Rien Quirynen

Inexact Newton Optimization with Iterated Sensitivities for Direct Optimal Control

Syscop Group Retreat

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Opening new horizons

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Outline

Newton-type scheme to solve $F(x) = 0$:[1]

$$\tilde{J}(\bar{x})\Delta x = -F(\bar{x}),$$

where $\tilde{J}(\bar{x}) \approx J(\bar{x}) := \frac{\partial F}{\partial x}(\bar{x})$ such that $\bar{x}^+ = \bar{x} + \Delta x$ (full step)

[1] Peter Deuflhard. *Newton methods for nonlinear problems: affine invariance and adaptive algorithms*. Vol. 35. Springer, 2011.

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Theorem: Local Newton-type contraction

Let us consider $F(x) \in C^2$ and the Jacobian approximation $\tilde{J}(\bar{x}) \in C$ is invertible in a neighborhood of the solution $x^\star$. The fixed point $x^\star$ is then asymptotically stable if and only if

$$\kappa^\star = \rho\left(\tilde{J}(x^\star)^{-1}J(x^\star) - \mathbb{1}\right) < 1.$$

When initialized close enough, the iterates $\bar{x}$ converge at least linearly to the point $x^\star$ with the asymptotic contraction rate $\kappa^\star$.

$\rho(\cdot)$: spectral radius or maximum absolute eigenvalue

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First order necessary conditions for optimality:

$$F(\bar{y}, \bar{\lambda}) := \begin{bmatrix} \nabla_y \mathcal{L}(\bar{y}, \bar{\lambda}) \\ g(\bar{y}) \end{bmatrix},$$

$$\mathcal{L}(y, \lambda) = f(y) + \lambda^\top g(y)$$

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$$\tilde{J}(\bar{y},\bar{\lambda}) \begin{bmatrix} \Delta y \\ \Delta \lambda \end{bmatrix} = -F(\bar{y},\bar{\lambda}),$$

$$\tilde{J}(\cdot) \approx \begin{bmatrix} \nabla_y^2 \mathcal{L}(\bar{y},\bar{\lambda}) & g_y^\top(\bar{y}) \\ g_y(\bar{y}) & \mathbb{0} \end{bmatrix}$$

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Local contraction iff $\kappa^\star = \rho\left(\tilde{J}(y^\star, \lambda^\star)^{-1} J(y^\star, \lambda^\star) - \mathbb{1}\right) < 1$

NLP problem formulation

$$\begin{aligned} \min_{z,\,w} \quad & f(z, w) \\ \text{s.t.} \quad & 0 = g(z, w) \end{aligned}$$

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Forward problem $z^{\star}(\bar{w})$:

$$g(z,\bar{w}) = 0, \quad \text{for a given } \bar{w}$$

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with
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Potschka's QP example[2]

$$\min_{y=(z,w)} \quad \frac{1}{2} y^\top H\, y$$
$$\text{s.t.} \quad 0 = [A_1, A_2]\, y,$$

with $M = \mathbb{1} \approx A_1$, $\tilde{H} = H$

[2] Andreas Potschka. "A direct method for the numerical solution of optimization problems with time-periodic PDE constraints". PhD thesis. University of Heidelberg, 2011. URL: http://www.iwr.uni-heidelberg.de/~Andreas.Potschka/publications.html.

Potschka's QP example[2]

$$H = \begin{bmatrix} 0.83 & 0.083 & 0.34 & -0.21 \\ 0.083 & 0.4 & -0.34 & -0.4 \\ 0.34 & -0.34 & 0.65 & 0.48 \\ -0.21 & -0.4 & 0.48 & 0.75 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 1.1 & 1.7 \\ 0 & 0.52 \end{bmatrix}, \; A_2 = \begin{bmatrix} -0.55 & -1.4 \\ -0.99 & -1.8 \end{bmatrix}$$

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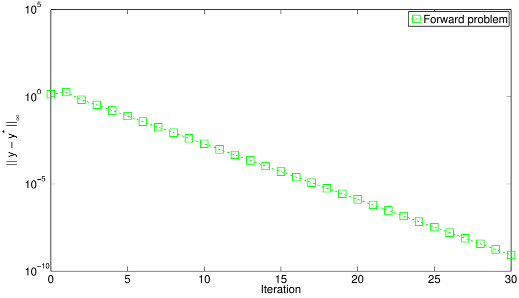

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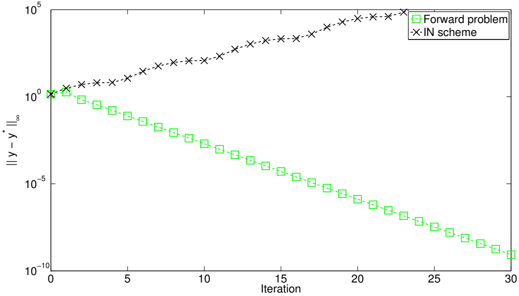

$$\kappa_{\mathrm{F}}^\star = \rho(A_1 - \mathbb{1}) = 0.48 < 1$$

$$\kappa_{\mathrm{IN}}^\star = \rho(\tilde{J}_{\mathrm{IN}}^{-1} J - \mathbb{1}) \approx 1.62 > 1$$

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Inexact Newton with Iterated Sensitivities (INIS)

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Sensitivity matrix update:

$$\Delta D = -M^{-1}(g_z \bar{D} - g_w)$$

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$$\underbrace{\begin{bmatrix} \tilde{H} & \begin{pmatrix} M^\top \\ \bar{D}^\top M^\top \end{pmatrix} & \mathbb{0} \\ \begin{pmatrix} M & M\bar{D} \end{pmatrix} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} \otimes M \end{bmatrix}}_{=:\tilde{J}_{\mathrm{INIS}}(\bar{y},\bar{\lambda},\bar{D})} \begin{bmatrix} \Delta z \\ \Delta w \\ \Delta\lambda \\ \mathrm{vec}(\Delta D) \end{bmatrix} = -\underbrace{\begin{bmatrix} \nabla_y \mathcal{L}(\bar{y},\bar{\lambda}) \\ g(\bar{y}) \\ \mathrm{vec}(g_z\bar{D} - g_w) \end{bmatrix}}_{=:F_{\mathrm{INIS}}(\bar{y},\bar{\lambda},\bar{D})},$$

$$\kappa_{\mathrm{INIS}}^{\star} = \rho(\tilde{J}_{\mathrm{INIS}}^{\star\,-1} J_{\mathrm{INIS}}^{\star} - \mathbb{1}) < 1?$$

Theorem: Local INIS-type contraction

The asymptotic rate of local convergence for the Inexact Newton method with Iterated Sensitivities (INIS) reads

$$\kappa^\star_{\mathrm{INIS}} = \rho(\tilde{J}^{\star^{-1}}_{\mathrm{INIS}} J^\star_{\mathrm{INIS}} - \mathbb{1}) = \max\left(\kappa^\star_{\mathrm{F}},\, \rho(\tilde{H}_z^{-1} H_z - \mathbb{1})\right),$$

where $Z^\top = \left[-g_w^\top g_z^{-\top},\, \mathbb{1}\right]$ such that $H_z = Z^\top H Z$ and $\tilde{H}_z = Z^\top \tilde{H} Z$.

Proof: see[3]

[3] R. Quirynen, S. Gros, and M. Diehl. "Inexact Newton-Type Optimization with Iterated Sensitivities". In: *SIAM Journal on Optimization (under review, preprint available at Optimization Online)* (2016).

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Proof: see[3]

Connection local contraction forward problem and INIS

- necessary and often sufficient
- exact Hessian $\Rightarrow$ $\kappa^\star_{\mathrm{INIS}} = \kappa^\star_{\mathrm{F}}$

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Theorem: Local INIS-type contraction

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Sketch of proof:

Theorem: Local INIS-type contraction

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Sketch of proof:

$$\det\left(\tilde{J}_{\mathrm{INIS}}^{-1} J_{\mathrm{INIS}} - (\gamma + 1)\mathbb{1}\right) = \det\left(J_{\mathrm{INIS}} - (\gamma + 1)\tilde{J}_{\mathrm{INIS}}\right) = 0$$

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$$J_{\mathrm{INIS}} - (\gamma+1)\,\tilde{J}_{\mathrm{INIS}} =$$

$$\begin{bmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \tilde{M} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} \end{bmatrix} \begin{bmatrix} H - (\gamma+1)\,\tilde{H} & \tilde{g}_y^\top & \mathbb{0} \\ \tilde{g}_y & \mathbb{0} & \mathbb{0} \\ X & \mathbb{0} & \mathbb{1} \otimes \tilde{M} \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \tilde{M} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} \end{bmatrix}^\top$$

where $\tilde{g}_y = g_z^{-1} g_y$ and $\tilde{M} = g_z - (\gamma+1)\,M$

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$$= \det\left(\tilde{M}\right)^{2+n_{\mathrm{w}}} \det\left(\begin{bmatrix} H - (\gamma+1)\,\tilde{H} & \tilde{g}_y^\top \\ \tilde{g}_y & \mathbb{0} \end{bmatrix}\right).$$

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where $\tilde{g}_y = g_z^{-1} g_y$ and $\tilde{M} = g_z - (\gamma+1)\,M$

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$$= \det\left(\tilde{M}\right)^{2+n_{\mathrm{w}}} \det\left(\begin{bmatrix} H - (\gamma+1)\,\tilde{H} & \tilde{g}_y^\top \\ \tilde{g}_y & \mathbb{0} \end{bmatrix}\right).$$

$$\Rightarrow\ \sigma\left(\tilde{J}_{\mathrm{INIS}}^{-1} J_{\mathrm{INIS}} - \mathbb{1}\right) = \sigma\left(M^{-1} g_z - \mathbb{1}\right) \cup \sigma\left(\tilde{H}_z^{-1} H_z - \mathbb{1}\right)$$

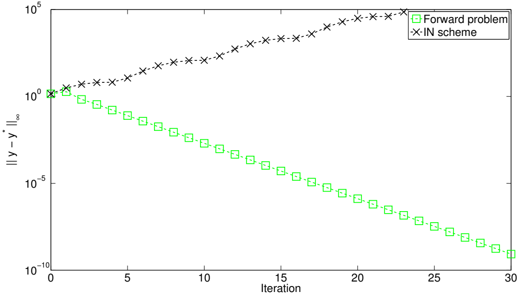
Potschka's QP example

$$\min_{y=(z,w)} \quad \frac{1}{2} y^\top H y$$
$$\text{s.t.} \quad 0 = [A_1, A_2]\, y,$$

$$H = \begin{bmatrix} 0.83 & 0.083 & 0.34 & -0.21 \\ 0.083 & 0.4 & -0.34 & -0.4 \\ 0.34 & -0.34 & 0.65 & 0.48 \\ -0.21 & -0.4 & 0.48 & 0.75 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 1.1 & 1.7 \\ 0 & 0.52 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.55 & -1.4 \\ -0.99 & -1.8 \end{bmatrix}$$

with $M = \mathbb{1} \approx A_1$, $\tilde{H} = H$



$$\kappa_F^\star = \rho(A_1 - \mathbb{1}) = 0.48 < 1$$

$$\kappa_{IN}^\star = \rho(\tilde{J}_{IN}^{-1} J - \mathbb{1}) \approx 1.62 > 1$$

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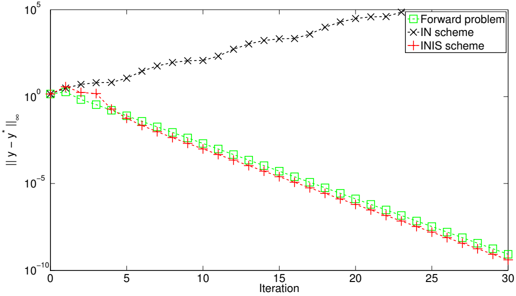

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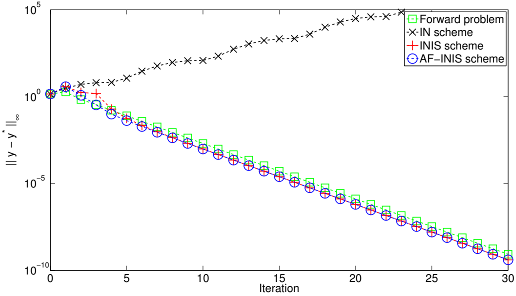

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AF-INIS: multiplier- and adjoint-free scheme (e.g. with Gauss-Newton)

Outline

Optimal Control Problem (OCP)

$$
\begin{aligned}
\min_{x(\cdot),u(\cdot)} \quad & \int_0^T \|F(x(t),u(t))\|_2^2 \,\mathrm{d}t + \|F_N(x(T))\|_2^2 \\
\text{s.t.} \quad x(0) =\ & \hat{x}_0 \\
0 =\ & f(\dot{x}(t), x(t), u(t)) \\
0 \geq\ & h(x(t), u(t)) \\
0 \geq\ & r(x(T)), \qquad \forall t \in [0, T]
\end{aligned}
$$

Optimal Control Problem (OCP) $\rightarrow$ *direct collocation*

$$
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& 0 = f(\dot{x}(t),x(t),u(t)) \\
& 0 \geq h(x(t),u(t)) \\
& 0 \geq r(x(T)), \qquad \forall t \in [0,T]
\end{aligned}
$$

$$
\begin{aligned}
\min_{X,U,K} \quad & \sum_{i=0}^{N-1} \|F_i(x_i,u_i)\|_2^2 + \|F_N(x_N)\|_2^2 \\
\text{s.t.} \quad & 0 = x_0 - \hat{x}_0 \\
& 0 = G(w_i,K_i) \\
& 0 = x_i + h\,B\,K_i - x_{i+1} \\
& 0 \geq h_i(x_i,u_i), \qquad i = 0,\ldots,N-1 \\
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& 0 \geq h_i(x_i, u_i), \qquad i = 0, \dots, N-1 \\
& 0 \geq r(x_N)
\end{aligned}$$

Choice of parametrization

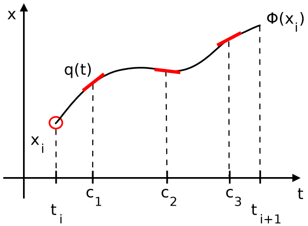
- control inputs: $u(\tau) = u_i$ for $\tau \in [t_i, t_{i+1})$
- discretization based on collocation polynomials
- collocation equations as NLP constraints

Collocation methods: Implicit Runge-Kutta

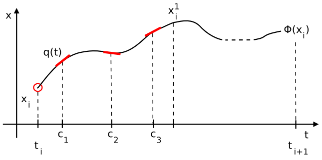
Collocation methods: Implicit Runge-Kutta

- continuous approximation
- polynomial $q(t)$ of degree S
- shooting interval $i = 0, \ldots, N-1$

$$g(\cdot) = \begin{bmatrix} f(k_{i,1}, x_i + h\sum_{r=1}^{S} a_{1r} k_{i,r}, u_i) \\ \vdots \\ f(k_{i,S}, x_i + h\sum_{r=1}^{S} a_{Sr} k_{i,r}, u_i) \end{bmatrix} = 0$$



Collocation methods: Implicit Runge-Kutta



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$$g^j(\cdot) = \begin{bmatrix} f(k_{i,1}^j, x_i^{j-1} + h\sum_{r=1}^S a_{1r} k_{i,r}^j, u_i) \\ \vdots \\ f(k_{i,S}^j, x_i^{j-1} + h\sum_{r=1}^S a_{Sr} k_{i,r}^j, u_i) \end{bmatrix} = 0$$

$$x_i^j = x_i^{j-1} + h\sum_{r=1}^S b_r k_{i,r}^j$$

- N_s fixed integration steps:
  $\phi(x_i, u_i) = x_i^{N_s} = x_i + h\,B\,K_i$

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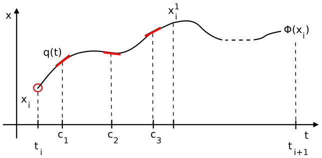

$$g^j(\cdot) = \begin{bmatrix} f(k^j_{i,1}, x_i^{j-1} + h\sum_{r=1}^{S} a_{1r} k^j_{i,r}, u_i) \\ \vdots \\ f(k^j_{i,S}, x_i^{j-1} + h\sum_{r=1}^{S} a_{Sr} k^j_{i,r}, u_i) \end{bmatrix} = 0$$

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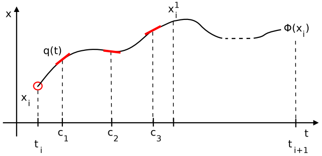

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- N_s fixed integration steps:
  $\phi(x_i, u_i) = x_i^{N_s} = x_i + h\,B\,K_i$

where $\frac{\partial G_i}{\partial K_i}$ is invertible

Sequential Quadratic Programming (SQP - GN)

Nonlinear Program (NLP)

$$
\begin{aligned}
\min_{X,\,U,\,K} \quad & \sum_{i=0}^{N-1} \|F_i(x_i, u_i)\|_2^2 + \|F_N(x_N)\|_2^2 \\
\text{s.t.} \quad & 0 = x_0 - \hat{x}_0 \\
& 0 = G(w_i, K_i) \\
& 0 = x_i + h\,B\,K_i - x_{i+1} \\
& 0 \geq h_i(x_i, u_i), \qquad i = 0, \ldots, N-1 \\
& 0 \geq r(x_N)
\end{aligned}
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Sequential Quadratic Programming (SQP - GN)

Nonlinear Program (NLP) $\rightarrow$ *quadratic subproblem*

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& 0 \geq h_i(x_i, u_i), \qquad i = 0, \dots, N-1 \\
& 0 \geq r(x_N)
\end{aligned}$$

$$\begin{aligned}
\min_{\Delta X, \Delta U, \Delta K} \quad & \sum_{i=0}^{N-1} \left\| F_i + J_i \begin{bmatrix} \Delta x_i \\ \Delta u_i \end{bmatrix} \right\|_2^2 + \|F_N + J_N \Delta x_N\|_2^2 \\
\text{s.t.} \quad & 0 = \bar{x}_0 - \hat{x}_0 + \Delta x_0 \\
& 0 = G_i + \frac{\partial G_i}{\partial w} \Delta w_i + \frac{\partial G_i}{\partial K} \Delta K_i \\
& 0 = d_i + \Delta x_i + h\,B\,\Delta K_i - \Delta x_{i+1} \\
& 0 \geq h_{\mathrm{lin,i}}(\Delta x_i, \Delta u_i), \qquad i = 0, \dots, N-1 \\
& 0 \geq r_{\mathrm{lin}}(\Delta x_N)
\end{aligned}$$

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Nonlinear Program (NLP) $\rightarrow$ *quadratic subproblem*

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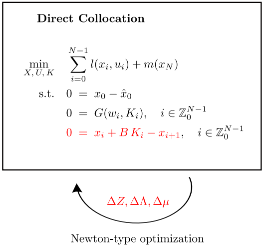
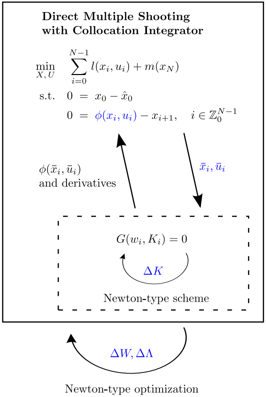
\end{aligned}
$$

Newton-type Optimization: Gauss-Newton (GN) but also exact Hessian $B_k = \nabla_W^2 \mathcal{L}(\cdot)$, Quasi-Newton, adjoint-based schemes, ...

Lifted collocation integrator[4]

"connection between collocation and multiple shooting"

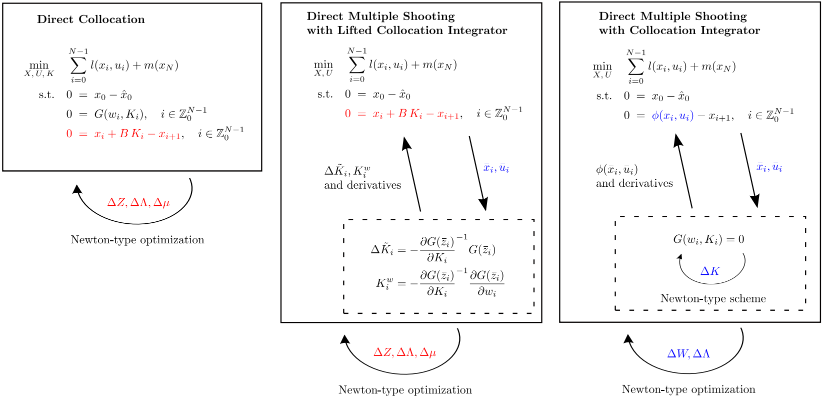




[4] R. Quirynen, S. Gros, and M. Diehl. "Lifted implicit integrators for direct optimal control". In: *Proceedings of the IEEE Conference on Decision and Control (CDC)*. 2015.

Lifted collocation integrator[4]

"connection between collocation and multiple shooting"



[4] Quirynen, Gros, and Diehl, "Lifted implicit integrators for direct optimal control".

Lifted collocation integrator (Gauss-Newton)

Bring direct collocation QP

$$
\begin{aligned}
\min_{\Delta Z} \quad & \sum_{i=0}^{N-1} \left(F_i^\top \Delta w_i + \frac{1}{2} \Delta w_i^\top H_i \, \Delta w_i \right) \\
\text{s.t.} \quad & 0 = \bar{x}_0 - \hat{x}_0 + \Delta x_0 \\
& 0 = G_i + \frac{\partial G_i}{\partial w} \Delta w_i + \frac{\partial G_i}{\partial K} \Delta K_i \qquad | \, \mu_i \\
& 0 = d_i + \Delta x_i + h \, B \, \Delta K_i - \Delta x_{i+1} \qquad | \, \lambda_i \\
& \ldots \qquad i = 0, \ldots, N-1
\end{aligned}
$$

into multiple shooting form:

$$
\begin{aligned}
\min_{\Delta W} \quad & \sum_{i=0}^{N-1} \left(F_i^\top \Delta w_i + \frac{1}{2} \Delta w_i^\top H_i \, \Delta w_i \right) \\
\text{s.t.} \quad & 0 = \bar{x}_0 - \hat{x}_0 + \Delta x_0 \\
& 0 = \tilde{d}_i + \frac{\mathrm{d}\phi}{\mathrm{d}w_i} \Delta w_i - \Delta x_{i+1} \qquad | \, \lambda_i \\
& \ldots \qquad i = 0, \ldots, N-1
\end{aligned}
$$

[5] R. Quirynen et al. “Lifted Collocation Integrators for Direct Optimal Control in ACADO Toolkit”. In: *Mathematical Programming Computation (under review, preprint available at Optimization Online)* (2016).

Lifted collocation integrator (Gauss-Newton)

Bring direct collocation QP

$$\min_{\Delta Z} \sum_{i=0}^{N-1} \left(F_i^\top \Delta w_i + \frac{1}{2} \Delta w_i^\top H_i \, \Delta w_i \right)$$

$$\begin{aligned} \text{s.t.} \quad 0 &= \bar{x}_0 - \hat{x}_0 + \Delta x_0 \\ 0 &= G_i + \frac{\partial G_i}{\partial w} \Delta w_i + \frac{\partial G_i}{\partial K} \Delta K_i && \mid \mu_i \\ 0 &= d_i + \Delta x_i + h\, B\, \Delta K_i - \Delta x_{i+1} && \mid \lambda_i \\ \ldots \quad & i = 0, \ldots, N-1 \end{aligned}$$

into multiple shooting form:

$$\min_{\Delta W} \sum_{i=0}^{N-1} \left(F_i^\top \Delta w_i + \frac{1}{2} \Delta w_i^\top H_i \, \Delta w_i \right)$$

$$\begin{aligned} \text{s.t.} \quad 0 &= \bar{x}_0 - \hat{x}_0 + \Delta x_0 \\ 0 &= \tilde{d}_i + \frac{\mathrm{d}\phi}{\mathrm{d}w_i} \Delta w_i - \Delta x_{i+1} && \mid \lambda_i \\ \ldots \quad & i = 0, \ldots, N-1 \end{aligned}$$

1. **Condensing** step for $i = 0, \ldots, N-1$

$$\Delta \tilde{K}_i = - \frac{\partial G_i}{\partial K}^{-1} G_i, \quad K_i^w = - \frac{\partial G_i}{\partial K}^{-1} \frac{\partial G_i}{\partial w}$$

$$\tilde{d}_i = d_i + h\, B\, \Delta \tilde{K}_i, \quad \frac{\mathrm{d}\phi}{\mathrm{d}w_i} = \begin{bmatrix} \mathbb{1} & \mathbb{0} \end{bmatrix} + h\, B\, K_i^w$$

[5] Quirynen et al., "Lifted Collocation Integrators for Direct Optimal Control in ACADO Toolkit".

Lifted collocation integrator (Gauss-Newton)

Bring direct collocation QP

$$\min_{\Delta Z} \sum_{i=0}^{N-1} \left(F_i^\top \Delta w_i + \frac{1}{2} \Delta w_i^\top H_i \Delta w_i \right)$$

$$\begin{aligned} \text{s.t.} \quad 0 &= \bar{x}_0 - \hat{x}_0 + \Delta x_0 \\ 0 &= G_i + \frac{\partial G_i}{\partial w} \Delta w_i + \frac{\partial G_i}{\partial K} \Delta K_i && \mid \mu_i \\ 0 &= d_i + \Delta x_i + h\, B\, \Delta K_i - \Delta x_{i+1} && \mid \lambda_i \\ \dots & \qquad i = 0, \dots, N-1 \end{aligned}$$

into multiple shooting form:

$$\min_{\Delta W} \sum_{i=0}^{N-1} \left(F_i^\top \Delta w_i + \frac{1}{2} \Delta w_i^\top H_i \Delta w_i \right)$$

$$\begin{aligned} \text{s.t.} \quad 0 &= \bar{x}_0 - \hat{x}_0 + \Delta x_0 \\ 0 &= \tilde{d}_i + \frac{\mathrm{d}\phi}{\mathrm{d}w_i} \Delta w_i - \Delta x_{i+1} && \mid \lambda_i \\ \dots & \qquad i = 0, \dots, N-1 \end{aligned}$$

1. **Condensing** step for $i = 0, \dots, N-1$

$$\Delta \tilde{K}_i = -\frac{\partial G_i}{\partial K}^{-1} G_i, \quad K_i^w = -\frac{\partial G_i}{\partial K}^{-1} \frac{\partial G_i}{\partial w}$$

$$\tilde{d}_i = d_i + h\, B\, \Delta \tilde{K}_i, \quad \frac{\mathrm{d}\phi}{\mathrm{d}w_i} = \begin{bmatrix} \mathbb{1} & \mathbb{0} \end{bmatrix} + h\, B\, K_i^w$$

2. QP solution: $\mathbf{\Delta w} = (\Delta w_0, \dots, \Delta w_N)$

[5] Quirynen et al., "Lifted Collocation Integrators for Direct Optimal Control in ACADO Toolkit".

Lifted collocation integrator (Gauss-Newton)

Bring direct collocation QP

$$\min_{\Delta Z} \sum_{i=0}^{N-1} \left(F_i^\top \Delta w_i + \frac{1}{2} \Delta w_i^\top H_i \, \Delta w_i \right)$$

$$\begin{aligned} \text{s.t.} \quad 0 &= \bar{x}_0 - \hat{x}_0 + \Delta x_0 \\ 0 &= G_i + \frac{\partial G_i}{\partial w} \Delta w_i + \frac{\partial G_i}{\partial K} \Delta K_i & \mid \mu_i \\ 0 &= d_i + \Delta x_i + h\, B\, \Delta K_i - \Delta x_{i+1} & \mid \lambda_i \\ \ldots & \quad i = 0, \ldots, N-1 \end{aligned}$$

into multiple shooting form:

$$\min_{\Delta \mathbf{W}} \sum_{i=0}^{N-1} \left(F_i^\top \Delta w_i + \frac{1}{2} \Delta w_i^\top H_i \, \Delta w_i \right)$$

$$\begin{aligned} \text{s.t.} \quad 0 &= \bar{x}_0 - \hat{x}_0 + \Delta x_0 \\ 0 &= \tilde{d}_i + \frac{\mathrm{d}\phi}{\mathrm{d}w_i} \Delta w_i - \Delta x_{i+1} & \mid \lambda_i \\ \ldots & \quad i = 0, \ldots, N-1 \end{aligned}$$

1. **Condensing** step for $i = 0, \ldots, N-1$

$$\Delta \tilde{K}_i = -\frac{\partial G_i}{\partial K}^{-1} G_i, \quad K_i^w = -\frac{\partial G_i}{\partial K}^{-1} \frac{\partial G_i}{\partial w}$$

$$\tilde{d}_i = d_i + h\, B\, \Delta \tilde{K}_i, \quad \frac{\mathrm{d}\phi}{\mathrm{d}w_i} = \begin{bmatrix} \mathbb{1} & \mathbb{0} \end{bmatrix} + h\, B\, K_i^w$$

2. QP solution: $\mathbf{\Delta W} = (\Delta w_0, \ldots, \Delta w_N)$

3. **Expansion** step for $i = 0, \ldots, N-1$

$$\Delta K_i = \Delta \tilde{K}_i + K_i^w \mathbf{\Delta w_i}$$

[5] Quirynen et al., "Lifted Collocation Integrators for Direct Optimal Control in ACADO Toolkit".

Lifted collocation integrator (Gauss-Newton)

Bring direct collocation QP

$$\min_{\Delta Z} \sum_{i=0}^{N-1} \left(F_i^\top \Delta w_i + \frac{1}{2} \Delta w_i^\top H_i \Delta w_i \right)$$

$$\text{s.t.} \quad 0 = \bar{x}_0 - \hat{x}_0 + \Delta x_0$$

$$0 = G_i + \frac{\partial G_i}{\partial w} \Delta w_i + \frac{\partial G_i}{\partial K} \Delta K_i \qquad | \, \mu_i$$

$$0 = d_i + \Delta x_i + h \, B \, \Delta K_i - \Delta x_{i+1} \qquad | \, \lambda_i$$

$$\ldots \qquad i = 0, \ldots, N-1$$

into multiple shooting form:

$$\min_{\mathbf{\Delta w}} \sum_{i=0}^{N-1} \left(F_i^\top \Delta w_i + \frac{1}{2} \Delta w_i^\top H_i \Delta w_i \right)$$

$$\text{s.t.} \quad 0 = \bar{x}_0 - \hat{x}_0 + \Delta x_0$$

$$0 = \tilde{d}_i + \frac{\mathrm{d}\phi}{\mathrm{d}w_i} \Delta w_i - \Delta x_{i+1} \qquad | \, \lambda_i$$

$$\ldots \qquad i = 0, \ldots, N-1$$

1. **Condensing** step for $i = 0, \ldots, N-1$

$$\Delta \tilde{K}_i = - \frac{\partial G_i}{\partial K}^{-1} G_i, \quad K_i^w = - \frac{\partial G_i}{\partial K}^{-1} \frac{\partial G_i}{\partial w}$$

$$\tilde{d}_i = d_i + h \, B \, \Delta \tilde{K}_i, \quad \frac{\mathrm{d}\phi}{\mathrm{d}w_i} = \begin{bmatrix} \mathbb{1} & \mathbb{0} \end{bmatrix} + h \, B \, K_i^w$$

2. QP solution: $\mathbf{\Delta w} = (\Delta w_0, \ldots, \Delta w_N)$

3. **Expansion** step for $i = 0, \ldots, N-1$

$$\Delta K_i = \Delta \tilde{K}_i + K_i^w \mathbf{\Delta w_i}$$

Proposition: Both iterations are *mathematically* equivalent.
Proof: see [5].

... but **not** numerically equivalent!

[5] Quirynen et al., "Lifted Collocation Integrators for Direct Optimal Control in ACADO Toolkit".

Inexact Newton type schemes for Implicit Runge-Kutta[6]

$$K^{[j]} = K^{[j-1]} - \frac{\partial \mathbf{G}(\mathbf{w}, \mathbf{K}^{[j-1]})}{\partial \mathbf{K}}^{-1} G(w, K^{[j-1]}), \quad j = 1, \ldots$$

[6] R. Quirynen, S. Gros, and M. Diehl. "Inexact Newton based Lifted Implicit Integrators for fast Nonlinear MPC". . In: *Proceedings of the IFAC Conference on Nonlinear Model Predictive Control (NMPC)*. 2015, pp. 32–38.

Inexact Newton type schemes for Implicit Runge-Kutta[6]

$$K^{[j]} = K^{[j-1]} - \mathbf{M}^{-1} G(w, K^{[j-1]}), \quad j = 1, \dots$$

where $M \approx \frac{\partial G(w, K^{[j-1]})}{\partial K}$ can be obtained as:

- reuse over integration steps

[6] Quirynen, Gros, and Diehl, "Inexact Newton based Lifted Implicit Integrators for fast Nonlinear MPC".

Inexact Newton type schemes for Implicit Runge-Kutta[6]

$$K^{[j]} = K^{[j-1]} - \mathbf{M}^{-1} G(w, K^{[j-1]}), \quad j = 1, \ldots$$

where $M \approx \frac{\partial G(w, K^{[j-1]})}{\partial K}$ can be obtained as:

- reuse over integration steps
- Simplified Newton

$$\begin{bmatrix} H_1 + h\, a_{11} J_1 & h\, a_{12} J_1 & h\, a_{13} J_1 \\ h\, a_{21} J_2 & H_2 + h\, a_{22} J_2 & h\, a_{23} J_2 \\ h\, a_{31} J_3 & h\, a_{32} J_3 & H_3 + h\, a_{33} J_3 \end{bmatrix}$$

$$\approx \begin{bmatrix} H + h\, a_{11} J & h\, a_{12} J & h\, a_{13} J \\ h\, a_{21} J & H + h\, a_{22} J & h\, a_{23} J \\ h\, a_{31} J & h\, a_{32} J & H + h\, a_{33} J \end{bmatrix}$$

$$= \mathbb{1}_3 \otimes H + h\, A \otimes J$$

[6] Quirynen, Gros, and Diehl, "Inexact Newton based Lifted Implicit Integrators for fast Nonlinear MPC".

Inexact Newton type schemes for Implicit Runge-Kutta[6]

$$K^{[j]} = K^{[j-1]} - \mathbf{M}^{-1} G(w, K^{[j-1]}), \quad j = 1, \dots$$

where $M \approx \frac{\partial G(w, K^{[j-1]})}{\partial K}$ can be obtained as:

- reuse over integration steps
- Simplified Newton
- Single Newton

$$\begin{bmatrix} H_1 + h\, a_{11} J_1 & h\, a_{12} J_1 & h\, a_{13} J_1 \\ h\, a_{21} J_2 & H_2 + h\, a_{22} J_2 & h\, a_{23} J_2 \\ h\, a_{31} J_3 & h\, a_{32} J_3 & H_3 + h\, a_{33} J_3 \end{bmatrix}$$

$$\approx \begin{bmatrix} H + h\, \mathbf{a}_{11} J & h\, \mathbf{a}_{12} J & h\, \mathbf{a}_{13} J \\ h\, \mathbf{a}_{21} J & H + h\, \mathbf{a}_{22} J & h\, \mathbf{a}_{23} J \\ h\, \mathbf{a}_{31} J & h\, \mathbf{a}_{32} J & H + h\, \mathbf{a}_{33} J \end{bmatrix}$$

$$= \mathbb{1}_3 \otimes H + h\, A \otimes J$$

$$\approx \begin{bmatrix} H + h\, \tilde{\mathbf{a}}_{11} J & h\, \tilde{\mathbf{a}}_{12} J & h\, \tilde{\mathbf{a}}_{13} J \\ h\, \tilde{\mathbf{a}}_{21} J & H + h\, \tilde{\mathbf{a}}_{22} J & h\, \tilde{\mathbf{a}}_{23} J \\ h\, \tilde{\mathbf{a}}_{31} J & h\, \tilde{\mathbf{a}}_{32} J & H + h\, \tilde{\mathbf{a}}_{33} J \end{bmatrix}$$

$$= \mathbb{1}_3 \otimes H + h\, \tilde{\mathbf{A}} \otimes J$$

[6] Quirynen, Gros, and Diehl, "Inexact Newton based Lifted Implicit Integrators for fast Nonlinear MPC".

Inexact Newton type schemes for Implicit Runge-Kutta[6]

$$K^{[j]} = K^{[j-1]} - \mathbf{M}^{-1} G(w, K^{[j-1]}), \quad j = 1, \ldots$$

where $M \approx \frac{\partial G(w, K^{[j-1]})}{\partial K}$ can be obtained as:

- reuse over integration steps
- Simplified Newton
- Single Newton
- fixed matrix M
- ...

$$\begin{bmatrix} H_1 + h\, a_{11} J_1 & h\, a_{12} J_1 & h\, a_{13} J_1 \\ h\, a_{21} J_2 & H_2 + h\, a_{22} J_2 & h\, a_{23} J_2 \\ h\, a_{31} J_3 & h\, a_{32} J_3 & H_3 + h\, a_{33} J_3 \end{bmatrix}$$

$$\approx \begin{bmatrix} H + h\, \mathbf{a}_{11} J & h\, \mathbf{a}_{12} J & h\, \mathbf{a}_{13} J \\ h\, \mathbf{a}_{21} J & H + h\, \mathbf{a}_{22} J & h\, \mathbf{a}_{23} J \\ h\, \mathbf{a}_{31} J & h\, \mathbf{a}_{32} J & H + h\, \mathbf{a}_{33} J \end{bmatrix}$$

$$= \mathbb{1}_3 \otimes H + h\, A \otimes J$$

$$\approx \begin{bmatrix} H + h\, \tilde{\mathbf{a}}_{11} J & h\, \tilde{\mathbf{a}}_{12} J & h\, \tilde{\mathbf{a}}_{13} J \\ h\, \tilde{\mathbf{a}}_{21} J & H + h\, \tilde{\mathbf{a}}_{22} J & h\, \tilde{\mathbf{a}}_{23} J \\ h\, \tilde{\mathbf{a}}_{31} J & h\, \tilde{\mathbf{a}}_{32} J & H + h\, \tilde{\mathbf{a}}_{33} J \end{bmatrix}$$

$$= \mathbb{1}_3 \otimes H + h\, \tilde{\mathbf{A}} \otimes J$$

[6] Quirynen, Gros, and Diehl, "Inexact Newton based Lifted Implicit Integrators for fast Nonlinear MPC".

Lifted collocation integrator (Gauss-Newton)

Bring direct collocation QP

$$\begin{aligned}\min_{\Delta Z} \quad & \sum_{i=0}^{N-1} \left(F_i^\top \Delta w_i + \frac{1}{2} \Delta w_i^\top H_i \, \Delta w_i \right) \\ \text{s.t.} \quad & 0 = \bar{x}_0 - \hat{x}_0 + \Delta x_0 \\ & 0 = G_i + \frac{\partial G_i}{\partial w} \Delta w_i + \frac{\partial G_i}{\partial K} \Delta K_i \qquad | \; \mu_i \\ & 0 = d_i + \Delta x_i + h \, B \, \Delta K_i - \Delta x_{i+1} \qquad | \; \lambda_i \\ & \ldots \qquad i = 0, \ldots, N-1\end{aligned}$$

into multiple shooting form:

$$\begin{aligned}\min_{\mathbf{\Delta W}} \quad & \sum_{i=0}^{N-1} \left(F_i^\top \Delta w_i + \frac{1}{2} \Delta w_i^\top H_i \, \Delta w_i \right) \\ \text{s.t.} \quad & 0 = \bar{x}_0 - \hat{x}_0 + \Delta x_0 \\ & 0 = \tilde{d}_i + \frac{\mathrm{d}\phi}{\mathrm{d}w_i} \Delta w_i - \Delta x_{i+1} \qquad | \; \lambda_i \\ & \ldots \qquad i = 0, \ldots, N-1\end{aligned}$$

1. **Condensing** step for $i = 0, \ldots, N-1$

$$\Delta \tilde{K}_i = - \frac{\partial G_i}{\partial K}^{-1} G_i, \quad K_i^w = - \frac{\partial G_i}{\partial K}^{-1} \frac{\partial G_i}{\partial w}$$

$$\tilde{d}_i = d_i + h \, B \, \Delta \tilde{K}_i, \quad \frac{\mathrm{d}\phi}{\mathrm{d}w_i} = \begin{bmatrix} \mathbb{1} & \mathbb{0} \end{bmatrix} + h \, B \, K_i^w$$

2. QP solution: $\mathbf{\Delta w} = (\Delta w_0, \ldots, \Delta w_N)$

3. **Expansion** step for $i = 0, \ldots, N-1$

$$\Delta K_i = \Delta \tilde{K}_i + K_i^w \, \mathbf{\Delta w_i}$$

Inexact lifted collocation integrator (Gauss-Newton)

Bring direct collocation QP

$$\min_{\Delta Z} \sum_{i=0}^{N-1} \left(F_i^\top \Delta w_i + \frac{1}{2} \Delta w_i^\top H_i \, \Delta w_i \right)$$

$$\text{s.t.} \quad 0 = \bar{x}_0 - \hat{x}_0 + \Delta x_0$$

$$0 = G_i + \frac{\partial G_i}{\partial w} \Delta w_i + \frac{\partial G_i}{\partial K} \Delta K_i \qquad | \; \mu_i$$

$$0 = d_i + \Delta x_i + h \, B \, \Delta K_i - \Delta x_{i+1} \qquad | \; \lambda_i$$

$$\ldots \qquad i = 0, \ldots, N-1$$

into multiple shooting form:

$$\min_{\mathbf{\Delta w}} \sum_{i=0}^{N-1} \left(\tilde{F}_i^\top \Delta w_i + \frac{1}{2} \Delta w_i^\top H_i \, \Delta w_i \right)$$

$$\text{s.t.} \quad 0 = \bar{x}_0 - \hat{x}_0 + \Delta x_0$$

$$0 = \tilde{d}_i + \frac{\mathrm{d}\phi}{\mathrm{d}w_i} \Delta w_i - \Delta x_{i+1} \qquad | \; \lambda_i$$

$$\ldots \qquad i = 0, \ldots, N-1$$

1. **Condensing** step for $i = 0, \ldots, N-1$

$$\Delta \tilde{K}_i = -M_i^{-1} G_i, \quad \tilde{K}_i^w = -M_i^{-1} \frac{\partial G_i}{\partial w}$$

$$\tilde{d}_i = d_i + h \, B \, \Delta \tilde{K}_i, \quad \frac{\mathrm{d}\phi}{\mathrm{d}w_i} = \begin{bmatrix} \mathbb{1} & \mathbb{0} \end{bmatrix} + h \, B \, \tilde{K}_i^w$$

$$\tilde{F}_i = F_i + \left(\frac{\partial G_i}{\partial w} + \frac{\partial G_i}{\partial K} \tilde{K}_i^w \right)^\top \bar{\mu}_i$$

2. QP solution: $\mathbf{\Delta w} = (\Delta w_0, \ldots, \Delta w_N)$, $\bar{\lambda}_{\mathbf{i}}^+$

3. **Expansion** step for $i = 0, \ldots, N-1$

$$\Delta K_i = \Delta \tilde{K}_i + \tilde{K}_i^w \mathbf{\Delta w_i}$$

$$\bar{\mu}_i^+ = \bar{\mu}_i - M_i^{-\top} \left(\frac{\partial G_i}{\partial K}^\top \bar{\mu}_i + B^\top \bar{\lambda}_{\mathbf{i}}^+ \right)$$

INIS lifted collocation integrator (Gauss-Newton)

Bring direct collocation QP

$$\min_{\Delta Z} \sum_{i=0}^{N-1} \left(F_i^\top \Delta w_i + \frac{1}{2} \Delta w_i^\top H_i \, \Delta w_i \right)$$

$$\begin{aligned} \text{s.t.} \quad & 0 = \bar{x}_0 - \hat{x}_0 + \Delta x_0 \\ & 0 = G_i + \frac{\partial G_i}{\partial w} \Delta w_i + \frac{\partial G_i}{\partial K} \Delta K_i & \mid \mu_i \\ & 0 = d_i + \Delta x_i + h\, B\, \Delta K_i - \Delta x_{i+1} & \mid \lambda_i \\ & \ldots \quad i = 0, \ldots, N-1 \end{aligned}$$

into multiple shooting form:

$$\min_{\mathbf{\Delta W}} \sum_{i=0}^{N-1} \left(\tilde{F}_i^\top \Delta w_i + \frac{1}{2} \Delta w_i^\top H_i \, \Delta w_i \right)$$

$$\begin{aligned} \text{s.t.} \quad & 0 = \bar{x}_0 - \hat{x}_0 + \Delta x_0 \\ & 0 = \tilde{d}_i + \frac{\mathrm{d}\phi}{\mathrm{d}w_i} \Delta w_i - \Delta x_{i+1} & \mid \lambda_i \\ & \ldots \quad i = 0, \ldots, N-1 \end{aligned}$$

1. **Condensing** step for $i = 0, \ldots, N-1$

$$\Delta \tilde{K}_i = -M_i^{-1} G_i, \quad \Delta K_i^w = -M_i^{-1} \left(\frac{\partial G_i}{\partial w} + \frac{\partial G_i}{\partial K} \bar{K}_i^w \right)$$

$$\tilde{d}_i = d_i + h\, B\, \Delta \tilde{K}_i, \quad \frac{\mathrm{d}\phi}{\mathrm{d}w_i} = \begin{bmatrix} \mathbb{1} & \mathbb{0} \end{bmatrix} + h\, B\, \bar{K}_i^w$$

$$\tilde{F}_i = F_i + \left(\frac{\partial G_i}{\partial w} + \frac{\partial G_i}{\partial K} \bar{K}_i^w \right)^\top \bar{\mu}_i$$

2. QP solution: $\mathbf{\Delta w} = (\Delta w_0, \ldots, \Delta w_N)$, $\bar{\lambda}_{\mathbf{i}}^+$

3. **Expansion** step for $i = 0, \ldots, N-1$

$$\Delta K_i = \Delta \tilde{K}_i + \bar{K}_i^w \, \mathbf{\Delta w_i}$$

$$\bar{K}_i^{w+} = \bar{K}_i^w + \Delta K_i^w$$

$$\bar{\mu}_i^+ = \bar{\mu}_i - M_i^{-\top} \left(\frac{\partial G_i}{\partial K}^\top \bar{\mu}_i + B^\top \bar{\lambda}_{\mathbf{i}}^+ \right)$$

AF-INIS lifted collocation integrator (Gauss-Newton)

Bring direct collocation QP

$$\min_{\Delta Z} \sum_{i=0}^{N-1} \left(F_i^\top \Delta w_i + \frac{1}{2} \Delta w_i^\top H_i \, \Delta w_i \right)$$

$$\begin{aligned} \text{s.t.} \quad 0 &= \bar{x}_0 - \hat{x}_0 + \Delta x_0 \\ 0 &= G_i + \frac{\partial G_i}{\partial w} \Delta w_i + \frac{\partial G_i}{\partial K} \Delta K_i && \mid \mu_i \\ 0 &= d_i + \Delta x_i + h\, B\, \Delta K_i - \Delta x_{i+1} && \mid \lambda_i \\ & \dots \quad i = 0, \dots, N-1 \end{aligned}$$

into multiple shooting form:

$$\min_{\mathbf{\Delta w}} \sum_{i=0}^{N-1} \left(F_i^\top \Delta w_i + \frac{1}{2} \Delta w_i^\top H_i \, \Delta w_i \right)$$

$$\begin{aligned} \text{s.t.} \quad 0 &= \bar{x}_0 - \hat{x}_0 + \Delta x_0 \\ 0 &= \tilde{d}_i + \frac{\mathrm{d}\phi}{\mathrm{d}w_i} \Delta w_i - \Delta x_{i+1} && \mid \lambda_i \\ & \dots \quad i = 0, \dots, N-1 \end{aligned}$$

1. **Condensing** step for $i = 0, \dots, N-1$

$$\Delta \tilde{K}_i = -M_i^{-1} G_i, \quad \Delta K_i^w = -M_i^{-1} \left(\frac{\partial G_i}{\partial w} + \frac{\partial G_i}{\partial K} \bar{K}_i^w \right)$$

$$\tilde{d}_i = d_i + h\, B\, \Delta \tilde{K}_i, \quad \frac{\mathrm{d}\phi}{\mathrm{d}w_i} = \begin{bmatrix} \mathbb{1} & \mathbb{0} \end{bmatrix} + h\, B\, \bar{K}_i^w$$

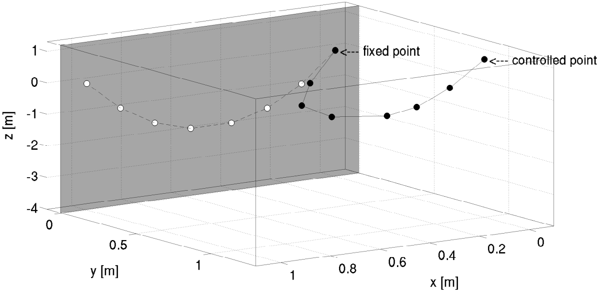
2. QP solution: $\mathbf{\Delta w} = (\Delta w_0, \dots, \Delta w_N)$

3. **Expansion** step for $i = 0, \dots, N-1$

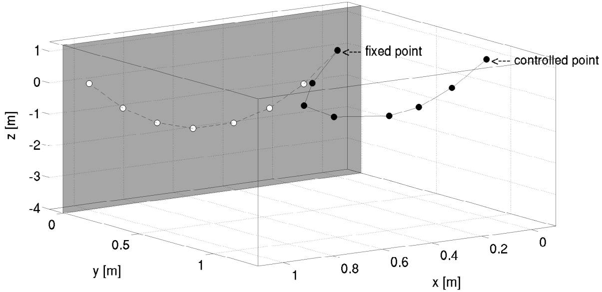
$$\Delta K_i = \Delta \tilde{K}_i + \bar{K}_i^w \mathbf{\Delta w_i}$$

$$\bar{K}_i^{w+} = \bar{K}_i^w + \Delta K_i^w$$

Numerical case study: chain of masses



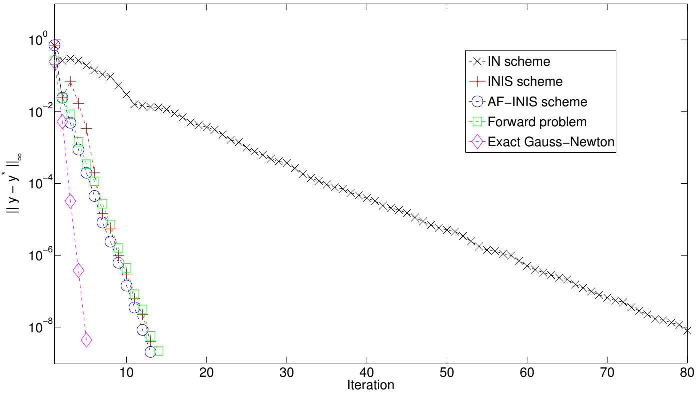
Numerical case study: chain of masses

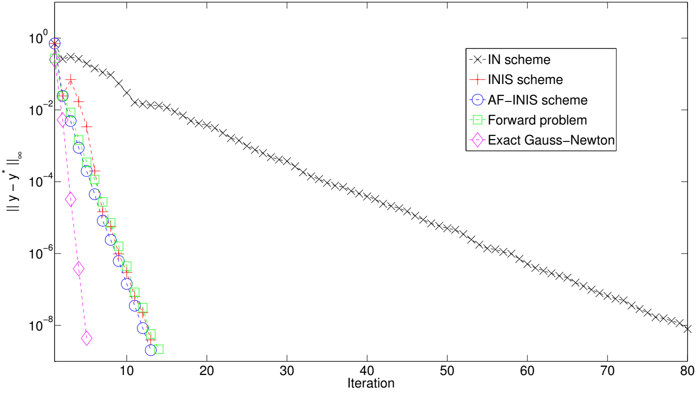


ACADO Toolkit

`www.acadotoolkit.org`

- n_{m} masses with springs
- one fixed, one controlled end
- $T = 5\ \mathrm{s}$, $N = 20$
- $6(n_{\mathrm{m}} - 1)$ states, 3 controls


IN scheme
INIS scheme
AF−INIS scheme
Forward problem
Exact Gauss−Newton
|| y − y* ||∞
Iteration
10^0
10^-2
10^-4
10^-6
10^-8
10
20
30
40
50
60
70
80



n_{m}	n_x	Gauss-Newton	IN	INIS	AF-INIS
4	18	14.79 ms	5.43 ms	4.76 ms	4.29 ms
5	24	34.04 ms	10.71 ms	9.39 ms	7.96 ms
6	30	62.08 ms	18.73 ms	14.88 ms	12.71 ms
7	36	106.57 ms	36.09 ms	21.93 ms	20.06 ms

Outline

Inexact Newton with Iterated Sensitivities [7]

- inexact Newton optimization
- local contraction theorem
- direct optimal control
- easy to implement

[7] Quirynen, Gros, and Diehl, "Inexact Newton-Type Optimization with Iterated Sensitivities".

Inexact Newton with Iterated Sensitivities [7]

- inexact Newton optimization
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within the **open-source ACADO** Toolkit

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Inexact Newton with Iterated Sensitivities [7]

- inexact Newton optimization
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within the **open-source ACADO** Toolkit

Outlook:

- further applications
- tailored implementations
- globalization strategies

[7] Quirynen, Gros, and Diehl, "Inexact Newton-Type Optimization with Iterated Sensitivities".

⚠ **Unique opportunity** (continued):

Numerical Simulation Methods for Embedded Optimization

1. Fast Nonlinear Model Predictive Control and Estimation
2. Numerical Simulation and Sensitivity Propagation
3. Symmetric Hessian Propagation Technique *(Robin)*
4. Structure Exploitation for Linear Subsystems
5. Compression Algorithm for Distributed Multiple Shooting
6. Lifted Newton-Type Collocation Integrators
7. Local Convergence of Inexact Newton with Iterated Sensitivities
8. Open-Source ACADO Code Generation Software *(Thor)*
9. Two-Stage Turbocharged Gasoline Engine