Exercises for Lecture Course on Numerical Optimization (NUMOPT)
Albert-Ludwigs-Universität Freiburg – Winter Term 2015-2016

Exercise 4: Estimation and Fitting Problems

(to be completed during exercise session on Nov 18, 2015 or sent by email to dimitris.kouzoupis@imtek.uni-freiburg.de before Nov 20, 2015)

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Aim of this exercise is to familiarize with linear least squares fitting problems.

Exercise Tasks

1. **Regularized linear least squares:** Given a matrix $J \in \mathbb{R}^{m\times n}$ with arbitrary dimensions, a symmetric positive definite matrix $Q \succ 0$, a vector of measurements $\eta \in \mathbb{R}^m$ and a point $\bar{x} \in \mathbb{R}^n$, calculate the limit:

$$\lim_{\substack{\alpha \to 0 \\ a > 0}} \arg\min_x \frac{1}{2}||\eta - Jx||_2^2 + \frac{\alpha}{2}(x-\bar{x})^\top Q(x-\bar{x}). \tag{1}$$

Hint: Use matrix square root and the Moore–Penrose pseudoinverse, i.e., SVD of a suitable matrix.

(3 points)

2. **Linear L_2 fitting (MATLAB):** Assume we have a set of N measurements $(x_i, y_i) \in \mathbb{R}^2$ onto which we would like to fit a line $y = ax + b$. This task can be expressed by the optimization problem:

$$\min_{a,b} \sum_{i=1}^{N}(ax_i + b - y_i)^2 = \min_{a,b} \left\| J \begin{pmatrix} a \\ b \end{pmatrix} - y \right\|_2^2. \tag{2}$$

As discussed in the lecture, the optimal solution of (2) can be calculated explicitly by solving the linear system:

$$J^T J \begin{pmatrix} a \\ b \end{pmatrix} = J^T y, \tag{3}$$

(a) Generate the problem data. Take $N = 30$ points in the interval $[0, 5]$ and generate the measurements $y_i = 3x_i + 4$. Add Gaussian noise with zero mean and standard deviation 1 to the measurements and plot the results. *Hint: lookup `linspace` and `randn` commands.*

(1 point)

(b) Write down matrix J and vector y for your fitting problem. Calculate the coefficients a, b in MATLAB using Equation (3) and plot the obtained line in the same graph as the measurements.

(2 points)

(c) Introduce 3 outliers in your measurements y and plot the new fitted line in your plot.

(1 point)

(d) Solve question 2(b) with YALMIP and compare the results.

(1 point)

You will need the measurements y (both with and without outliers) and the matrix J for the next task.

3. **Linear L_1 fitting (MATLAB):** In this task we want to fit a line to the same set of measurements, but we use a different cost function:

$$\min_{a,b} \sum_{i=1}^{N} |(ax_i + b - y_i)|. \tag{4}$$

This objective is not differentiable, so we will need auxiliary variables to form an equivalent problem. We introduce the so-called slack variables $s_1, \ldots, s_N$ and solve instead:

$$\begin{aligned} &\min_{a,b,s} \sum_i s_i && && \text{(5a)} \\ &\text{s.t. } -s_i \leq ax_i + b - y_i \leq s_i, && i = 1, \ldots, N, && \text{(5b)} \\ &\quad\;\; -s_i \leq 0, && i = 1, \ldots, N. && \text{(5c)} \end{aligned}$$

(a) Problem (5) is a Linear Program. In order to solve it with `linprog`, the native LP solver of MATLAB, we need to bring it to the form:

$$\begin{aligned} &\min_z f^T z && \text{(6a)} \\ &\text{s.t } Az \leq b && \text{(6b)} \\ &\quad Cz = d && \text{(6c)} \\ &\quad l_z \leq z \leq u_z, && \text{(6d)} \end{aligned}$$

Write matrix A and vectors f, b on paper. Order your variables as $z = [a, b, s_1, \ldots, s_N]$. Use matrix J from the previous exercise to define A.

(2 points)

(b) Solve the problem using the measurements y from the previous exercise (both with and without outliers) and compare the results with the L2 fitting.

(1 point)

(c) Solve Problem (5) with YALMIP and compare the results.

(1 point)

This sheet gives in total 12 points.