) = 0, \\ & \mathbf{h}\left(\mathbf{w}\right) \leq 0 \end{aligned}$$

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- Input profile restricted to a finite-dimensional space (e.g. piecewise-constant)
- Easy to treat all types of constraints

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The big distinction in Optimal Control...

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"First optimise then discretize" (HJB & PMP)

- First write the **continuous equations** describing the solution to the problem
- ... then **discretize the equations** & solve

Note: the PMP "family" is referred to as *Indirect* optimal control here

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vs.

"First discretize then optimise " (DP & DOC)

- **First discretize** the continuous OCP into a discrete one...
- ... then write the **discrete equations** describing the solution & solve

Pontryagin Maximum Principle

Simple continuous problem:

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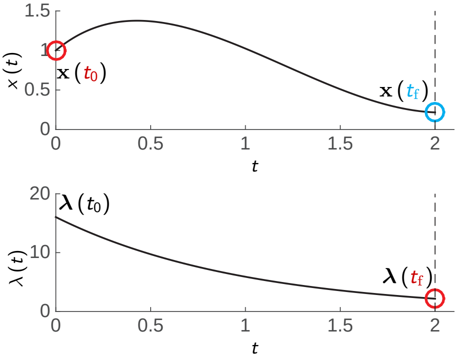

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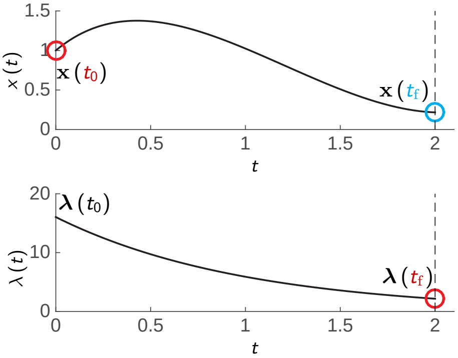

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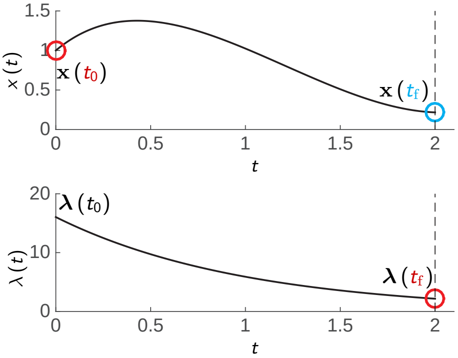

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Two Points Boundary Value Problem

- Integrate forward ? We have $\mathbf{x}(t_0) = \mathbf{x}_0$, but we don't have $\boldsymbol{\lambda}(t_0)$...

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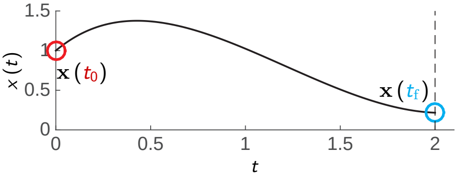

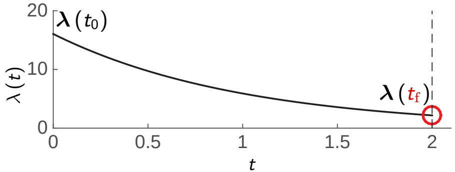

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Two Points Boundary Value Problem

- Integrate forward ? We have $\mathbf{x}(t_0) = \mathbf{x}_0$, but we don't have $\boldsymbol{\lambda}(t_0)$...
- Integrate forward-backward ? Integrate state forward from $\mathbf{x}(t_0) = \mathbf{x}_0$ then backward from $\boldsymbol{\lambda}(t_\mathrm{f}) = \nabla_{\mathbf{x}}\phi\left(\mathbf{x}\left(t_\mathrm{f}\right)\right)$... but we don't know $\mathbf{u}$...

Pontryagin Maximum Principle

Simple continuous problem:

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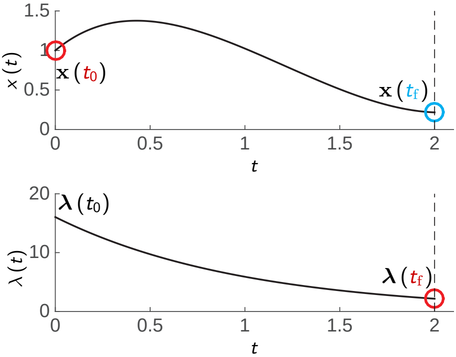

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Two Points Boundary Value Problem

Note that the **entire** solution is "described by" $\boldsymbol{\lambda}(t_0)$

Solving the PMP equations / TPBVP

Input: Initial conditions $\mathbf{x}_0$, guess $\boldsymbol{\lambda}_0$
while $\|\mathbf{r}\| > \mathrm{tol}$ **do**

Integrate with $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$
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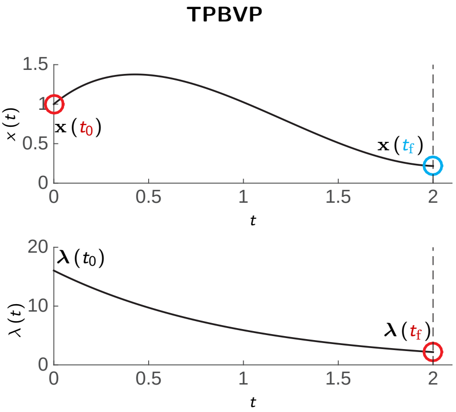

Solving the PMP equations / TPBVP

Input: Initial conditions $\mathbf{x}_0$, guess $\boldsymbol{\lambda}_0$

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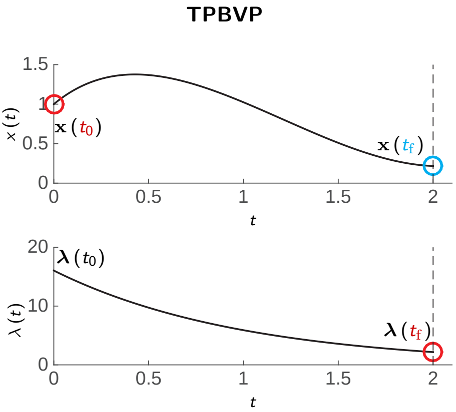

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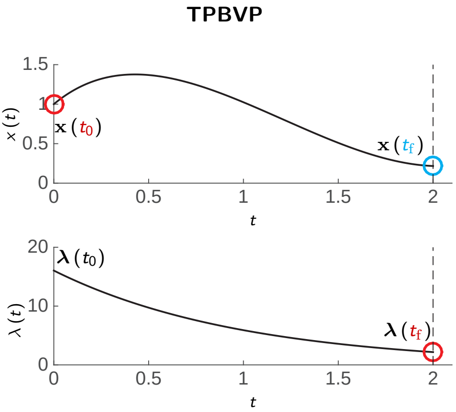

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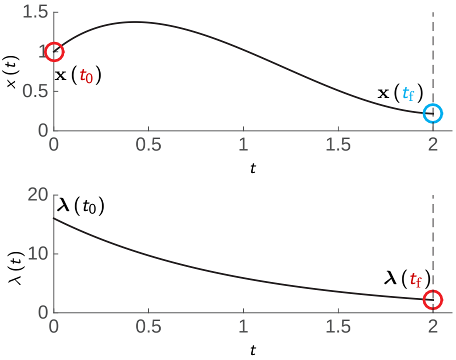

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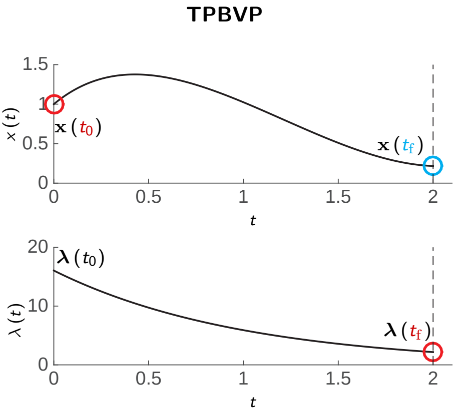

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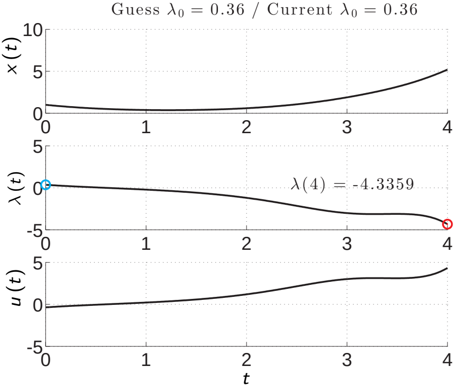

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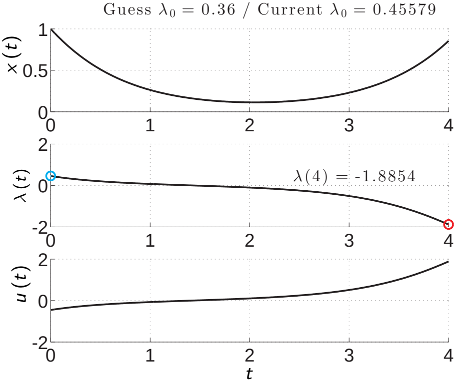

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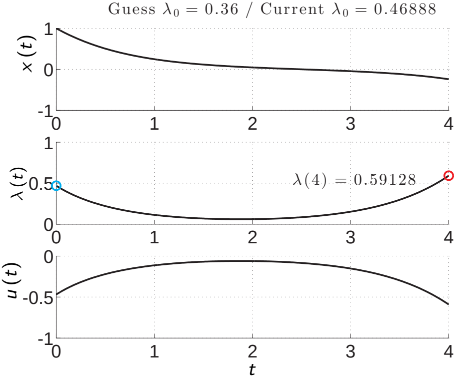

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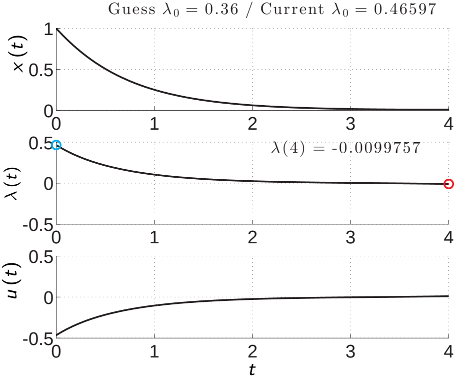

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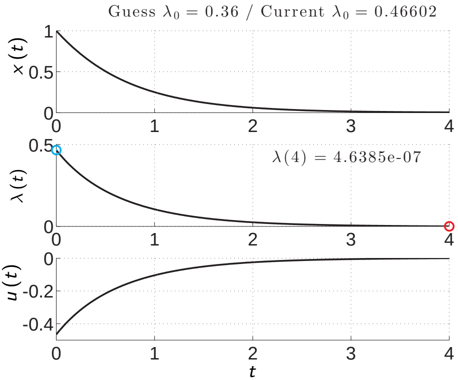

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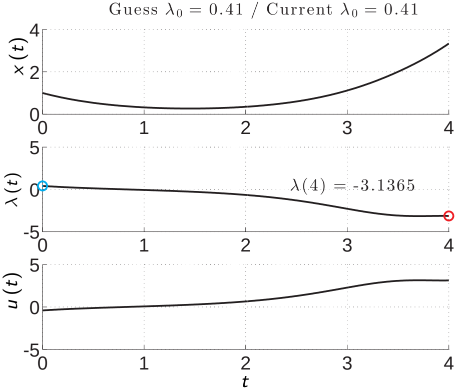

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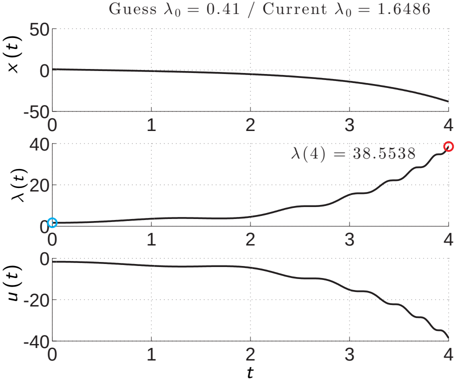

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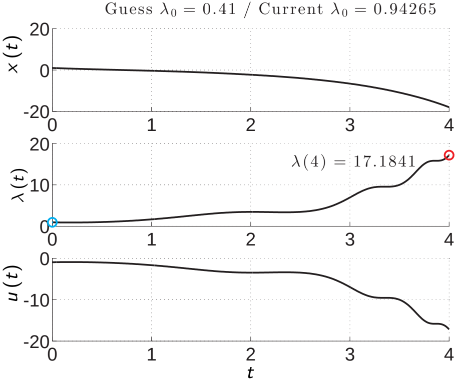

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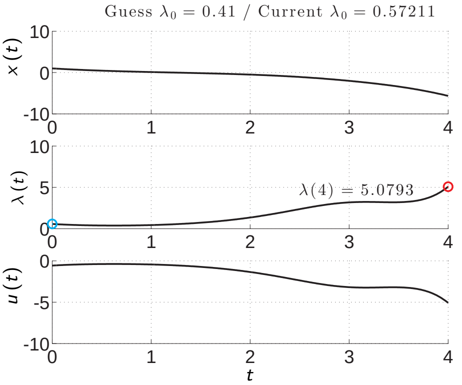

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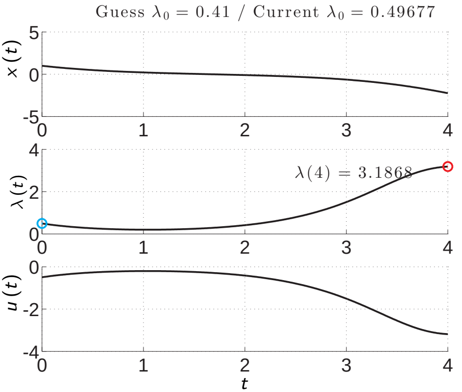

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Find λ_0 to enforce $\lambda_f = \lambda(4) = 0$!!

Example:

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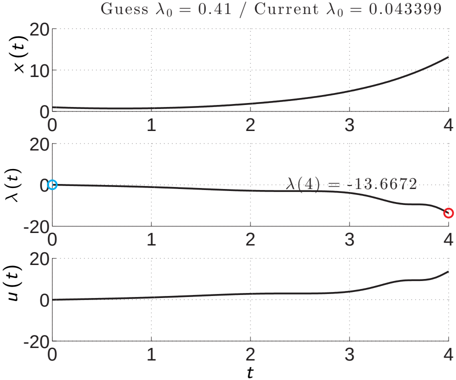

Solving the PMP equations / TPBVP

Input: Initial conditions $\mathbf{x}_0$, guess $\boldsymbol{\lambda}_0$
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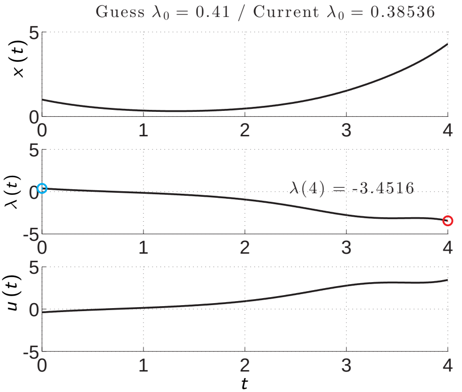

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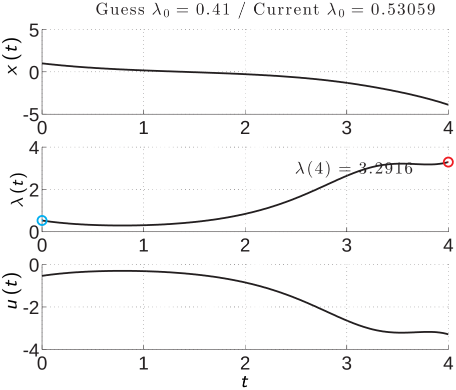

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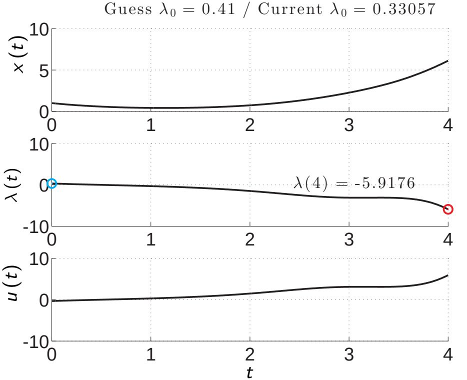

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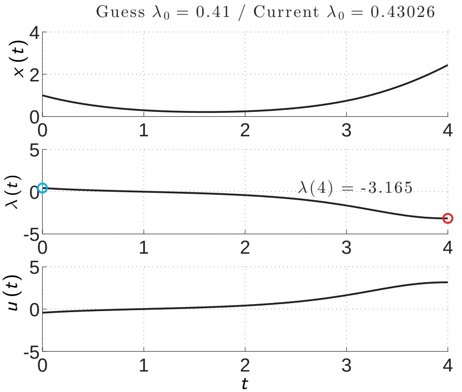

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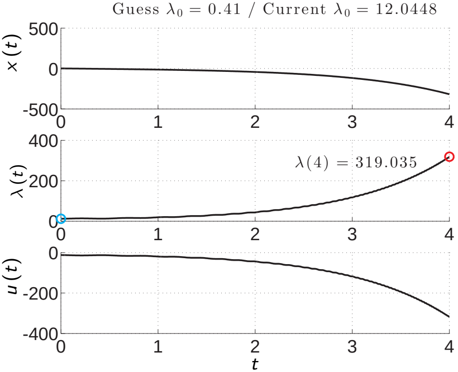

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What is going on ?!?

Let's try to understand the relationship
$\lambda_0 \to \lambda(t)$

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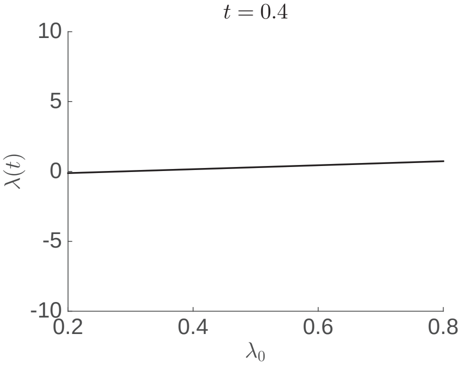

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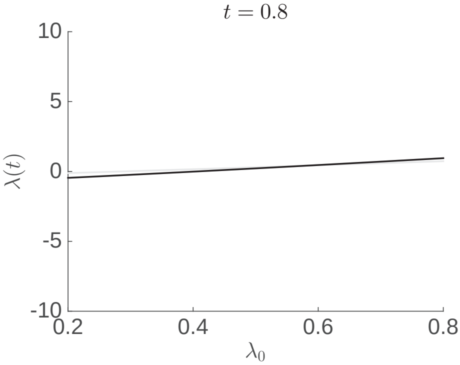

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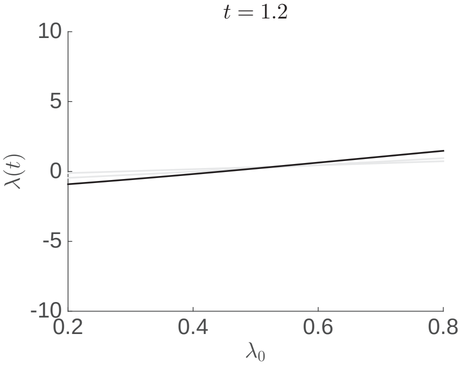

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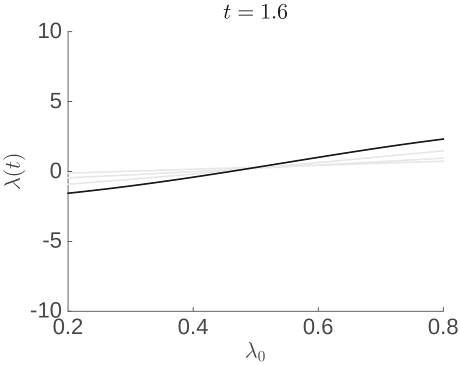

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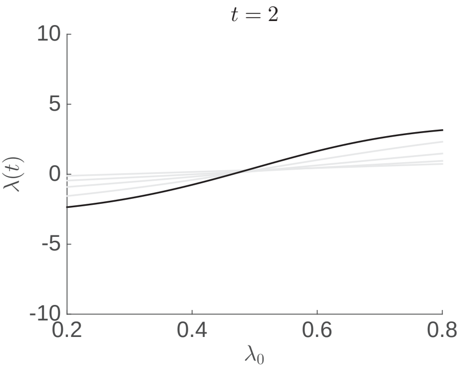

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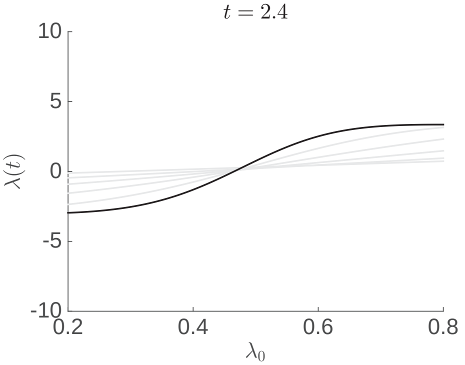

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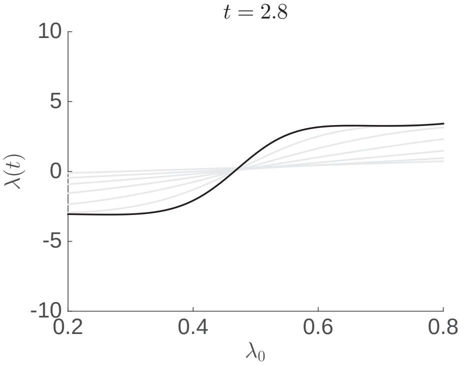

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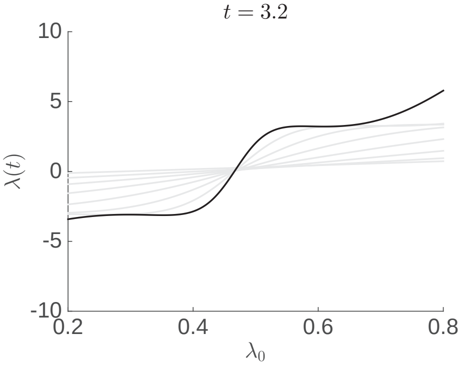
Find λ_0 to enforce $\lambda_\mathrm{f} = \lambda(4) = 0$!!

Example:

$$\min_{\mathbf{x},\mathbf{u}} \quad \frac{1}{2}\int_0^4 \left(x^2 + u^2\right) dt$$

$$\dot{x} = u - \sin(x), \quad x(0) = 1$$



Solving the PMP equations / TPBVP

Input: Initial conditions $\mathbf{x}_0$, guess $\boldsymbol{\lambda}_0$
while $\|\mathbf{r}\| > \mathrm{tol}$ **do**

Integrate with $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$
$$\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \qquad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$$

Compute:

$$\mathbf{r} = \boldsymbol{\lambda}(t_\mathrm{f}) - \nabla_{\mathbf{x}}\phi(\mathbf{x}(t_\mathrm{f})) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_0}$$

Newton step: $\boldsymbol{\lambda}_0 \leftarrow \boldsymbol{\lambda}_0 - \frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_0}^{-1}\mathbf{r}$

$$H(x, u, \lambda) = \frac{1}{2}\left(x^2 + u^2\right) + \lambda(u - \sin(x))$$

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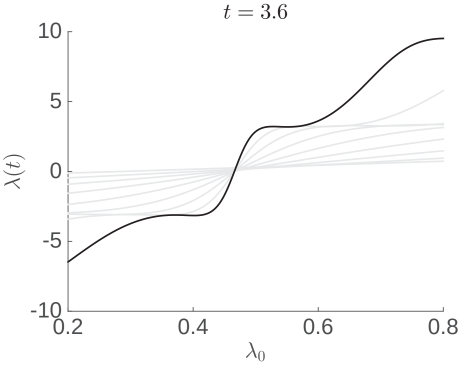

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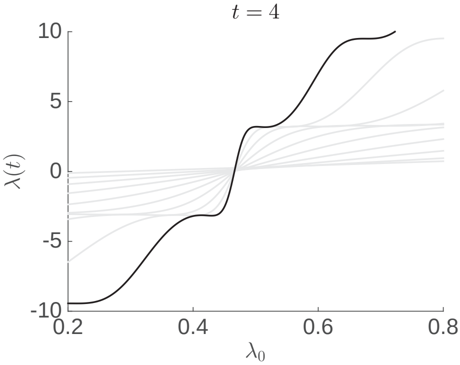

Conservation of volumes in the $\mathbf{x}, \boldsymbol{\lambda}$ space

Consider a compact domain $D(t)$ in the $\mathbf{x}(t)$, $\boldsymbol{\lambda}(t)$ space, with boundary $\partial D(t)$. The *volume* of $D(t)$ say $\rho(D(t)) \in \mathbb{R}$ is given by:

$$\rho(D(t)) = \int_{D(t)} \mathrm{d}D$$

and evolves with time as the integration of $\mathbf{x}(t)$, $\boldsymbol{\lambda}(t)$ proceeds.

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where is $\mathbf{n}_{\partial D}$ the normal to the surface ∂D

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$$\mathrm{div}\left(\left[\begin{array}{c} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\lambda}} \end{array} \right] \right) = \mathrm{Tr}\left(\frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}} + \frac{\partial \dot{\boldsymbol{\lambda}}}{\partial \boldsymbol{\lambda}} \right) = \mathrm{Tr}\left(\frac{\partial \mathbf{F}}{\partial \mathbf{x}} - \frac{\partial}{\partial \boldsymbol{\lambda}} \frac{\partial H}{\partial \mathbf{x}} \right) = 0$$

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The volume of $D(t)$ is constant throughout the integration of $\mathbf{x}$, $\boldsymbol{\lambda}$!!

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* See Liouville's Theorem on phase-space distribution functions in Hamiltonian mechanics

Conservation of volumes in the $\mathbf{x}, \boldsymbol{\lambda}$ space (cont')

Problem:

$$\min_{\mathbf{x},\mathbf{u}} \quad \frac{1}{2}\int_0^{t_\mathrm{f}} \left(x^2 + u^2\right) dt$$
$$\dot{x} = u - \sin(x), \quad x(0) = 1$$

Yields

$$H(x, u, \lambda) = \frac{1}{2}\left(x^2 + u^2\right) + \lambda\left(u - \sin(x)\right)$$

Hamiltonian is minimised by $u = -\lambda$.

The dynamics become

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Find λ_0 to enforce $\lambda_\mathrm{f} = \lambda(t_\mathrm{f}) = 0$

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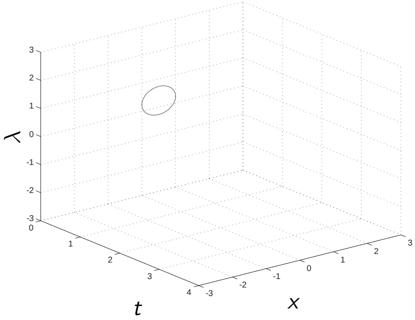

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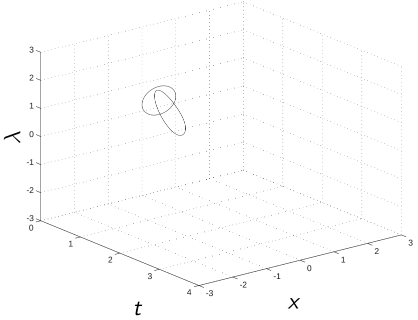

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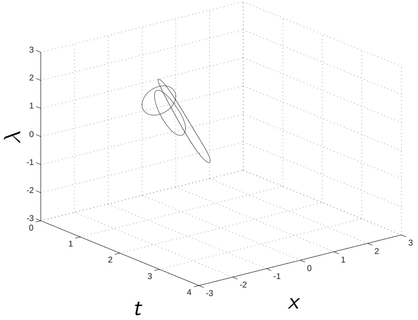

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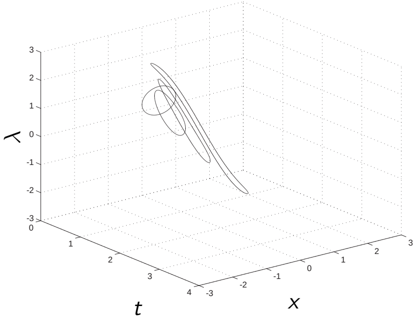

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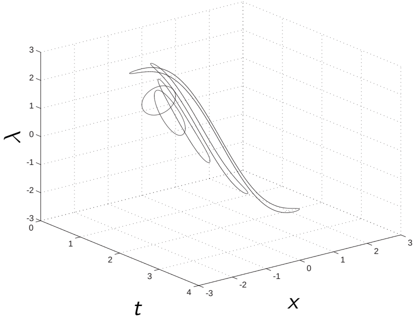

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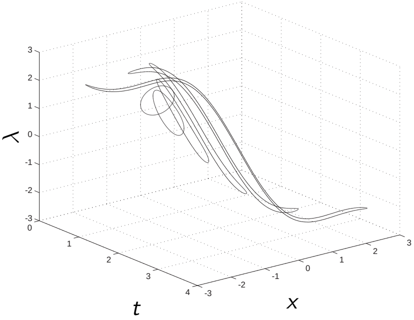

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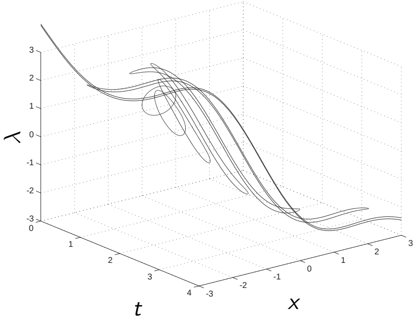

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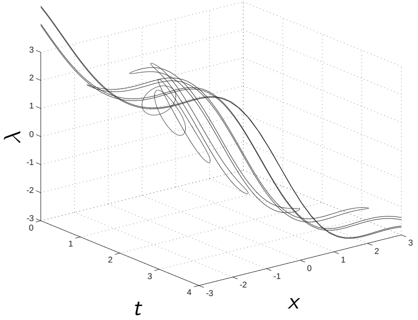

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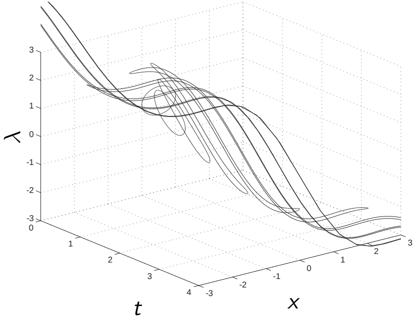

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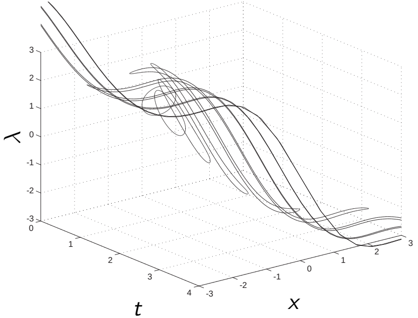

The punchline:

Conservation of volumes in the $\mathbf{x}, \boldsymbol{\lambda}$ space (cont')

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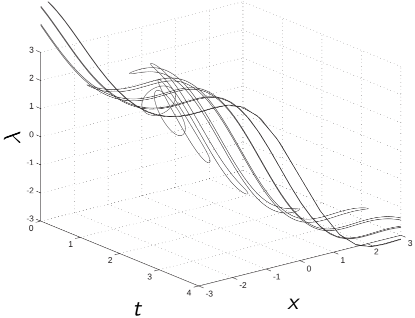


The punchline:
the states $\mathbf{x}$ and costates $\boldsymbol{\lambda}$ cannot be both stable, as it would imply $\rho(D(t)) \to 0$!!

Conservation of volumes in the $\mathbf{x}, \boldsymbol{\lambda}$ space (cont')

Problem:

$$\min_{\mathbf{x},\mathbf{u}} \quad \frac{1}{2}\int_0^{t_\mathrm{f}} \left(x^2 + u^2\right) dt$$

$$\dot{x} = u - \sin(x), \quad x(0) = 1$$

Yields

$$H(x, u, \lambda) = \frac{1}{2}\left(x^2 + u^2\right) + \lambda\left(u - \sin(x)\right)$$

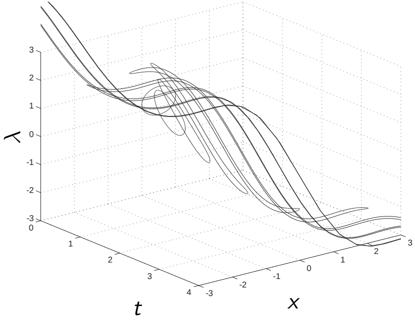
Hamiltonian is minimised by $u = -\lambda$.

The dynamics become

$$\dot{x} = -\lambda - \sin(x)$$

$$\dot{\lambda} = \lambda \cos(x) - x$$

Find λ_0 to enforce $\lambda_\mathrm{f} = \lambda(t_\mathrm{f}) = 0$



The punchline:
the states $\mathbf{x}$ and costates $\boldsymbol{\lambda}$ cannot be both stable, as it would imply $\rho(D(t)) \to 0$!!

Important consequence: the condition

$$\boldsymbol{\lambda}(t_\mathrm{f}) = \frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{x}(t_\mathrm{f}))$$

is often very sensitive to $\boldsymbol{\lambda}_0$!!

How to actually use PMP then ?

PMP equations: $\mathbf{u}^\star = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ with:

$$\begin{aligned} \text{States}: \quad & \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), & \mathbf{x}(t_0) = \bar{\mathbf{x}}_0 \\ \text{Costates}: \quad & \dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), & \boldsymbol{\lambda}(t_\mathrm{f}) = \nabla_{\mathbf{x}} \phi(\mathbf{x}(t_\mathrm{f})) \end{aligned}$$

Key idea: apply the multiple-shooting principle to the PMP equations

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Key idea: apply the multiple-shooting principle to the PMP equations

State-costate

$$\mathbf{s} = \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix}$$

Optimal control

$$\mathbf{u}(\mathbf{s}) = \arg\min_{\mathbf{u}} H(\mathbf{u}, \mathbf{s})$$

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Integrator $\boldsymbol{\xi}(\mathbf{s}_k)$ over the time intervals $t \in [t_k,\ t_{k+1}]$ with:

$$\dot{\mathbf{s}} = \begin{bmatrix} \mathbf{F}(\mathbf{x}, \mathbf{u}) \\ -\nabla_{\mathbf{x}} H(\mathbf{u}, \mathbf{s}) \end{bmatrix}, \quad \text{with} \quad \mathbf{s}(t_k) = \mathbf{s}_k$$

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Root-finding problem over the variables $\mathbf{s}_{0,\ldots,N}$:

$$\mathbf{r}(\mathbf{s}) = \begin{bmatrix} \mathbf{x}_0 - \bar{\mathbf{x}}_0 \\ \boldsymbol{\xi}(\mathbf{s}_0) - \mathbf{s}_1 \\ \vdots \\ \boldsymbol{\xi}(\mathbf{s}_{N-1}) - \mathbf{s}_N \\ \boldsymbol{\lambda}_N - \nabla_{\mathbf{x}} \phi(\mathbf{x}_N) \end{bmatrix} = 0$$

Indirect multiple-shooting

Problem:

$$\min_{\mathbf{x},\mathbf{u}} \quad \frac{1}{2}\int_0^{t_\mathrm{f}} \left(x^2 + u^2\right) dt$$

$$\dot{x} = u - \sin(x), \quad x(0) = 1$$

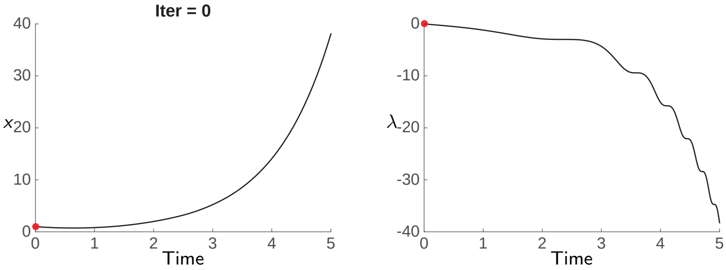
PMP equations with $u = -\lambda$:

$$\dot{x} = -\lambda - \sin(x), \qquad x(0) = 1$$

$$\dot{\lambda} = \lambda\cos(x) - x, \qquad \lambda(5) = 0$$

Initial guess: $\mathbf{x} = 1, \quad \boldsymbol{\lambda} = 0,$ **Final time**: $t_\mathrm{f} = 5,$ **full Newton steps**

$\#$ of shooting interval $= 1$ (single shooting)



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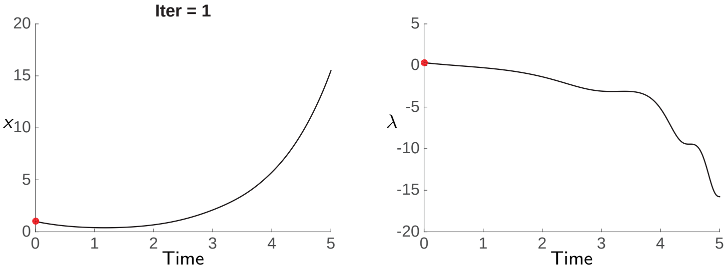

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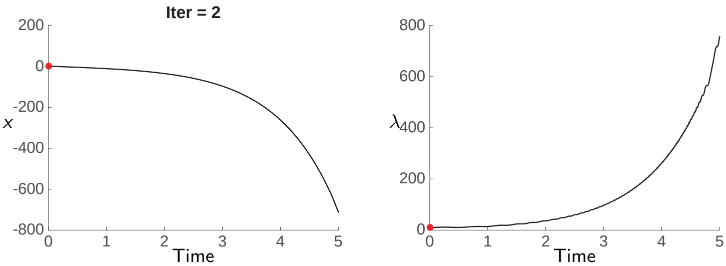

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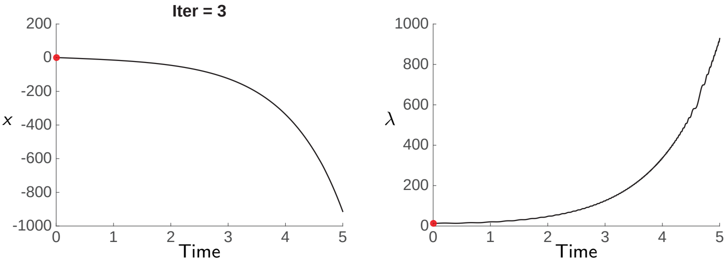

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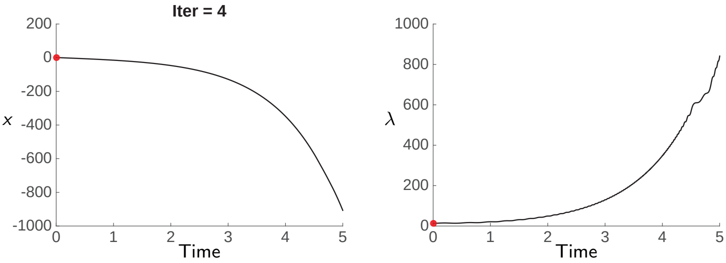

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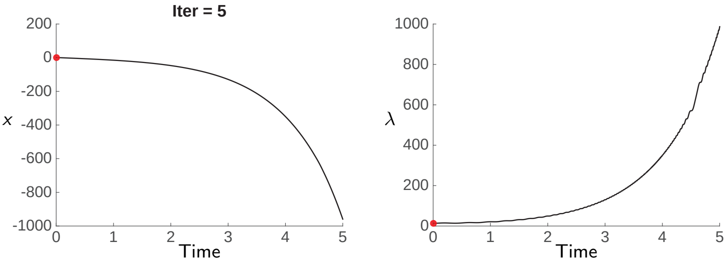

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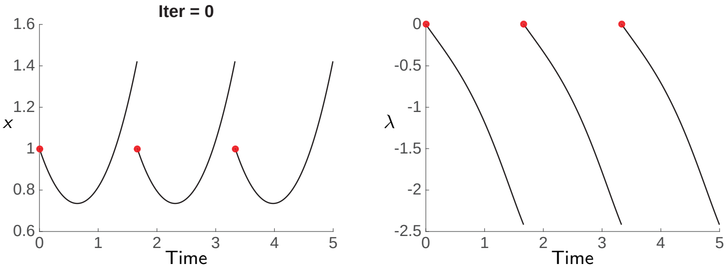
$\#$ of shooting interval $= 3$



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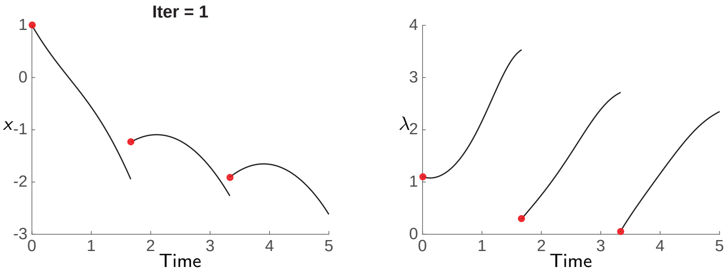

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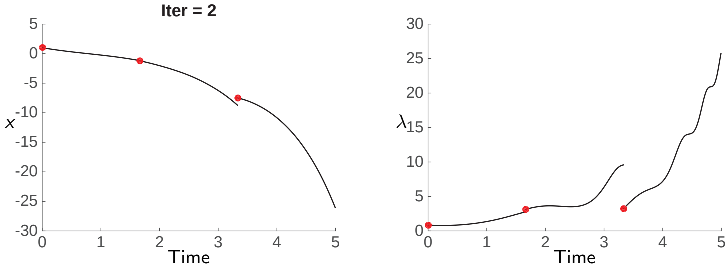

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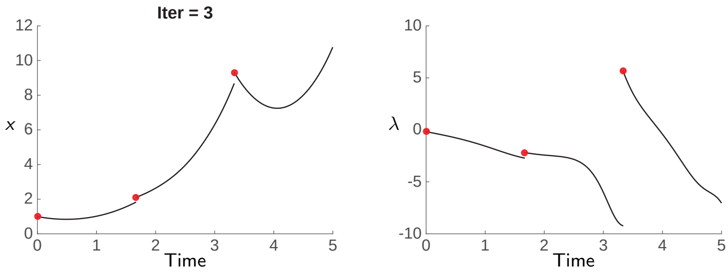

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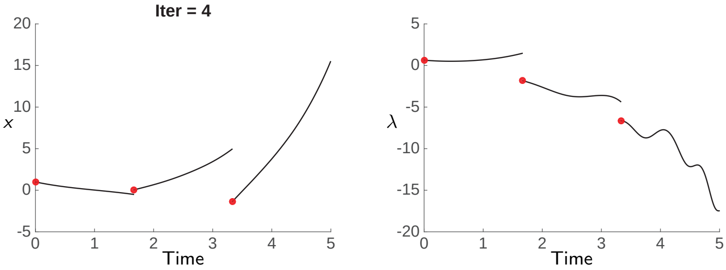

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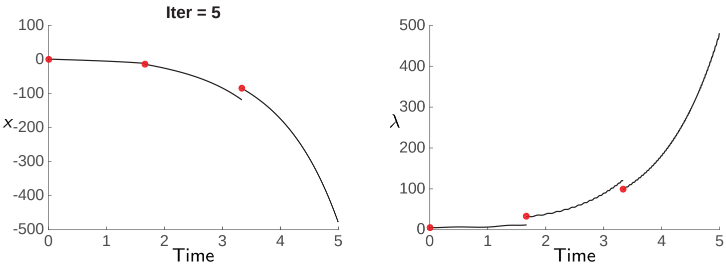

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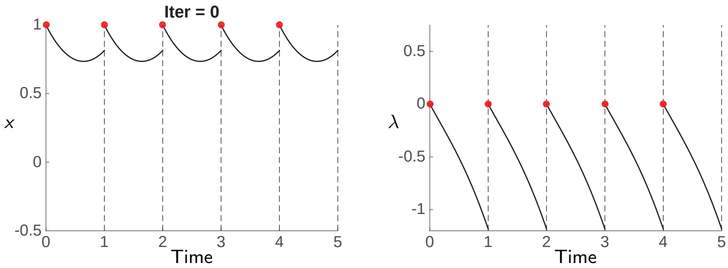

Indirect multiple-shooting

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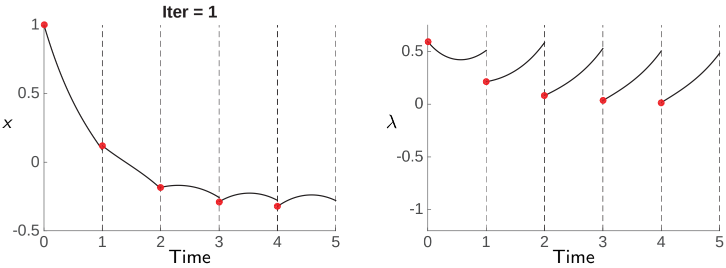

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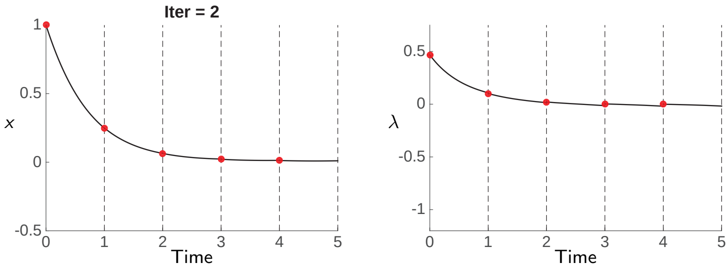

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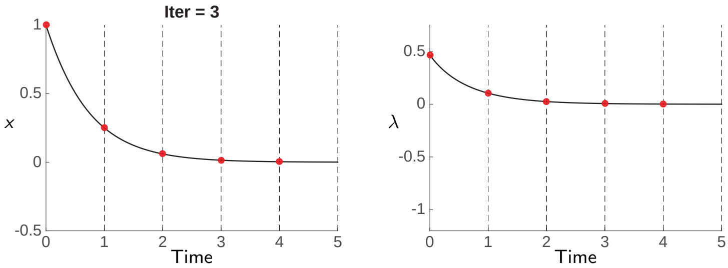

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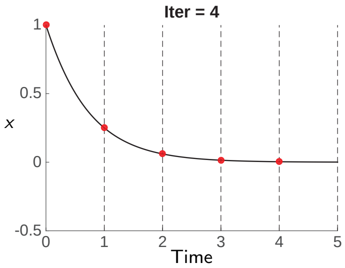

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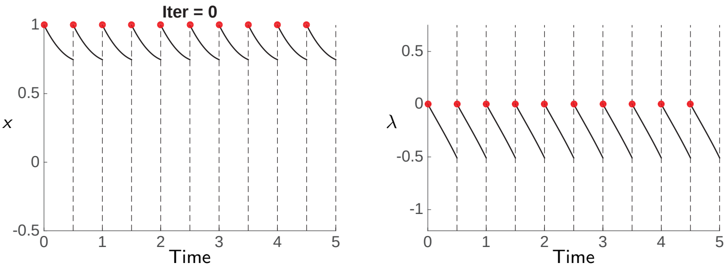
$\#$ of shooting interval $= 10$



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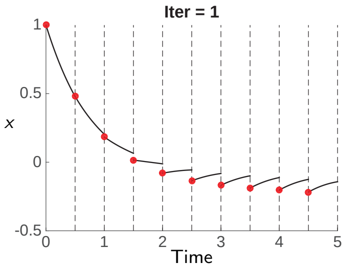

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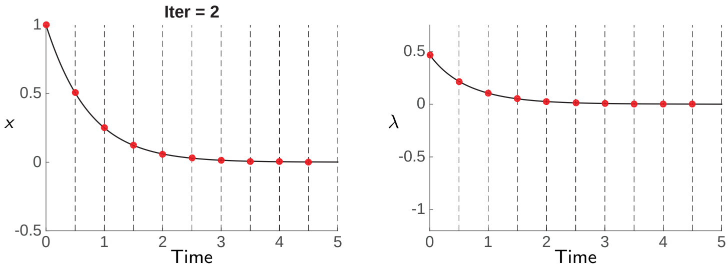


Indirect multiple-shooting

Problem:

$$\min_{\mathbf{x},\mathbf{u}} \quad \frac{1}{2}\int_0^{t_\mathrm{f}} \left(x^2 + u^2\right) dt$$

$$\dot{x} = u - \sin(x), \quad x(0) = 1$$

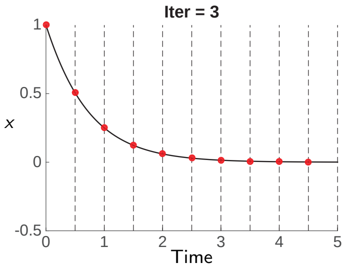
PMP equations with $u = -\lambda$:

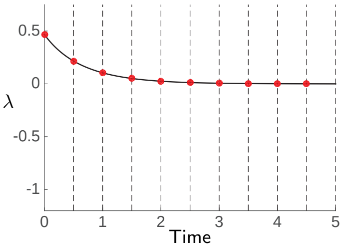
$$\dot{x} = -\lambda - \sin(x), \qquad x(0) = 1$$

$$\dot{\lambda} = \lambda\cos(x) - x, \qquad \lambda(5) = 0$$

Initial guess: $\mathbf{x} = 1, \quad \boldsymbol{\lambda} = 0,$ **Final time**: $t_\mathrm{f} = 5,$ **full Newton steps**

$\#$ of shooting interval $= 10$





Outline

Interpretation of $H_{\mathbf{u}}$

Consider the **functional**:

$$J\left[\mathbf{u}\left(.\right)\right] = \quad \phi\left(\mathbf{x}\left(t_{\mathrm{f}}\right)\right) + \int_{t_0}^{t_{\mathrm{f}}} L\left(\mathbf{x}, \mathbf{u}\right) \mathrm{d}t$$

$$\text{s.t.} \quad \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right), \quad \mathbf{x}\left(t_0\right) = \mathbf{x}_0$$

that maps an input profile $\mathbf{u}\left(.\right)$ into the corresponding cost.

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Gâteaux derivative (see Calculus of Variations)

$$\delta J[\mathbf{u}(.), \boldsymbol{\xi}(.)] = \lim_{\tau \to 0} \frac{J[\mathbf{u}(.) + \tau\boldsymbol{\xi}(.)] - J[\mathbf{u}(.)]}{\tau}$$

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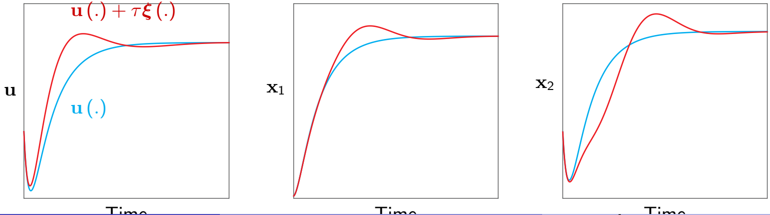

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Optimality:

$$\delta J\left[\mathbf{u}^{\star}\left(.\right), \boldsymbol{\xi}\left(.\right)\right] = 0, \quad \forall \boldsymbol{\xi}\left(.\right)$$

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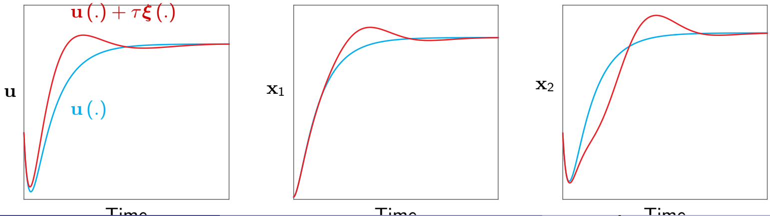

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What if $\mathbf{u}(.)$ is restricted to some (Banach) space ? E.g. piecewise-constant...

... **then $\boldsymbol{\xi}(.)$ is restricted to the same space !**

Interpretation of $H_{\mathbf{u}}$ (cont')

Consider a piecewise-constant parametrization of $\mathbf{u}(.)$...

... akin to a restriction of $\mathbf{u}(.)$ to that Banach space !

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Piecewise-constant parametrization

$$\mathbf{u}(t) = \mathbf{u}_k \qquad \forall\, t \in [t_k,\ t_{k+1}]$$
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$$J[\mathbf{u}(.)] \equiv \bar{J}(\mathbf{u}_1, ..., \mathbf{u}_{N-1})$$

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Then optimality requires:

$$\delta J\left[\mathbf{u}^\star\left(.\right), \boldsymbol{\xi}\left(.\right)\right] = \int_0^{t_\mathrm{f}} H_{\mathbf{u}}\left(\mathbf{x}(t), \boldsymbol{\lambda}\left(t\right), \mathbf{u}^\star\left(t\right)\right) \cdot \boldsymbol{\xi}\left(t\right) \mathrm{d}t =$$
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$$\frac{\partial \bar{J}}{\partial \mathbf{u}_k} = \int_{t_k}^{t_{k+1}} H_{\mathbf{u}}(\mathbf{x}(t), \boldsymbol{\lambda}(t), \mathbf{u}_k)\,\mathrm{d}t$$

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When solving an OCP using Direct Optimal Control, one can see the NLP solver as trying to get $\int_{t_k}^{t_{k+1}} H_{\mathbf{u}}(\mathbf{x}(t), \boldsymbol{\lambda}(t), \mathbf{u}_k)\,\mathrm{d}t = 0$!!

Outline

Input bounds in indirect optimal control

OCP with input bounds:

$$\begin{aligned} \min_{\mathbf{x},\mathbf{u}} \quad & \phi(\mathbf{x}(t_\mathrm{f})) + \int_{t_0}^{t_\mathrm{f}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) dt \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right), \quad \mathbf{x}\left(t_0\right) = \mathbf{x}_0 \\ & \mathbf{u}_{\min} \leq \mathbf{u} \leq \mathbf{u}_{\max} \end{aligned}$$

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Define the Hamiltonian function

$$H\left(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u}\right) = L\left(\mathbf{x}, \mathbf{u}\right) + \boldsymbol{\lambda}^\top \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right) + \boldsymbol{\mu}^\top \underbrace{\left[\begin{array}{c} \mathbf{u}_{\min} - \mathbf{u} \\ \mathbf{u} - \mathbf{u}_{\max} \end{array}\right]}_{\leq 0}$$

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When $\mathbf{u}$ hits the bounds, $\boldsymbol{\mu}$ "creates a gradient" in H to enforce feasibility

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$$\begin{aligned} \min_{\mathbf{x},\mathbf{u}} \quad & \phi(\mathbf{x}(t_\mathrm{f})) + \int_{t_0}^{t_\mathrm{f}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) dt \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right), \quad \mathbf{x}\left(t_0\right) = \mathbf{x}_0 \\ & \mathbf{u}_{\min} \leq \mathbf{u} \leq \mathbf{u}_{\max} \end{aligned}$$

An equivalent but simpler approach: define the Lagrange function

$$\mathcal{L}\left(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}\right) = L\left(\mathbf{x}, \mathbf{u}\right) + \boldsymbol{\lambda}^\top \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right)$$

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Note: optimality reads as...

$$\int_0^{t_\mathrm{f}} \mathcal{L}_{\mathbf{u}}\left(\mathbf{x}(t), \boldsymbol{\lambda}\left(t\right), \mathbf{u}\left(t\right)\right) \cdot \boldsymbol{\xi}\left(t\right) \mathrm{d}t \geq 0 \quad \text{for any } \textbf{feasible direction } \boldsymbol{\xi}(.)$$

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Note: $\mathbf{u}^\star$ is now a **non-smooth** function of $\mathbf{x}$, $\boldsymbol{\lambda}$. Must be handled carefully when solving TPBVP via Newton !!

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Outline

Singular Optimal Control problems

Consider

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PMP equations with $\mathcal{L}\left(\mathbf{x},\mathbf{u},\boldsymbol{\lambda}\right) = \boldsymbol{\lambda}^\top \mathbf{F}$

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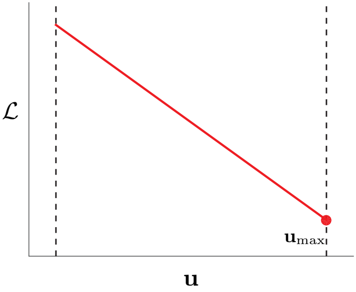

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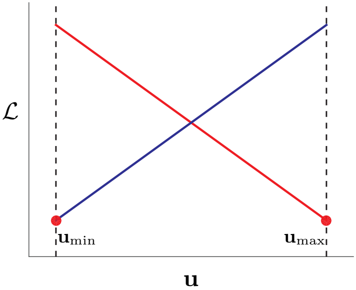

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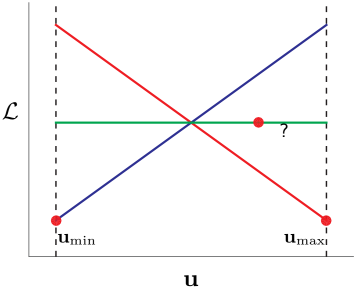

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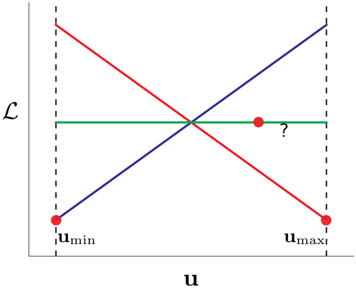

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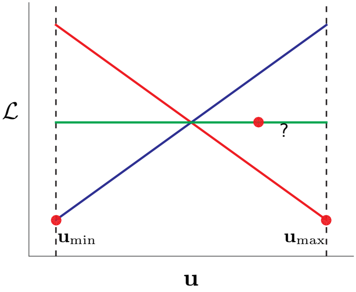

Find $\mathbf{u}$ when $\boldsymbol{\lambda}^\top \mathbf{g}\left(\mathbf{x}\right) = 0$?

$$\text{Use} \quad \frac{\mathrm{d}^i}{\mathrm{d}t^i}\mathcal{L}_\mathbf{u} = 0, \quad i > 0$$

for some i, the input $\mathbf{u}$ appears in $\frac{\mathrm{d}^i}{\mathrm{d}t^i}\mathcal{L}_\mathbf{u}$. Then solve $\frac{\mathrm{d}^i}{\mathrm{d}t^i}\mathcal{L}_\mathbf{u} = 0$ for $\mathbf{u}$!!

Singular Optimal Control - Example

$$
\begin{aligned}
\min_{\mathbf{x}(.),\mathbf{u}(.)} \quad & \frac{1}{2}\int_0^1 \mathbf{x}_1^2 \,\mathrm{d}t \\
\text{s.t.} \quad & \dot{\mathbf{x}} = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]\mathbf{x} + \left[\begin{array}{c} 0 \\ 1 \end{array}\right]\mathbf{u}, \quad \mathbf{x}(0) = \left[\begin{array}{c} 0 \\ 1 \end{array}\right] \\
& -5 \leq \mathbf{u} \leq 5
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$$\mathcal{L}_{\mathbf{u}} = \boldsymbol{\lambda}_2 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\mathbf{u}} = -\boldsymbol{\lambda}_1 \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{L}_{\mathbf{u}} = \mathbf{x}_1 \qquad \frac{\mathrm{d}^3}{\mathrm{d}t^3}\mathcal{L}_{\mathbf{u}} = \mathbf{x}_2 \qquad \frac{\mathrm{d}^4}{\mathrm{d}t^4}\mathcal{L}_{\mathbf{u}} = \mathbf{u}$$

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$$\mathcal{L}_{\mathbf{u}} = \boldsymbol{\lambda}_2 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\mathbf{u}} = -\boldsymbol{\lambda}_1 \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{L}_{\mathbf{u}} = \mathbf{x}_1 \qquad \frac{\mathrm{d}^3}{\mathrm{d}t^3}\mathcal{L}_{\mathbf{u}} = \mathbf{x}_2 \qquad \frac{\mathrm{d}^4}{\mathrm{d}t^4}\mathcal{L}_{\mathbf{u}} = \mathbf{u}$$

Optimal input:

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- We will have 2 switching times, to have $2 + 2 = 4$

Singular Optimal Control - Example

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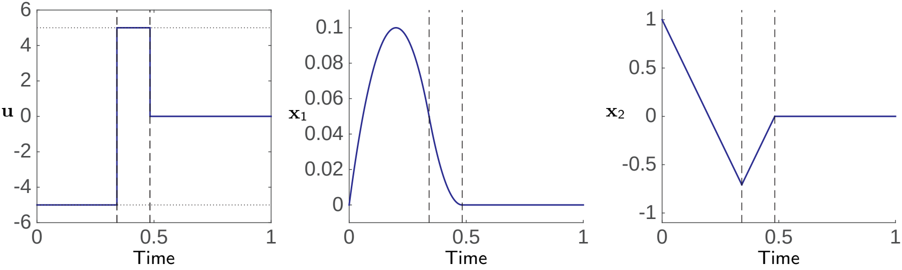
Optimal solution



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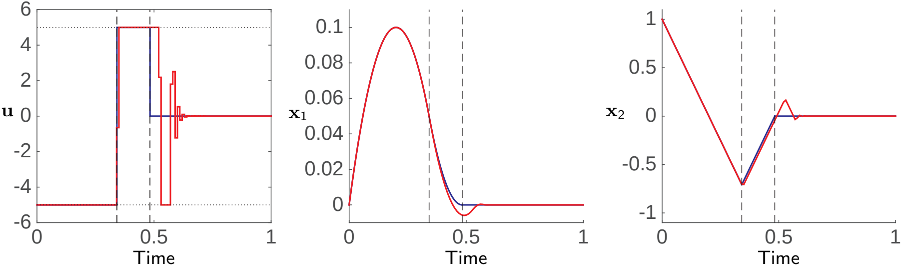
Optimal solution vs. solution from multiple-shooting $(t_{k+1} - t_k = 0.01)$



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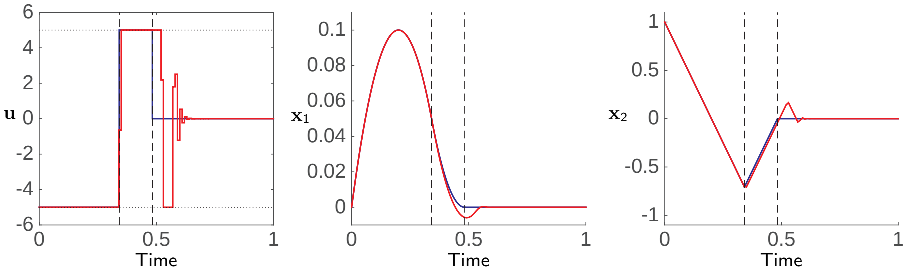
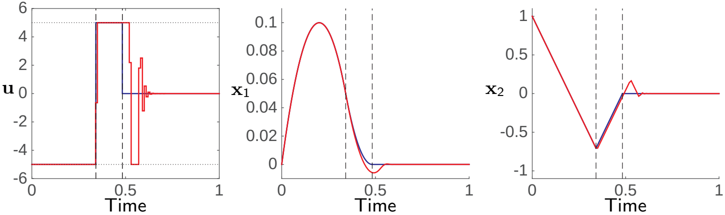


Very common behavior of direct optimal control for singular problems.
What is going on ?!?

Singular optimal control with direct methods



Consider the problem:

$$
\begin{aligned}
J(\mathbf{u}) = \quad & \phi\left(\mathbf{x}\left(.\right), \mathbf{u}\left(.\right)\right) \\
\text{s.t.} \quad & \dot{\mathbf{x}}(t) = \mathbf{F}\left(\mathbf{x}(t), \mathbf{u}(t)\right) \\
& \mathbf{x}\left(t_0\right) = \mathbf{x}_0 \\
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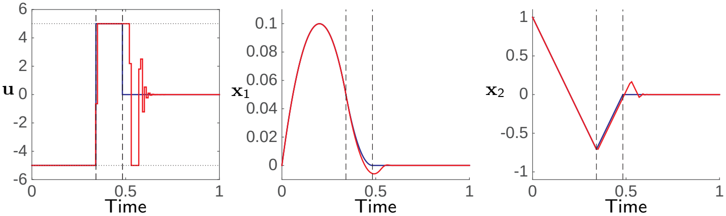
with $\mathbf{u}\left(t\right) = \mathbf{u}_k$ for $t \in [t_k,\ t_{k+1}]$.

Singular optimal control with direct methods



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with $\mathbf{u}(t) = \mathbf{u}_k$ for $t \in [t_k,\ t_{k+1}]$. Then

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\frac{\partial J}{\partial \mathbf{u}_k} = \int_{t_k}^{t_{k+1}} \mathcal{L}_{\mathbf{u}}\left(\mathbf{x}(t), \boldsymbol{\lambda}(t), \mathbf{u}_k\right) \mathrm{d}t
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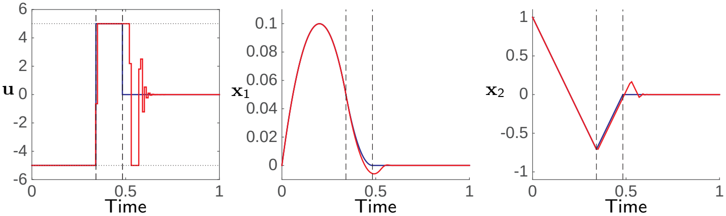
is zero when $\mathbf{u}_k$ is off the bounds.

Singular optimal control with direct methods



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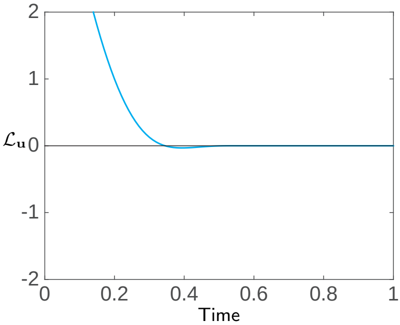
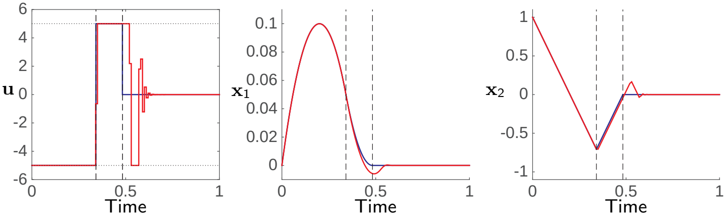


Singular optimal control with direct methods



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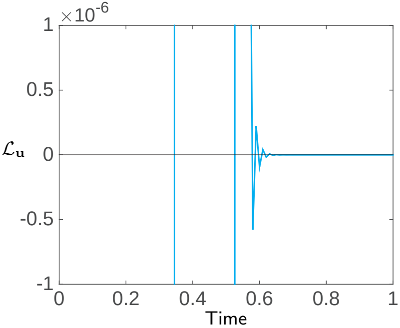

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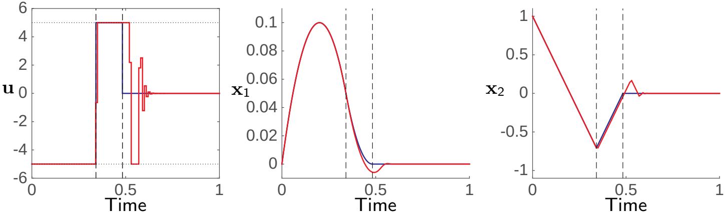

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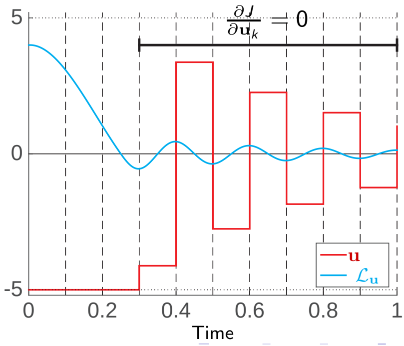

Outline

1. Introduction to the Pontryagin Maximum Principle (PMP)

2. Interpretation of $H_{\mathbf{u}}$

3. Input bounds in Indirect Optimal Control

4. Singular Optimal Control problems

5. General constraints in Indirect Optimal Control

State constraints in indirect optimal control

OCP with state (mixed) constraints:

$$\begin{aligned}
\min_{\mathbf{x},\mathbf{u}} \quad & \phi(\mathbf{x}(t_\mathrm{f})) + \int_{t_0}^{t_\mathrm{f}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) dt \\
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Define the Hamiltonian function

$$H\left(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u}\right) = L\left(\mathbf{x}, \mathbf{u}\right) + \boldsymbol{\lambda}^\top \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right) + \boldsymbol{\mu}^\top \mathbf{h}\left(\mathbf{x}, \mathbf{u}\right)$$

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$$\begin{aligned} \text{States}: \quad & \dot{\mathbf{x}} = \mathbf{F}, & \mathbf{x}(t_0) = \mathbf{x}_0 \\ \text{Costates}: \quad & \dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H, & \boldsymbol{\lambda}(t_\mathrm{f}) = \nabla_{\mathbf{x}} \phi\left(\mathbf{x}\left(t_\mathrm{f}\right)\right) \end{aligned}$$

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$$H\left(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u}\right) = L\left(\mathbf{x}, \mathbf{u}\right) + \boldsymbol{\lambda}^\top \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right) + \boldsymbol{\mu}^\top \mathbf{h}\left(\mathbf{x}, \mathbf{u}\right)$$

Get the optimal control solution from: $\mathbf{u}^\star = \underset{\mathbf{u}}{\mathrm{argmin}}\; H\left(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u}\right)$, with:

$$\begin{aligned} \text{States}: \quad & \dot{\mathbf{x}} = \mathbf{F}, & \mathbf{x}(t_0) = \mathbf{x}_0 \\ \text{Costates}: \quad & \dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H, & \boldsymbol{\lambda}(t_\mathrm{f}) = \nabla_{\mathbf{x}} \phi\left(\mathbf{x}\left(t_\mathrm{f}\right)\right) \\ \text{Feasibility}: \quad & \mathbf{h}\left(\mathbf{x}, \mathbf{u}\right) \leq 0, & \boldsymbol{\mu} \geq 0 \end{aligned}$$

State constraints in indirect optimal control

OCP with state (mixed) constraints:

$$\begin{aligned}
\min_{\mathbf{x},\mathbf{u}} \quad & \phi(\mathbf{x}(t_\mathrm{f})) + \int_{t_0}^{t_\mathrm{f}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) dt \\
\text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right), \quad \mathbf{x}\left(t_0\right) = \mathbf{x}_0 \\
& \mathbf{h}\left(\mathbf{x}, \mathbf{u}\right) \leq 0
\end{aligned}$$

Define the Hamiltonian function

$$H\left(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u}\right) = L\left(\mathbf{x}, \mathbf{u}\right) + \boldsymbol{\lambda}^\top \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right) + \boldsymbol{\mu}^\top \mathbf{h}\left(\mathbf{x}, \mathbf{u}\right)$$

Get the optimal control solution from: $\mathbf{u}^\star = \underset{\mathbf{u}}{\operatorname{argmin}}\ H\left(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u}\right)$, with:

$$\begin{aligned}
\text{States}: \quad & \dot{\mathbf{x}} = \mathbf{F}, & \mathbf{x}(t_0) = \mathbf{x}_0 \\
\text{Costates}: \quad & \dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H, & \boldsymbol{\lambda}(t_\mathrm{f}) = \nabla_{\mathbf{x}} \phi\left(\mathbf{x}\left(t_\mathrm{f}\right)\right) \\
\text{Feasibility}: \quad & \mathbf{h}\left(\mathbf{x}, \mathbf{u}\right) \leq 0, & \boldsymbol{\mu} \geq 0 \\
\text{Complementary slack.}: \quad & \boldsymbol{\mu}^\top \mathbf{h}\left(\mathbf{x}, \mathbf{u}\right) = 0 &
\end{aligned}$$

State constraints in indirect optimal control

OCP with state (mixed) constraints:

$$\begin{aligned} \min_{\mathbf{x},\mathbf{u}} \quad & \phi(\mathbf{x}(t_\mathrm{f})) + \int_{t_0}^{t_\mathrm{f}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) dt \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right), \quad \mathbf{x}\left(t_0\right) = \mathbf{x}_0 \\ & \mathbf{h}\left(\mathbf{x}, \mathbf{u}\right) \leq 0 \end{aligned}$$

Define the Hamiltonian function

$$H\left(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u}\right) = L\left(\mathbf{x}, \mathbf{u}\right) + \boldsymbol{\lambda}^\top \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right) + \boldsymbol{\mu}^\top \mathbf{h}\left(\mathbf{x}, \mathbf{u}\right)$$

Get the optimal control solution from: $\mathbf{u}^\star = \underset{\mathbf{u}}{\operatorname{argmin}} \; H\left(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u}\right)$, with:

$$\begin{aligned} \text{States}: \quad & \dot{\mathbf{x}} = \mathbf{F}, & \mathbf{x}(t_0) = \mathbf{x}_0 \\ \text{Costates}: \quad & \dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H, & \boldsymbol{\lambda}(t_\mathrm{f}) = \nabla_{\mathbf{x}} \phi\left(\mathbf{x}\left(t_\mathrm{f}\right)\right) \\ \text{Feasibility}: \quad & \mathbf{h}\left(\mathbf{x}, \mathbf{u}\right) \leq 0, & \boldsymbol{\mu} \geq 0 \\ \text{Complementary slack.}: \quad & \boldsymbol{\mu}^\top \mathbf{h}\left(\mathbf{x}, \mathbf{u}\right) = 0 & \end{aligned}$$

The PMP equations can be hard to solve in general. No good PMP-based general-purpose solver available.

State constraints in indirect optimal control

OCP with state (mixed) constraints:

$$\begin{aligned}\min_{\mathbf{x},\mathbf{u}} \quad & \phi(\mathbf{x}(t_\mathrm{f})) + \int_{t_0}^{t_\mathrm{f}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) dt \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right), \quad \mathbf{x}\left(t_0\right) = \mathbf{x}_0 \\ & \mathbf{h}\left(\mathbf{x}, \mathbf{u}\right) \leq 0\end{aligned}$$

Tentative solutions based on IP method

$$\begin{aligned}\text{Stationarity}: \quad & H_{\mathbf{u}}\left(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u}\right) = 0 \\ \text{States}: \quad & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right), & \mathbf{x}(t_0) = \mathbf{x}_0 \\ \text{Costates}: \quad & \dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H\left(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u}\right), & \boldsymbol{\lambda}(t_\mathrm{f}) = \nabla_{\mathbf{x}}\phi\left(\mathbf{x}\left(t_\mathrm{f}\right)\right) \\ \text{Feasibility}: \quad & \mathbf{h}\left(\mathbf{x}, \mathbf{u}\right) \leq 0, & \boldsymbol{\mu} \geq 0 \\ \text{Complementary slack.}: \quad & \boldsymbol{\mu}^\top \mathbf{h}\left(\mathbf{x}, \mathbf{u}\right) = \tau\end{aligned}$$

- Handle dynamics + constraints $H_{\mathbf{u}} = 0$ and $\boldsymbol{\mu}^\top \mathbf{h} = \tau$ as a DAE (c.f. next week)
- Handle $\mathbf{h}\left(\mathbf{x}, \mathbf{u}\right) \leq 0$ and $\boldsymbol{\mu} \geq 0$ via step length (c.f. IP lecture)
- Also done using Primal IP approach (move $\mathbf{h}$ in the cost using log barrier)