Numerical Optimal Control with DAEs
Lecture 11: High-Index DAEs

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AWESCO PhD course

Objectives of the lecture

- Why are DAEs not always "easy"to solve ?
- What is a DAE index ? How does it make the DAE "easy" or not ?
- What to do about it ?
- What happens with DAE models for mechanical systems ?
- Some possible additional numerical problems

Outline

1. "Easy" & "Hard" DAEs
2. Differential Index
3. Index Reduction
4. Constraints drift

DAE - Easy & Hard DAEs

DAE:

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}\right) = 0$$

"At any time instant, for given $\mathbf{x}, \mathbf{u}$, the DAE equation provides $\dot{\mathbf{x}}, \mathbf{z}$, generating the trajectories."

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What is a "well-behaved" DAE ?

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Consider $\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}\right) = 0$ as a **root-finding problem** in $\dot{\mathbf{x}}$, $\mathbf{z}$. When can we find $\dot{\mathbf{x}}$, $\mathbf{z}$, e.g. using Newton ?

"Easy" DAEs

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$$\dot{\mathbf{x}}, \mathbf{z} = \boldsymbol{\xi}\left(\mathbf{x}, \mathbf{u}\right)$$

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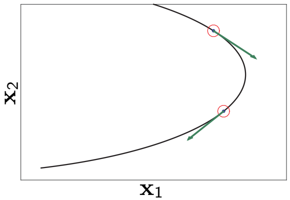

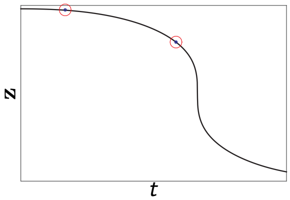

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Semi-explicit DAE:

$$\tilde{\mathbf{F}} = \begin{bmatrix} \dot{\mathbf{x}} - \mathbf{F}\left(\mathbf{z}, \mathbf{x}, \mathbf{u}\right) \\ \mathbf{G}\left(\mathbf{z}, \mathbf{x}, \mathbf{u}\right) \end{bmatrix} = 0$$

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Then the matrix:

$$\begin{bmatrix} \frac{\partial \tilde{\mathbf{F}}}{\partial \dot{\mathbf{x}}} & \frac{\partial \tilde{\mathbf{F}}}{\partial \mathbf{z}} \end{bmatrix} = \begin{bmatrix} I & \frac{\partial \tilde{\mathbf{F}}}{\partial \mathbf{z}} \\ 0 & \frac{\partial \mathbf{G}}{\partial \mathbf{z}} \end{bmatrix}$$

is full-rank if $\frac{\partial \mathbf{G}}{\partial \mathbf{z}}$ is full rank

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I.e. we can compute $\dot{\mathbf{x}}$, $\mathbf{z}$ from $\mathbf{x}$

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Then:

$$\begin{bmatrix} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}_{1,2}} & \frac{\partial \mathbf{F}}{\partial \mathbf{z}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is rank-deficient.

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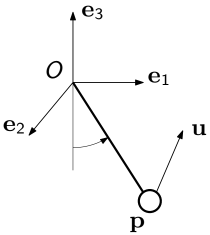
is rank-deficient. This is a "not easy" DAE !! We cannot write $\dot{\mathbf{x}}_{1,2}$ and $\mathbf{z}$ as functions of $\mathbf{x}_{1,2}$...

DAE - 3D pendulum

Model is a **semi-explicit** DAE with $\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \end{bmatrix}$

$$\begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{v}} \end{bmatrix} = \dot{\mathbf{x}} = \overbrace{\begin{bmatrix} \mathbf{v} \\ \frac{\mathbf{u}}{m} - g\mathbf{e}_3 - \frac{z}{m}\mathbf{p} \end{bmatrix}}^{\mathbf{F}(\mathbf{x},z,\mathbf{u})}$$

$$0 = \underbrace{\mathbf{p}^\top \mathbf{p} - L^2}_{\mathbf{G}(\mathbf{x})}$$



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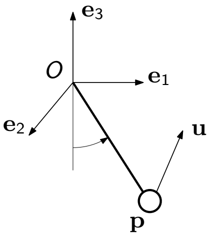

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Consider the root-finding problem to be solved in $\dot{\mathbf{x}}, z$:

$$\mathbf{r}(\dot{\mathbf{x}}, \mathbf{x}, z, \mathbf{u}) = \begin{bmatrix} \dot{\mathbf{x}} - \mathbf{F}(\mathbf{x}, z, \mathbf{u}) \\ \color{blue}{\mathbf{G}(\mathbf{x})} \end{bmatrix} = 0$$

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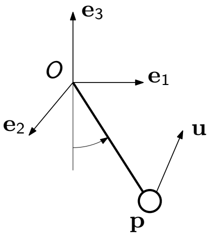

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Then:

$$\nabla_{\dot{\mathbf{x}},z} \mathbf{r}^\top = \begin{bmatrix} I & 0 & 0 \\ 0 & I & \mathbf{p} \\ 0 & 0 & 0 \end{bmatrix}$$ is rank-deficient. The Newton step does not exist !!

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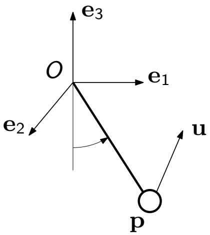

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Note that $\frac{\partial \mathbf{G}(\mathbf{x})}{\partial z} = 0$ **!!**

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DAE - Delta Robot

Lagrange model yields a semi-explicit DAE with:

$$\mathbf{G}(\mathbf{x}) = \begin{bmatrix} \|\mathbf{p}-\mathbf{p}_1\|^2 - L^2 \\ \|\mathbf{p}-\mathbf{p}_2\|^2 - L^2 \\ \|\mathbf{p}-\mathbf{p}_3\|^2 - L^2 \end{bmatrix}$$

where

$$\mathbf{p}_k = R_k^z = \begin{bmatrix} \cos\gamma_k & \sin\gamma_k & 0 \\ -\sin\gamma_k & \cos\gamma_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L\cos\alpha_k \\ 0 \\ L\sin\alpha_k \end{bmatrix}$$

using $\gamma_{1,2,3} = \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$.

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Algebraic variables $\mathbf{z}$ for the forces in the arms:

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Such that the DAE:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, z, \mathbf{u})$$
$$0 = \mathbf{G}(\mathbf{x})$$

... cannot be solved for $\mathbf{z}$, because $\frac{\partial \mathbf{G}(\mathbf{x})}{\partial \mathbf{z}} = 0$!!

DAE from Lagrange Mechanics

Is that a general problem in Lagrange mechanics ? Pretty much ...

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The difficulty comes from having *holonomic* (aka purely position-dependent) constraints:

$$\mathbf{G}(\mathbf{q}) = 0$$

which "hold the system together".

DAE from Lagrange Mechanics

Is that a general problem in Lagrange mechanics ? Pretty much ...

The difficulty comes from having *holonomic* (aka purely position-dependent) constraints:

$$\mathbf{G}\left(\mathbf{q}\right) = 0$$

which "hold the system together". Then the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0$$

deliver an explicit ODE for the accelerations $\ddot{\mathbf{q}}$, involving the algebraic variables $\mathbf{z}$.

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What is going on ?!?

Outline

DAE - Differential Index

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Definition:

The DAE **differential index** is the minimum i such that:

$$\frac{\mathrm{d}^i}{\mathrm{d}t^i}\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}\right) = 0$$

is a pure ODE

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Example:

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}\right) = \left[\begin{array}{c} \mathbf{x}_1 - \dot{\mathbf{x}}_1 + 1 \\ \dot{\mathbf{x}}_1 \mathbf{x}_2 + 2 \end{array}\right] = 0$$

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For $i=1$ reads as:

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$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}) = \begin{bmatrix} \mathbf{x}_1 - \dot{\mathbf{x}}_1 + 1 \\ \dot{\mathbf{x}}_1 \mathbf{x}_2 + 2 \end{bmatrix} = 0$$

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Using (to write a 1^{st}-order ODE)

$$\mathbf{s} \equiv \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \dot{\mathbf{x}}_1 \end{bmatrix} \qquad \text{we have} \qquad \dot{\mathbf{F}}(\dot{\mathbf{s}}, \mathbf{s}) = \begin{bmatrix} \dot{\mathbf{s}}_1 - \mathbf{s}_3 \\ \mathbf{s}_3 - \dot{\mathbf{s}}_3 \\ \dot{\mathbf{s}}_3 \mathbf{s}_2 + \mathbf{s}_3 \dot{\mathbf{s}}_2 \end{bmatrix}$$

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And

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{s}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & \mathbf{s}_3 & \mathbf{s}_2 \end{bmatrix}, \qquad \text{with} \qquad \det\left(\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{s}}}\right) = \mathbf{s}_3 \quad \Rightarrow \quad \text{now we have an ODE}$$

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F is an index-1 DAE

And

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How does the differential index relate to the DAE being "easy" to solve ??

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For an **index-1 DAE**:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}\right) = \dot{\mathbf{F}} = 0$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}=\dot{\mathbf{F}}=\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}\ddot{\mathbf{x}}+\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\dot{\mathbf{x}}+\frac{\partial \mathbf{F}}{\partial \mathbf{z}}\dot{\mathbf{z}}+\frac{\partial \mathbf{F}}{\partial \mathbf{u}}\dot{\mathbf{u}}=0$$

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Then the ODE reads as (use $\mathbf{v}=\dot{\mathbf{x}}$):

$$\dot{\mathbf{x}}=\mathbf{v}$$

$$\begin{bmatrix}\dot{\mathbf{v}}\\ \dot{\mathbf{z}}\end{bmatrix}=-\begin{bmatrix}\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \mathbf{z}}\end{bmatrix}^{-1}\left(\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\dot{\mathbf{x}}+\frac{\partial \mathbf{F}}{\partial \mathbf{u}}\dot{\mathbf{u}}\right)$$

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An index-1 DAE has

$$\begin{bmatrix}\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \mathbf{z}}\end{bmatrix}$$

full rank and is therefore "easy" to solve !!

Semi-explicit DAEs - Differential Index

For a **semi-explicit DAE** the differential index is the minimum i such that:

$$\dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$

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Remark: for an index-1 semi-explicit DAE:

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yields a pure ODE. We have:

$$\dot{\mathbf{z}} = -\frac{\partial \mathbf{G}}{\partial \mathbf{z}}^{-1}\left(\frac{\partial \mathbf{G}}{\partial \mathbf{x}}\mathbf{F} + \frac{\partial \mathbf{G}}{\partial \mathbf{u}}\dot{\mathbf{u}}\right)$$

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Example is an **index-2** DAE

Then
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{G} = \mathbf{x}_1\mathbf{x}_2 + \mathbf{x}_2 z = 0$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{G} = \dot{\mathbf{x}}_1\mathbf{x}_2 + \mathbf{x}_1\dot{\mathbf{x}}_2 + \dot{\mathbf{x}}_2 z + \mathbf{x}_2\dot{z} = 0$$

Differential Index - 3D pendulum

Example: 3D pendulum

$$m\ddot{\mathbf{p}} = \mathbf{u} - mg\mathbf{e}_3 - z\mathbf{p}$$

$$0 = \underbrace{\frac{1}{2}\left(\mathbf{p}^\top\mathbf{p} - L^2\right)}_{\mathbf{G}(\mathbf{x})}$$

For a **semi-explicit DAE** the differential index is the minimum i such that:

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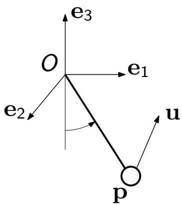

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Perform **two** time differentiations on $\mathbf{G}$ yields:

$$\ddot{\mathbf{G}} = \frac{1}{2}\left(\ddot{\mathbf{p}}^\top \mathbf{p} + \mathbf{p}^\top \ddot{\mathbf{p}} + 2\dot{\mathbf{p}}^\top \dot{\mathbf{p}}\right) = \mathbf{p}^\top \ddot{\mathbf{p}} + \dot{\mathbf{p}}^\top \dot{\mathbf{p}} = 0$$

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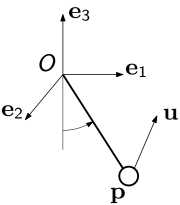

Differential Index - 3D pendulum

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$$m\ddot{\mathbf{p}} = \mathbf{u} - mg\mathbf{e}_3 - z\mathbf{p}$$
$$0 = \underbrace{\frac{1}{2}\left(\mathbf{p}^\top \mathbf{p} - L^2\right)}_{\mathbf{G}(\mathbf{x})}$$

Perform **two** time differentiations on $\mathbf{G}$ yields:

$$\ddot{\mathbf{G}} = \frac{1}{2}\left(\ddot{\mathbf{p}}^\top \mathbf{p} + \mathbf{p}^\top \ddot{\mathbf{p}} + 2\dot{\mathbf{p}}^\top \dot{\mathbf{p}}\right) = \mathbf{p}^\top \ddot{\mathbf{p}} + \dot{\mathbf{p}}^\top \dot{\mathbf{p}} = 0$$

Substitute $\ddot{\mathbf{p}}$ from $m\ddot{\mathbf{p}} = \mathbf{u} - mg\mathbf{e}_3 - z\mathbf{p}$ yields:

$$\mathbf{p}^\top \left(\frac{1}{m}\mathbf{u} - g\mathbf{e}_3 - \frac{1}{m}z\mathbf{p}\right) + \dot{\mathbf{p}}^\top \dot{\mathbf{p}} = 0$$

For a **semi-explicit DAE** the differential index is the minimum i such that:

$$\dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$
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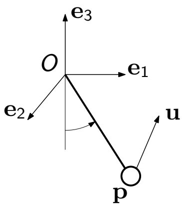
is an ODE



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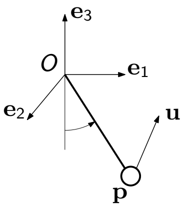

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A **third** time differentiation yields an ODE for z:

$$\dot{z} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\mathbf{p}^\top\mathbf{p}}\left(\mathbf{p}^\top\mathbf{u} - mg\mathbf{p}^\top\mathbf{e}_3 + m\dot{\mathbf{p}}^\top\dot{\mathbf{p}}\right)\right]$$

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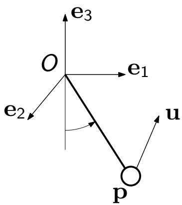

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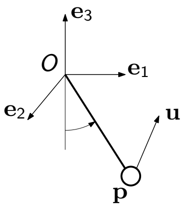

The 3D pendulum in Lagrange is an index-3 DAE !!

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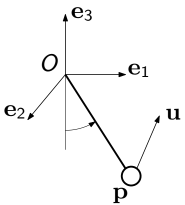

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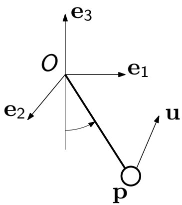

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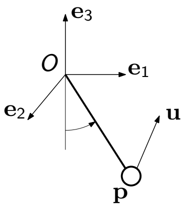

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This is an index-1 (i.e. "easy") DAE !!

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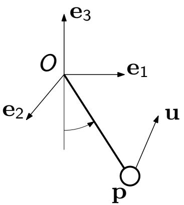

We have converted the index-3 DAE into an index-1 DAE !!

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Transforming a high-index DAE into an equivalent lower-index one is labelled **index reduction**

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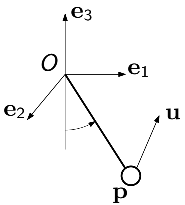

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Outline

1. "Easy" & "Hard" DAEs
2. Differential Index
3. **Index Reduction**
4. Constraints drift

DAEs from Lagrange Mechanics

Index-3 DAE from Lagrange:

$$\frac{\mathrm{d}}{\mathrm{d}}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{F}_{\mathrm{g}}$$

$$\mathbf{c}(\mathbf{q}) = 0$$

with $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{z}) = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q}) - \mathbf{z}^\top \mathbf{c}(\mathbf{q})$

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For most mechanical applications:

$$T\left(\mathbf{q}, \dot{\mathbf{q}}\right) = \frac{1}{2}\dot{\mathbf{q}}^\top M(\mathbf{q})\dot{\mathbf{q}}$$

such that:

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Index-1 DAE model:

$$\begin{bmatrix} M(\mathbf{q}) & \nabla_{\mathbf{q}}\mathbf{c}(\mathbf{q}) \\ \nabla_{\mathbf{q}}\mathbf{c}(\mathbf{q})^\top & 0 \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{q}} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{\mathrm{g}} - \dot{M}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \nabla_{\mathbf{q}}\left(T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q})\right) \\ -\nabla_{\mathbf{q}}\left(\nabla_{\mathbf{q}}\mathbf{c}(\mathbf{q})^\top \dot{\mathbf{q}}\right)^\top \dot{\mathbf{q}} \end{bmatrix}$$

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$$\frac{\mathrm{d}}{\mathrm{d}}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}^\top = M(\mathbf{q})\ddot{\mathbf{q}} + \dot{M}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$$

Index-1 DAE model:

$$\begin{bmatrix} M(\mathbf{q}) & \nabla \mathbf{c}(\mathbf{q}) \\ \nabla \mathbf{c}(\mathbf{q})^\top & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{\mathrm{g}} - \dot{M}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \nabla_{\mathbf{q}}\left(T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q})\right) \\ -\nabla_{\mathbf{q}}\left(\nabla \mathbf{c}(\mathbf{q})^\top \dot{\mathbf{q}}\right)^\top \dot{\mathbf{q}} \end{bmatrix}$$

Models based on Lagrange mechanics typically are **index-3 DAEs**, making them intrinsically difficult to use. The best approach to treat them is usually to proceed with an **index reduction** down to **index 1** for which very classical integration tools work well.

Index reduction for semi-explicit DAEs - A general view

High-index semi-explicit DAE

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{F}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \\ 0 &= \mathbf{G}(\mathbf{x}, \mathbf{z}, \mathbf{u})\end{aligned}$$

Algorithm (*see "Nonlinear Programming", L.T. Biegler*)

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Algorithm (*see "Nonlinear Programming", L.T. Biegler*)

1. Check if the DAE system is index 1 (i.e. $\frac{\partial \mathbf{G}}{\partial \mathbf{z}}$ full rank). If yes, stop.
2. Identify a subset of algebraic equations that **can be solved** for a subset of algebraic variables.
3. Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ on the remaining algebraic equations that contain the differential variables $\mathbf{x}_j$.
4. Terms $\dot{\mathbf{x}}_j$ will appear in these differentiated equations.
5. Substitute the $\dot{\mathbf{x}}_j$ with $\mathbf{F}_j(\mathbf{x}, \mathbf{z}, \mathbf{u})$. This leads to new algebraic equations.
6. With this new DAE system, go to step 1.

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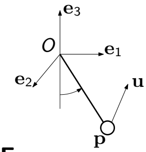
Writing a general-purpose "Index-reduction algorithm" can be very tricky, as one of the steps is not easily automated

DAE Consistency - 3D pendulum

Does the index reduction really yield equivalent models ?



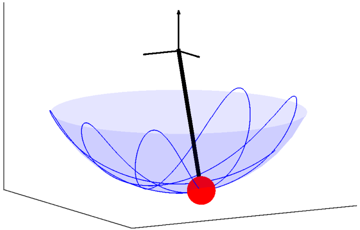
Index-3 DAE

$$m\ddot{\mathbf{p}} = \mathbf{u} - mg\mathbf{e}_3 - z\mathbf{p}$$

$$0 = \frac{1}{2}\left(\mathbf{p}^\top \mathbf{p} - L^2\right)$$

Index-1 DAE

$$\begin{bmatrix} mI & \mathbf{p} \\ \mathbf{p}^\top & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{p}} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{u} - mg\mathbf{e}_3 \\ -\dot{\mathbf{p}}^\top \dot{\mathbf{p}} \end{bmatrix}$$



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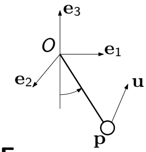

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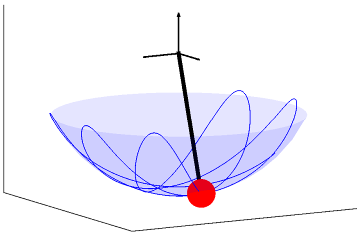
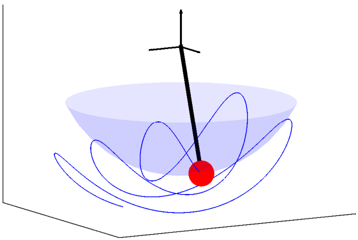
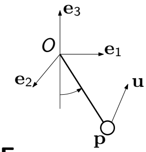




DAE Consistency - 3D pendulum



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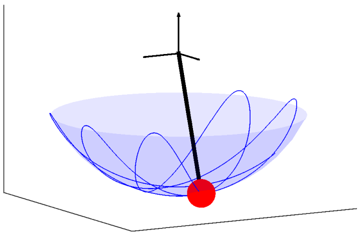

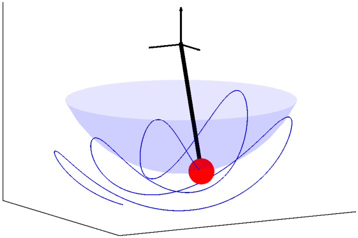

What is going on ??

DAE Consistency - 3D pendulum

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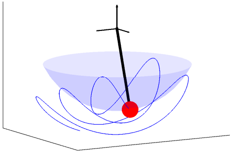

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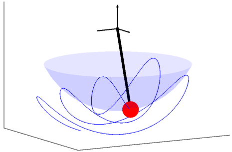

Index reduction

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DAE Consistency - 3D pendulum

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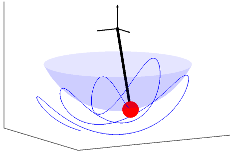

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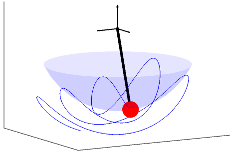
DAE Consistency - 3D pendulum



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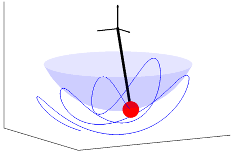

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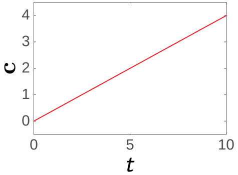

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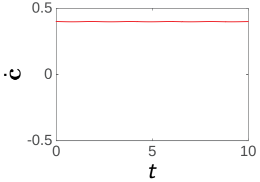

DAE Consistency - 3D pendulum

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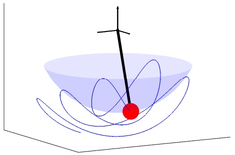

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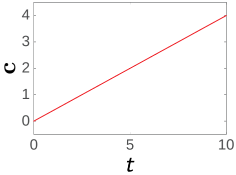

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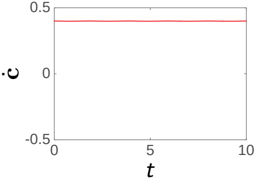

... is built to impose $\ddot{\mathbf{c}} = 0$ at all time. But it does not ensure

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How can we address that ??

DAE Consistency - 3D pendulum

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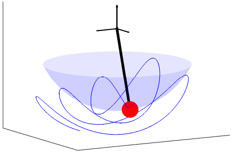

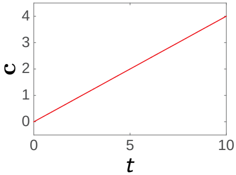

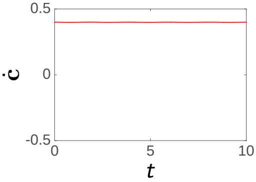

DAE Consistency - 3D pendulum

Index-1 DAE

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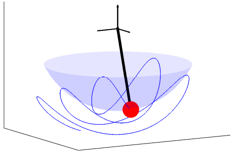
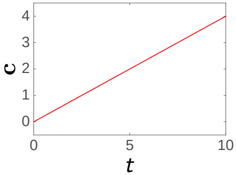
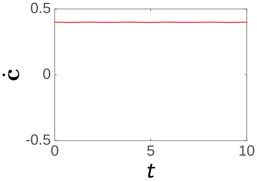
Then if $\mathbf{c} = 0$ and $\dot{\mathbf{c}} = 0$ are satisfied at **any** time on the trajectory, then they are satisfied at **all** time.







DAE Consistency - 3D pendulum

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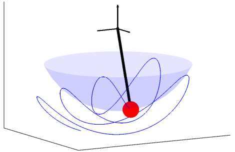
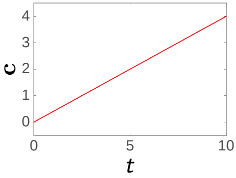
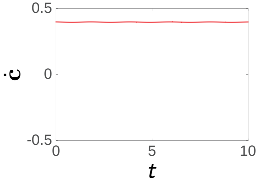
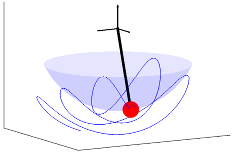






DAE Consistency - 3D pendulum



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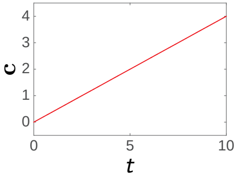
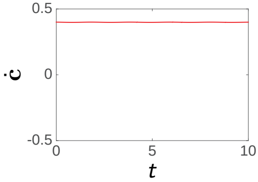
with the consistency conditions:

$$\mathbf{c} = \frac{1}{2}\left(\mathbf{p}^\top \mathbf{p} - L^2\right) = 0, \qquad \dot{\mathbf{c}} = \mathbf{p}^\top \dot{\mathbf{p}} = 0$$

... to be satisfied e.g. at t_0.





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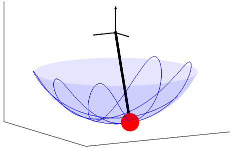

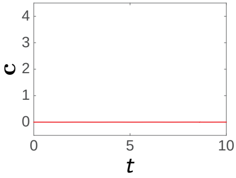

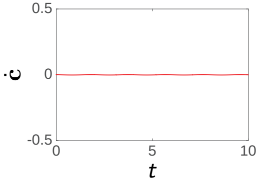

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with the consistency conditions:

$$\mathbf{c} = \frac{1}{2}\left(\mathbf{p}^\top \mathbf{p} - L^2\right) = 0, \qquad \dot{\mathbf{c}} = \mathbf{p}^\top \dot{\mathbf{p}} = 0$$

... to be satisfied e.g. at t_0.

Consistency of DAEs from Lagrange Mechanics

Index-3 DAE from Lagrange:

$$\frac{\mathrm{d}}{\mathrm{d}}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{F}_{\mathrm{g}}$$
$$\mathbf{c}(\mathbf{q}) = 0$$

with $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{z}) = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q}) - \mathbf{z}^\top \mathbf{c}(\mathbf{q})$

For most mechanical applications:

$$T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^\top M(\mathbf{q})\dot{\mathbf{q}}$$

Index reduction based on:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{c}(\mathbf{q}) = \nabla \mathbf{c}(\mathbf{q})^\top \dot{\mathbf{q}} \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{c}(\mathbf{q}) = \nabla \mathbf{c}(\mathbf{q})^\top \ddot{\mathbf{q}} + \nabla_{\mathbf{q}}\left(\nabla \mathbf{c}(\mathbf{q})^\top \dot{\mathbf{q}}\right)^\top \dot{\mathbf{q}}$$

Index-1 DAE model:

$$\begin{bmatrix} M(\mathbf{q}) & \nabla_{\mathbf{q}}\mathbf{c}(\mathbf{q}) \\ \nabla_{\mathbf{q}}\mathbf{c}(\mathbf{q})^\top & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{\mathrm{g}} - \dot{M}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \nabla_{\mathbf{q}}\left(T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q})\right) \\ -\nabla_{\mathbf{q}}\left(\nabla \mathbf{c}(\mathbf{q})^\top \dot{\mathbf{q}}\right)^\top \dot{\mathbf{q}} \end{bmatrix}$$

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Outline

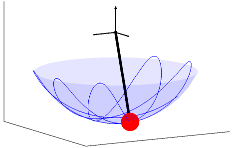
Constraints drift - 3D pendulum

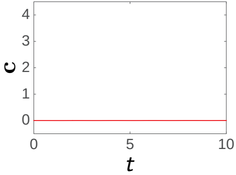
Index-1 DAE

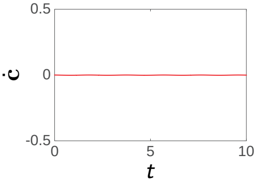
$$\begin{bmatrix} mI & \mathbf{p} \\ \mathbf{p}^\top & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{p}} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{u} - mg\mathbf{e}_3 \\ -\dot{\mathbf{p}}^\top \dot{\mathbf{p}} \end{bmatrix}$$

... is built to impose $\ddot{\mathbf{c}} = 0$ at all time.

Then if $\mathbf{c} = 0$ <u>and</u> $\dot{\mathbf{c}} = 0$ are satisfied at **any** time on the trajectory, then they are satisfied at **all** time.







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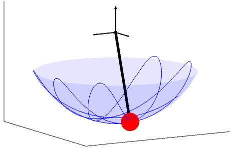

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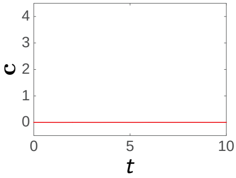

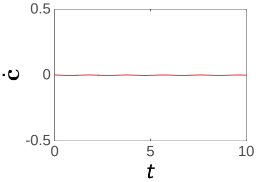

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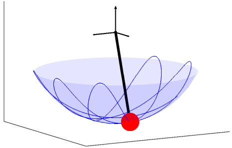

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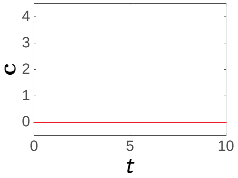

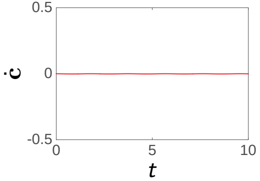

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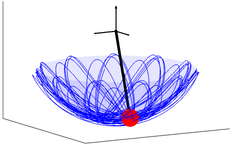

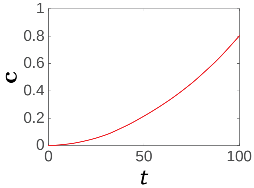

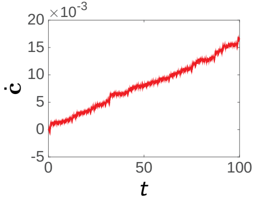

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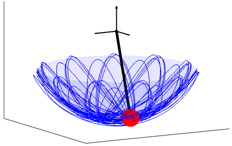

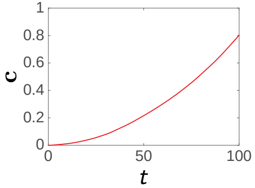

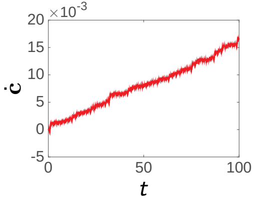

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... imposed at e.g. t_0.

With consistent initial conditions, $\mathbf{c} = 0$ and $\dot{\mathbf{c}} = 0$ **would be** satisfied at all time if we had no numerical error in the integration !!

Baumgartne stabilization of the constraints drift

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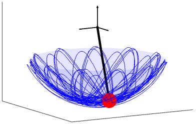
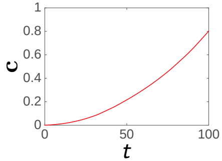
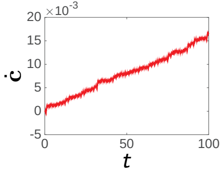
Why does the drift happen: $\ddot{\mathbf{c}} = 0$ has marginally stable dynamics ($\mathbf{c}$ is two integrations of $\ddot{\mathbf{c}}$ hence two poles at 0).







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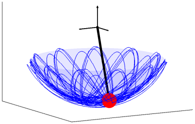
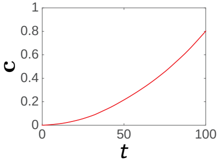
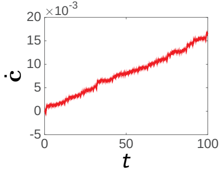
Key idea: impose a stable dynamics to the constraints







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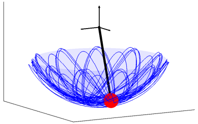
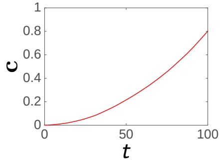
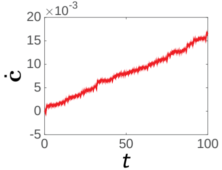
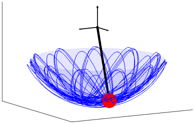
$$\ddot{\mathbf{c}} + \gamma_1 \dot{\mathbf{c}} + \gamma_0 \mathbf{c} = 0$$







Baumgartne stabilization of the constraints drift



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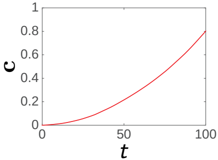

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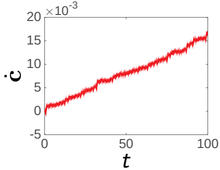


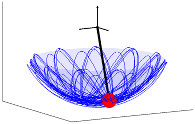
Key idea: impose a stable dynamics to the constraints, build the index-1 DAE to impose:

$$\ddot{\mathbf{c}} + \gamma_1 \dot{\mathbf{c}} + \gamma_0 \mathbf{c} = 0$$

E.g. for the 3D pendulum:

$$\mathbf{p}^\top \ddot{\mathbf{p}} + \dot{\mathbf{p}}^\top \dot{\mathbf{p}} + \gamma_1 \mathbf{p}^\top \dot{\mathbf{p}} + \frac{\gamma_2}{2}\left(\mathbf{p}^\top \mathbf{p} - L^2\right) = 0$$

Baumgartne stabilization of the constraints drift



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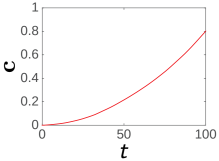

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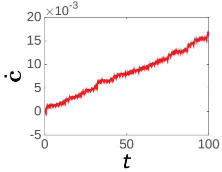

Index-1 DAE with stabilization:

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Baumgartne stabilization of the constraints drift

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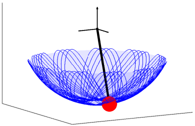
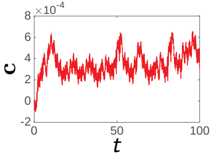
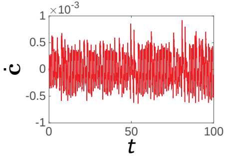






E.g. poles at -1

Baumgartne stabilization of the constraints drift

Index-1 DAE:

$$\left[\begin{array}{cc} ml & \mathbf{p} \\ \mathbf{p}^\top & 0 \end{array}\right]\left[\begin{array}{c} \ddot{\mathbf{p}} \\ z \end{array}\right] = \left[\begin{array}{c} \mathbf{u} - mg\mathbf{e}_3 \\ -\dot{\mathbf{p}}^\top\dot{\mathbf{p}} \end{array}\right]$$

with the **consistency conditions**:

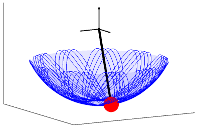
$$\mathbf{c} = \frac{1}{2}\left(\mathbf{p}^\top\mathbf{p} - L^2\right) = 0, \qquad \dot{\mathbf{c}} = \mathbf{p}^\top\dot{\mathbf{p}} = 0$$

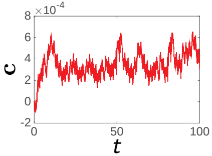
... imposed at t_0.

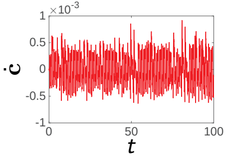
Why does the drift happen: $\ddot{\mathbf{c}} = 0$ has marginally stable dynamics ($\mathbf{c}$ is two integrations of $\ddot{\mathbf{c}}$ hence two poles at 0).

The Baumgartne stabilization must be used carefully !

- Fast poles introduce stiffness in the dynamics
- The interaction between the stabilization and the integrator error is non-trivial...







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A related problem - Invariants in ODEs

Consistency & drift are not DAE-specific.

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$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u})$$

model a physical reality. Some ODEs are representative only when some consistency conditions:

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Not covered in this course but good to know: Symplectic integrators (for handling drift)