Fizzy's guide to vorticity and vortex methods

R. Leuthold

May 2, 2018

I'd rather study fluids than swim in them.

-Fizzy

Contents

1 defining dilation and vorticity

Let's define two thermodynamic state variables that describe the evolution of a flow field.

The first of these state variables is the *dilation* $\vartheta := \nabla \cdot \boldsymbol{u}$. The dilation is the divergence of the velocity at any point within the flow field. The dilation describes the change of the fluid velocity along a set of axes, or - in alternate terms, the normal strain experienced by the fluid element. In doing so, it describes how much the fluid's volume is compressed into and expanded out-of itself. That is, the dilation describes the compressibility of the fluid. This can be seen, by considering an incompressible context when the flow has constant density, such that conservation of mass without sources or sinks $\nabla \cdot (\rho \boldsymbol{u}) = 0$ requires that $\vartheta = 0$.

The second of these state variables is the *vorticity* $\boldsymbol{\omega} := \nabla \times \boldsymbol{u}$. The vorticity is the curl of the velocity at any point within the flow field. That is, the vorticity is twice the angular velocity of an infinitessimally small[1] fluid element. As a fluid element is not a rigid body, its angular velocity contains pure-rotation about some arbitrary point, as well as spin about the fluid element iself.

The fluid can be considered as a giant multibody problem, where each of the individual fluid elements within the multibody problem has its rotation both forced by and damped by the friction of viscosity. The damping process may seem intuitive - after all, doesn't friction typically act to damp velocities? - while the forcing process may seem counter-intuitive to our macro-scale lifes. But, we should remember that this also happens on our scales. Consider that when spinning a top[2], the torque to accelerate the top's rotation comes from friction with our hands, and the torque to decelerate the top comes from viscous skin-friction with the air.

Let's consider a fluid under a shearing velocity gradient. Across a shear-layer, a fluid element will experience a shearing deformation that elongates the edge in the faster stream respective to the edge in the slower stream. If this fluid is viscous, and there is no counter-acting shear deformation orthogonal to the axis that will oppose rotation, the fluid element will resist this deformation by rotating locally. Then, shear-layers and their consequent shear-stresses will tend to be associated with non-zero vorticities.

Returning to the picture of a fluid as a multibody problem of fluid elements, a fluid element would be vorticity-free if, either, the fluid has no rotation at all, or if the spin of the element perfectly balances the rotation of the element about any external point. That is, a flow with curved streamlines can be vorticity-free, as in the case of fluid travelling down a drain, where the individual fluid elements keep their orientation in space.[15] In contrast, a flow with straight streamlines can have non-zero vorticity, as in the case of a boundary layer, as evidenced by the shear-strain on the fluid layers.

Using an explanation expanded from Branlard[5], Helmholtz's vorticity equation[3] can be used to classify the

[1] Remember, all flow modelling derived from or based on the Navier Stokes equations has already assumed that fluids are continuoums, and not collections of molecules.

[2] a the little wooden body-of-rotation frequently used to demonstrate gyroscopic effects

[3] To get Helmholtz's vorticity equation, start with the compressible Navier-Stokes

$$\left(\frac{D\boldsymbol{u}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \boldsymbol{F} - \frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \tau \right),$$

and take the curl:

$$\left(\nabla \times \frac{D\boldsymbol{u}}{Dt} = \nabla \times \frac{\partial \boldsymbol{u}}{\partial t} + \nabla \times (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = \nabla \times \boldsymbol{F} - \nabla \times \left(\frac{1}{\rho} \nabla p \right) + \nabla \times \left(\frac{1}{\rho} \nabla \cdot \tau \right) \right).$$

Then, apply two identities that $\nabla \times (a\boldsymbol{v}) = a(\nabla \times \boldsymbol{v}) + (\nabla a) \times \boldsymbol{v}$ and $\boldsymbol{a} \cdot \nabla \boldsymbol{a} = \frac{1}{2} \nabla (\boldsymbol{a} \cdot \boldsymbol{a}) - \boldsymbol{a} \times (\nabla \times \boldsymbol{a})$ until:

$$\left(\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times \left(\frac{1}{2} \nabla (\boldsymbol{u} \cdot \boldsymbol{u}) - \boldsymbol{u} \times \boldsymbol{\omega} \right) = \nabla \times \boldsymbol{F} - \frac{1}{\rho} \nabla \times \nabla p - \nabla \left(\frac{1}{\rho} \right) \times \nabla p + \nabla \times \left(\frac{1}{\rho} \nabla \cdot \tau \right) \right).$$

Because the curl of a gradient is zero, this becomes:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) = \nabla \times \boldsymbol{F} - \nabla \left(\frac{1}{\rho} \right) \times \nabla p + \nabla \times \left(\frac{1}{\rho} \nabla \cdot \tau \right).$$

With two additional identities that $\nabla (a/b) = (b(\nabla a) - (\nabla b)a)/b^2$ and $\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a}(\nabla \cdot \boldsymbol{b}) - \boldsymbol{b}(\nabla \cdot \boldsymbol{a}) + (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b}$, this gives:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{u}(\nabla \cdot \boldsymbol{\omega}) + \boldsymbol{\omega}(\nabla \cdot \boldsymbol{u}) - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} = \nabla \times \boldsymbol{F} + \frac{\nabla \rho}{\rho^2} \times \nabla p + \nabla \times \left(\frac{1}{\rho} \nabla \cdot \tau \right).$$

Finally, the divergence of a curl $(\nabla \cdot \boldsymbol{\omega})$ vanishes, and we reshuffle sides.

processes in a fluid that drive and damp the vorticity:

$$\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + \underbrace{(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega}}_{\text{convection}} = \underbrace{(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}}_{\text{stretching}} - \underbrace{\boldsymbol{\omega}\vartheta}_{\text{dilation}} + \underbrace{\nabla\times\boldsymbol{F}}_{\text{n.c. forces}} + \underbrace{\frac{1}{\rho^2}\nabla\rho\times\nabla p}_{\text{baroclinicity}} + \underbrace{\nabla\times\left(\frac{1}{\rho}\nabla\cdot\boldsymbol{\tau}\right)}_{\text{viscous shear stress}}, \tag{1}$$

where $\frac{D}{Dt}$ is the material derivative, $\boldsymbol{F}$ is a forcing term, ρ is the fluid density, p the static pressure, and τ is the stress caused by viscosity when a fluid element is deformed. With respect to the labels:

- 'convection' indicates the rate of change of the vorticity as it travels with the flow;
- 'stretching', that vorticity becomes stronger if the flow accelerates in the direction of the vorticity and becomes weaker if the flow decelerates along the vorticity;
- 'dilation', that changing the density of fluids can create vorticity;
- 'n.c. forces', that because the curl of conservative forces is necessarily zero, only non-conservative forcing contributes to this term;
- 'baroclinicity', the degree of mixing in the fluid due to skew[4] between the pressure gradient and density gradient; and
- 'viscous shear stress', the rotation described previously to occur for imbalanced counter-acting shear stresses.

2 defining a vortex

You may have noticed that the word "vorticity" resembles the word "vortex" that is colloquially used to refer to spinning fluid structures, such as tornados. This resemblance is not coincidental. A vortex is defined[5] as a structure of concentrated vorticity.

This is admittedly a fairly tautalogical definition, analogous to saying that some conductor is a structure of concentrated current. That is, defining a tube-like vortex based on its vorticity distribution is equivalent to defining a wire by the current that is running through it.

Notice that the current running through a given conductor may be strong, or may be 0A; it may be variable in time or unevenly distributed, as in circuits with multiple closed loops. Similarly, the vorticity within a conductor can have various strengths and be distributed unsteadily in time and non-uniformly in space.

Just as the current within the wire is invisible and only the wire is visible - vorticity itself is not visible. If we can on occasion see a vortex, it is because the fluid within that specific vortex has collected a high concentration of suspended particles such as water droplets or dust particles.

Further, as metal conductors can take whatever shape in which they are manufactured, vortices can also take arbitrary shapes. That we tend to think of vortices as tube-like structures or rings constructed from tube-like structures has more to do[6] with which vortices will accumulate enough particles to make themselves visible, then any characteristic tube-like shape.

All of this is to say that we should not have preconcieved notions about what defines a *vortex*, except that it contains vorticity, which is by definition related to circulatory motion within fluids.

3 using dilation and vorticity to define some flow assumptions

One benefit we get from having defined the dilation and the vorticity, is that certain typical flow assumptions can be written in their perspective. While these flow assumptions are not needed immediately, their presentation can

[4] For an example of vorticity caused by baroclinicity, picture the air-water interface on the ocean's surface. Strong horizontal winds are caused by a driving pressure gradient, which is orthogonal to the vertical density gradient over the water. This produces rotation in the fluid elements near the ocean surface, which shows up as waves. Ocean waves are examples of the classic Kelvin-Helmholtz instability, that can occasionally also be seen atmospherically.

[5] See [23] for an explanation of why there is not yet a formal mathematical definition for a vortex.

[6] also a small amount to do with the fact that the tube-like elementary solution is easy to compute, see section 13...

be used as examples of what ϑ and $\boldsymbol{\omega}$ mean practically speaking. Further, the presentation of these assumptions here allows for their uninterrupted use later on.

The typical flow assumptions mentioned here are: *irrotationality*, where the vorticity is zero; *incompressibility*, where the dilation is zero; and *inviscidness*, where the fluid viscosity is zero. When all three of these assumptions is made, the fluid is said to be *potential*.

Mathematically, this is stated as:

$$\text{potential:} \quad \begin{cases} \text{irrotational:} & \boldsymbol{\omega} = \mathbf{0}, \\ \text{incompressible:} & \vartheta = 0, \\ \text{inviscid:} & \nu = 0, \end{cases} \tag{2}$$

where ν is the kinematic viscosity of the fluid.

While these characteristics can apply to any given fluid element within a domain, they are usually assumed to apply only to the *external flow* far from any solid boundaries within the flow. For example, flow is assumed to be *externally potential* if the flow far from solid boundaries is laminar and not circulating, slow compared to the speed of sound, and has behavior driven mainly by inertial forces.

From an engineering perspective, the flow can be assumed to be externally incompressible and inviscid when, respectively, the Mach number is less than 0.6 and the Reynolds number approaches infinity. Arguing for external irrotationality is more difficult, and is usually done by arguing that the sources of vorticity described in (1) are negligible.

The *Mach number* Ma is explicitly defined as the ratio between the characteristic flow speed u_* and the speed of sound a of the medium. That is:

$$\text{Ma} = \frac{u_*}{a} = \frac{u_*}{\sqrt{\gamma_{\text{air}} R_{\text{air}} T_*}},$$

where γ_{air} is the adiabatic constant of the fluid, R_{air} is the specific gas constant, and T_* is the characteristic temperature of the flow. For standard, dry-air, $\gamma_{\text{air}} = 1.4$ and $R_{\text{air}} = 287\text{J/kg/K}$.

The *Reynolds number* Re is explicitly defined as the ratio between the intertial forces and the viscous forces in a flow. That is, for a characteristic length ℓ_*, a characteristic flow speed u_*, a characterisic flow kinematic visocity ν_*, the characteristic flow dynamic viscosity μ_*, the characteristic mean-free-path of a fluid molecule $\overline{\ell}_*$, K_μ a constant for homogenous gases, and the characteristic root-mean-squared velocity of a fluid molecule $\overline{u}_*$ [1]:

$$\text{Re} = \frac{u_* \ell_*}{\nu_*} = \frac{u_* \ell_* \rho_*}{\mu_*} = \frac{u_* \ell_*}{K_\mu \overline{u}_* \overline{\ell}_*}.$$

A potential flow solution has the curious property of being found to be a minimum kinetic energy state. This is formally written[7] in Kelvin's minimum kinetic energy theorem.

[7] The derivation of Kelvin's minium kinetic energy theorem can be found in multiple sources, but is here reproduced from [7].

Assume that there are two incompressible velocity fields that satisfy the same normal boundary conditions in a simply connected domain V: the irrotational $\boldsymbol{u}_0 = \nabla\phi$ and the rotational $\boldsymbol{u}_\omega$. We can specify that $\boldsymbol{u}_\omega = \boldsymbol{u}_0 + \boldsymbol{u}_\Delta$, for some arbitrary difference $\boldsymbol{u}_\Delta$. Since both $\boldsymbol{u}_0$ and $\boldsymbol{u}_\omega$ have the same normal boundary conditions, then the normal component of $\boldsymbol{u}_\Delta$ must be zero along the boundary ∂V.

Then, the kinetic energy of the irrotational T_0, rotational T_ω and difference T_Δ flow fields can be found as:

$$T_0 = \frac{\rho}{2}\int_V \boldsymbol{u}_0 \cdot \boldsymbol{u}_0 dV, \qquad T_\omega = \frac{\rho}{2}\int_V \boldsymbol{u}_\omega \cdot \boldsymbol{u}_\omega dV, \qquad T_\Delta = \frac{\rho}{2}\int_V \boldsymbol{u}_\Delta \cdot \boldsymbol{u}_\Delta dV.$$

We can substitute the addition expression of $\boldsymbol{u}_\omega$ here to give:

$$T_\omega = \frac{\rho}{2}\int_V (\boldsymbol{u}_0 + \boldsymbol{u}_\Delta)\cdot(\boldsymbol{u}_0 + \boldsymbol{u}_\Delta)\, dV = \frac{\rho}{2}\int_V \boldsymbol{u}_0 \cdot \boldsymbol{u}_0 \, dV + \frac{\rho}{2}\int_V \boldsymbol{u}_\Delta \cdot \boldsymbol{u}_\Delta \, dV + \rho\int_V \boldsymbol{u}_\Delta \cdot \boldsymbol{u}_0 \, dV = T_0 + T_\Delta + \rho\int_V \boldsymbol{u}_\Delta \cdot (\nabla\phi)\, dV.$$

Now, the chain rule says that $\nabla\cdot(\phi\boldsymbol{u}_\Delta) = \boldsymbol{u}_\Delta\cdot(\nabla\phi) + \phi\nabla\cdot\boldsymbol{u}_\Delta$. Further, incompressible conservation of mass says that $\nabla\cdot\boldsymbol{u}_0 = \nabla\cdot\boldsymbol{u}_\omega = \nabla\cdot\boldsymbol{u}_\Delta = 0$. This gives:

$$T_\omega = T_0 + T_\Delta + \rho\int_V \nabla\cdot(\phi\boldsymbol{u}_\Delta) - \phi\nabla\cdot\boldsymbol{u}_\Delta \, dV = T_0 + T_\Delta + \rho\int_V \nabla\cdot(\phi\boldsymbol{u}_\Delta)\, dV.$$

We can here apply the divergence theorem:

$$T_\omega = T_0 + T_\Delta + \rho\int_{\partial V} \phi\,(\boldsymbol{u}_\Delta\cdot\hat{\boldsymbol{n}})\, ds,$$

where we see that the remaining integral must vanish, because we know that the normal component of $\boldsymbol{u}_\Delta$ is zero along the boundary ∂V.

Since T_Δ is a kinetic energy, it must be positive. That means that $T_\omega = T_0 + T_\Delta \geq T_0$.

Theorem 1 (Kelvin's minimum kinetic energy theorem)**.** *Of all incompressible velocity fields that satisfy the same normal boundary conditions in a simply connected domain, the irrotational velocity field will have the lowest kinetic energy.*

Note that we will not assume any of the three potential flow assumptions unless explicitly stated.

4 the boundary layer, or where we might apply the previously-defined flow assumptions

You may have noticed that the above definition of 'external flow' was vague about what it means to be 'far' from solid boundaries. This section intends to expand on the statement of 'far'.

Let's make a thought experiment of an infinite domain that contains only one solid object of finite dimension.

If we take a molecular[8] view of the solid boundary, we see that the surface must be jagged, at least where the body's molecules stick together. Then, the molecules of the fluid travelling immediately next to the boundary will get caught by the jagged molecules of the boundary. The fluid molecules immediately next to the boundary can have no velocity[9] with respect to the solid boundary's velocity. This is called the *no-slip condition*, and indicates the significant influence of friction on the flow near the wall.

However, infinitey far away from the solid body, the fluid flow must have recovered to its free-stream values. For fluids that can be modelled as incompressible continua, we see that the influence of the body on the flow must decrease continuously between the boundary and infinitely far away.

At some distance δ normal to the wall, the flow will have 'mostly' recovered to the freestream velocity, such that:

$$\boldsymbol{u}(\delta) = f_\delta \boldsymbol{u}_\infty, \tag{3}$$

using the freestream velocity $\boldsymbol{u}_\infty$ and an arbitrarily large $f_\delta \in [0,1]$ for example $f_\delta = 0.99$. Empirical approximation methods for δ can be found in almost every general aerodynamics textbook, including [16].

The distance δ is called the *boundary layer thickness*, and separates the flow between the wall and 'infinitely-far-away' into two regions. The closer region is called the *boundary layer* wherein friction effects are definitely not negligible. The region outside of δ is called the external flow, and is where external flow assumptions such as external-irrotationality or external-inviscidness may or may not be reasonable.

This separation of our domain into a boundary layer and an external flow is relevant to the state variable vorticity because viscous shear stress is one of the creation mechanisms of vorticity. Remember that shear stress $\tau(y)$ is directly related to the shear gradient of the flow velocity:

$$\tau(y) = \mu \frac{\partial u}{\partial y}, \tag{4}$$

for $\hat{\boldsymbol{x}}$ parallel to a wall, $\hat{\boldsymbol{y}}$ normal to a wall pointing out, y the height above the wall along $\hat{\boldsymbol{y}}$, and u the $\hat{\boldsymbol{x}}$ component of $\boldsymbol{u}$.

Then, as noted by Prandtl[13], thin boundary layers require large shear stresses on the boundary layer fluid elements, to allow the flow to recover over the distance δ.

> "If, however, the viscosity is very slight and the path of the flow along the surface is not too long, then the velocity will have its normal value in immediate proximity to the surface. In the thin transition [boundary] layer, the great velocity differences will then produce noticeable effects in spite of the small viscosity constants." [13]

However, for fluids with low viscosity, the shear stress may be negligible within the external flow. In many engineering applications, the other mechanisms of vorticity creation may be similarly negligible in the external flow.

[8] Molecular arguments are rare in fluid dynamics as models deriving from the Navier-Stokes equations all inherently assume that fluids behave as continua.

[9] disregarding boundary layer suction or blowing

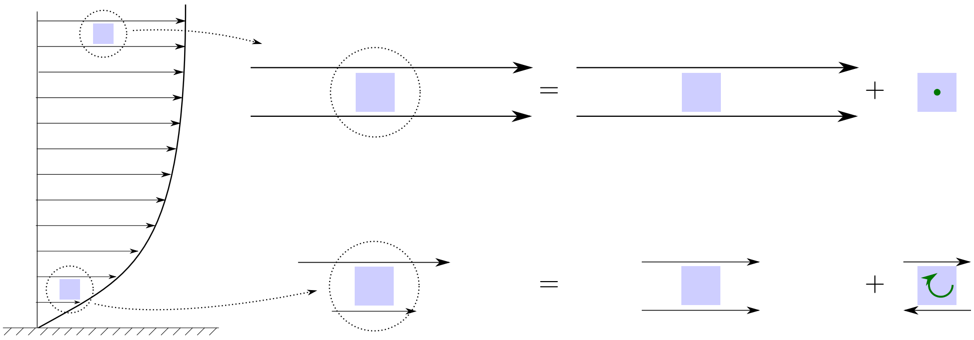

Figure 1: a sketch showing negligible vorticity in the external flow and non-zero vorticity in the boundary layer

Effectively, then, the separation of the domain into a potential external flow and a boundary layer may split the domain into a region with negligible viscosity and vorticity, and a region with significant viscosity and vorticity. This is *Prandtl's boundary layer theory*[10].

Prandtl's boundary layer is not immediately necessary to the next sections, but will show up again in section 11.

5 the vortex force, or why practical people are interested in vorticity

Vorticity is the curl of the velocity, and so must accelerate fluid elements perpendicular to the vorticity's vector. And, Newton's laws requires that an acceleration of fluid elements implies the existence of a force on the surrounding fluid due to the vorticity. This force must be balanced by an equal and opposite force from the surrounding fluid on the fluid element at the location of the vorticity.

The derivation and interpretations given in this section are inspired by section 2.3.7 of [5] and part B of [12].

Consider first, the differential form of the incompressible and invscid momentum conservation equation[11] for homogenous fluids, where the total pressure $p_t = p + \frac{1}{2}\rho\,||\boldsymbol{u}||_2^2$:

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = \rho \boldsymbol{k} - \nabla p_{\mathrm{t}} + \rho \boldsymbol{u} \times \boldsymbol{\omega} \tag{5}$$

Notice here that $\boldsymbol{k}$ has units of m/s^2, such that $\boldsymbol{f} = \rho \boldsymbol{k}$ is a force density.

Let's simplify this problem by assuming that our fluid element is in equilibrium. That is, the flow is steady such that $\frac{\partial \boldsymbol{u}}{\partial t}$ vanishes. Luckily, Bernoulli's equation ($p_{\mathrm{t}} =$ constant) is known to be valid for steady, incompressible, and inviscid flows[12] - and we have already made all of these assumptions previously.

Then, we can consider a type of 'free-body diagram' of the force densities now acting on the fluid element. Notice that there are only two remaining types of force densities. We have: a non-conservative force density $\boldsymbol{f}$ that we have to apply externally if we want to preserve equilibrium, and a remaining force density that must be due to

[10] A historical perspective on the boundary layer theory and its clear importance in fluid dynamics is given in [2].

[11] Since $\nabla(\boldsymbol{a}\cdot\boldsymbol{b}) = (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} + (\boldsymbol{a}\cdot\nabla)\boldsymbol{b} + \boldsymbol{a}\times(\nabla\times\boldsymbol{b}) + \boldsymbol{b}\times(\nabla\times\boldsymbol{a})$, and the total pressure $p_t = p + \frac{1}{2}\rho\,||\boldsymbol{u}||_2^2$, we can re-write the incompressible and inviscid momentum conservation equations as:

$$\begin{aligned}
\frac{D\boldsymbol{u}}{Dt} &= \boldsymbol{k} - \frac{\nabla p}{\rho} \\
\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} &= \rho\boldsymbol{k} - \nabla p \\
\rho\frac{\partial \boldsymbol{u}}{\partial t} &= \rho\boldsymbol{k} - \nabla p - \frac{1}{2}\rho\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) + \rho\boldsymbol{u}\times\boldsymbol{\omega} \\
&= \rho\boldsymbol{k} - \nabla\left(p + \frac{1}{2}\rho\,||\boldsymbol{u}||_2^2\right) + \rho\boldsymbol{u}\times\boldsymbol{\omega} \\
&= \rho\boldsymbol{k} - \nabla p_{\mathrm{t}} + \rho\boldsymbol{u}\times\boldsymbol{\omega}
\end{aligned}$$

[12] For confirmation that this does not also require irrotationality, see section 2.1, subsection 'Particular case of B1 and B2: homogeneous, incompressible, perfect fluid' of [5].

vorticity $\boldsymbol{f}_\text{v}$:

$$\boldsymbol{f}_\text{v} = \rho \boldsymbol{u} \times \boldsymbol{\omega}. \tag{6}$$

Though the naming may lead to confusion about the dimensions of the force density $\boldsymbol{f}_\text{v}$, this expression is known as the *vortex force*. We can think of the vortex force as being the (negative) pressure gradient imposed by the entire flow on the particular fluid element, due to the element's own vorticity.

You'll notice that, due to the cross product, the force density produced is perpendicular to the flow velocity at a given point. This means that the vortex force is an expression of the 'lift' on the fluid element, and does not include drag effects other than the induced drag that is purely a re-direction of the lift vector.

This is particularly useful to us, if we consider the vorticity of the flow directly along the surface of a solid body, the place where viscous shear-stresses in the boundary layer are most active at creating vorticity. The force exerted by the surrounding fluid on the surface-layer fluid elements would then be transfered to the body as lift.

That is, the total lift force acting on the body due to the surface-layer vorticity - also called the *bound vorticity* - can then be found as:

$$\boldsymbol{F} = \int_V \boldsymbol{f}_\text{v} \, dV = \rho \int_V \boldsymbol{u} \times \boldsymbol{\omega} \, dV, \tag{7}$$

provided that V is a control volume containing purely the body and the surface-layer fluid elements.

The take-away message here, is that the bound vorticity is responsible for the lift on the body. If the bound vorticity changes, so will the lift, and vice-versa.

6 the flow field, or why impractical people are interested in vorticity and dilation

These two state variables, dilation and vorticity, are relevant because they describe the nature of the flow, using Helmholtz decomposition.

The following description and interpretation of Helmholtz and Helmholtz-Hodge decomposition is heavily inspired by the review paper [4], in which can be found the proofs of the various theorems given in this section except for the multiply connected Helmholtz-Hodge decomposition.

Theorem 2 (Helmholtz Decomposition)**.** *Consider a flow field within an infinite domain $\boldsymbol{x} \in \mathbb{R}^3$, where the velocity decays to zero towards the edges of this infinite domain. Then, the velocity in the flow field $\boldsymbol{u}(\boldsymbol{x})$ is uniquely determined by the dilation of the fluid $\vartheta(\boldsymbol{x})$ and the vorticity of the fluid $\boldsymbol{\omega}(\boldsymbol{x})$.*

The Helmholtz decomposition takes some universe that only contains fluid. This fluid is generally still, but is moving in the interior of this universe. Then, by knowing the dilation and vorticity within this all-fluid universe, we have completely described the velocity of the fluid. This means that with some help from the Navier-Stokes relations and thermodynamic relations, the dilation and the vorticity uniquely contain all of the information needed to describe this all-fluid universe.

However, we're not only interested in all-fluid universes, because solid constructions such as humans cannot inhabit all-fluid universes. Luckily, there is an addendum to the Helmholtz decomposition that is valid in bounded domains.

Theorem 3 (Helmholtz Decomposition in a Bounded Domain)**.** *Consider a flow field within a simply-connected domain $\boldsymbol{x} \in \Omega \subset \mathbb{R}^3$ with a boundary $\partial\Omega$, whose inwards facing normal unit vector is called $\hat{\boldsymbol{n}}$. Then, the velocity in the flow field $\boldsymbol{u}(\boldsymbol{x})$ is uniquely determined by the dilation of the fluid $\vartheta(\boldsymbol{x})$, the vorticity of the fluid $\boldsymbol{\omega}(\boldsymbol{x})$, and the velocity normal to the boundary at the boundary $\hat{\boldsymbol{n}} \cdot \boldsymbol{v}(\boldsymbol{x}_\partial)$ for $\boldsymbol{x}_\partial \in \partial\Omega$.*

Now, we have some finite domain, which is simply connected but might wrap around a pocket containing a relevant geometry such as a lifting surface as in Figure 2. If we know the dilation and the vorticity within the domain, and the normal component of the velocity everywhere on the boundary of the domain, then we have uniquely described the velocity within the domain.

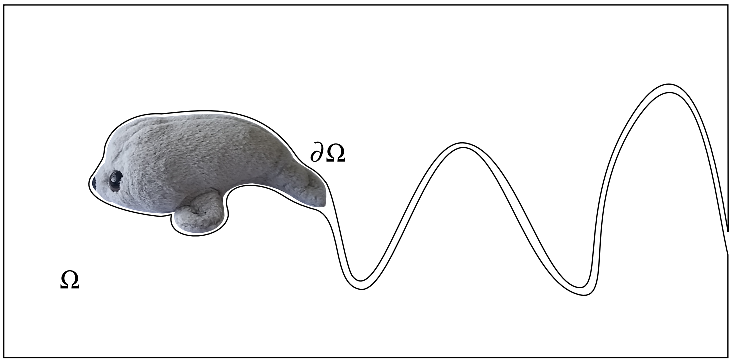

Figure 2: A simply-connected domain Ω around Fizzy, with its boundary $\partial\Omega$ where Helmholtz decomposition could be applied. The gaps between Fizzy and the bound boundary, as well as between the two sides of the shed boundary, have been exaggerated for visibility. In reality, these gaps should be infinitessimally small.

As mentioned, we need to know the normal velocity component along each of the three segments of $\partial\Omega$. First, the normal velocity component has to be known along the external boundary of the domain, which might be far enough upstream or downstream of the lifting surface to have recovered to the freestream. Second, the normal component must be known along the boundary of the pocket, which we can call the *bound boundary* for consistency with future notation. If the bound boundary is wrapped around some solid object without suction or blowing, then the *no-normal condition* will hold along the bound boundary so that the normal component of the velocity is zero.

There is a third section of $\partial\Omega$ where the normal velocity component must be known: the portion of the boundary behind the bound boundary where the simply connected domain abuts itself. Again, for reasons that will later become apparent, we will call this portion of the boundary the *shed boundary*. The placement of the shed boundary is an engineering decision, as there are an infinite number of possible locations.

A natural criteria for the selection of this location is that the normal velocity component should be zero. If there is no normal component to the velocity, then the velocity must be parallel to the boundary, implying that the shed boundary lays on a stream-surface of the flow. It then becomes apparent that the bound boundary is not so much determined by the surface of the object within the pocket, as by the stream-surface around the object. Then, the shed boundary stream-surface is the continuation of the bound boundary stream-surface. Notice that this implies that the connection point between the bound boundary and the shed boundary is not inherently located at the trailing edge, but wherever the suction-surface stream-surface first meets the pressure-surface stream-surface.

If we are interested in the case of multiple airfoils within our modelled universe, as is obviously the case for a multiple-kite system, then we need to be sure that this decomposition will also work for multiply-connected domains. That Helmholtz decomposition does hold for multiply-connected domains is shown in [8], with a theorem that is summarized as follows.

Theorem 4 (Helmholtz Decomposition for Multiply Connected Domains)**.** *Consider a continuous flow field within a multiply-connected domain $\boldsymbol{x} \in \Omega \subset \mathbb{R}^3$ with a boundary $\partial\Omega$ that is piecewise smooth. Then, Ω can be divided into a finite number of simply-connected domains Ω_i, each with their own flow fields. Then, the velocity in each flow field $\boldsymbol{u}_((\boldsymbol{x}_i)$ is uniquely determined by the dilation of the fluid $\vartheta(\boldsymbol{x}_i)$, the vorticity of the fluid $\boldsymbol{\omega}(\boldsymbol{x}_i)$, and the velocity normal to the boundary at the various boundary $\hat{\boldsymbol{n}} \cdot \boldsymbol{v}(\boldsymbol{x}_{\partial,i})$ for $\boldsymbol{x}_{\partial,i} \in \partial\Omega_i$. Together, the flow fields of the various sub-domains will give the full flow field.*

To this point, we have described the various Helmholtz decompositions as relevant purely to aerodynamics. But, it is important to realize that these decompositions are not possible because of some magical property of fluid flows, but are true of all vector fields. This is stated below.

Theorem 5 (Helmholtz Hodge Decomposition (HHD))**.** *Consider a vector field $\boldsymbol{\xi}(\boldsymbol{x})$ within a simply-connected domain $\boldsymbol{x} \in \Omega$ with a boundary $\partial\Omega$, whose inwards facing normal unit vector is called $\hat{\boldsymbol{n}}$. The divergence $\nabla \cdot \boldsymbol{\xi}$ and curl $\nabla \times \boldsymbol{\xi}$ of the vector field are known, as is either the normal component along the boundary $\xi_n(\boldsymbol{x}_\partial) = \hat{\boldsymbol{n}} \cdot \boldsymbol{\xi}(\boldsymbol{x}_\partial)$ or the tangential compontent along the boundary $\xi_t(\boldsymbol{x}_\partial) = \hat{\boldsymbol{n}} \times \boldsymbol{\xi}(\boldsymbol{x}_\partial)$ for $\boldsymbol{x}_\partial \in \partial\Omega$. Then, the vector field is uniquely determined as the sum of irrotational vector field $\boldsymbol{\Phi}$, ie. $\nabla \times \boldsymbol{\Phi} = \boldsymbol{0}$, and an incompressible vector field $\boldsymbol{\Psi}$, ie. $\nabla \cdot \boldsymbol{\Psi} = 0$:*

$$\boldsymbol{\xi} = \boldsymbol{\Phi} + \boldsymbol{\Psi}.$$

Further, the irrotational vector field is the gradient of a scalar potential ϕ such that $\mathbf{\Phi} = \nabla\phi$, and the incompressible vector field is the curl of a vector potential $\boldsymbol{\psi}$ such that $\Psi = \nabla \times \boldsymbol{\psi}$.

We can decompose any "sufficiently smooth" vector field into a curl-free and a divergence-free part. If we assume that the velocity field $\boldsymbol{u}$ is sufficiently smooth, then this Helmholtz decomposition defines a *scalar potential* ϕ and a *vector potential* $\boldsymbol{\psi}$, such that:

$$\boldsymbol{u} = \nabla\phi + \nabla \times \boldsymbol{\psi} \tag{8}$$

The velocity at a given point has three degrees of freedom, and the scalar potential has one degree of freedom, so we typically also remove one degree of freedom from the vector potential. This additional condition is called the *gauge condition*, where $\nabla \cdot \boldsymbol{\psi} = 0$.

We can take the divergence and the curl, separately, of (8), to get:

$$\nabla \cdot \boldsymbol{u} = \nabla^2\phi = \vartheta \tag{9}$$
$$\nabla \times \boldsymbol{u} = \nabla \times \nabla \times \boldsymbol{\psi} = \nabla(\nabla \cdot \boldsymbol{\psi}) - \nabla^2\boldsymbol{\psi} = -\nabla^2\boldsymbol{\psi} = \boldsymbol{\omega} \tag{10}$$

Based on Helmholtz-Hodge decomposition, these combination of these equations - the Poisson equation on dilation (10), the Poisson equation on vorticity (10), the gauge condition, and the known normal velocity at the boundary of the domain - will uniquely describe the flow field within the domain Ω:

$$\nabla^2\phi\,(\boldsymbol{x}) = \vartheta \qquad \text{for } \boldsymbol{x} \in \Omega, \tag{11a}$$
$$-\nabla^2\boldsymbol{\psi}\,(\boldsymbol{x}) = \boldsymbol{\omega} \qquad \text{for } \boldsymbol{x} \in \Omega, \tag{11b}$$
$$\nabla \cdot \boldsymbol{\psi}\,(\boldsymbol{x}) = 0 \qquad \text{for } \boldsymbol{x} \in \Omega, \tag{11c}$$
$$\hat{\boldsymbol{n}} \cdot (\nabla\phi(\boldsymbol{x}) + \nabla \times \boldsymbol{\psi}(\boldsymbol{x})) = u_{\mathrm{n}}(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \partial\Omega, \tag{11d}$$

7 assuming incompressibility gives a Poisson equation for vorticity

If we assume that the flow is incompressible ($\vartheta = 0$), then the scalar potential ϕ will disappear from (11) to give the form of:

$$-\nabla^2\boldsymbol{\psi}\,(\boldsymbol{x}) = \boldsymbol{\omega} \qquad \text{for } \boldsymbol{x} \in \Omega, \tag{12a}$$
$$\hat{\boldsymbol{n}} \cdot (\boldsymbol{u}_{\mathrm{sol}}(\boldsymbol{x}) + \nabla \times \boldsymbol{\psi}(\boldsymbol{x})) = u_{\mathrm{n}}(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \partial\Omega, \tag{12b}$$

provided that $\boldsymbol{u}_{\mathrm{sol}}$ is some solenoidal velocity field. This solenoidal velocity field is typically used to satisfy the boundary conditions on the external boundaries of the domain, which - for example - may be far enough away from the obstacle in the flow that uniform freestream flow conditions apply.

The induced velocity has a singularity when the Biot-Savart expression is evaluated on a point within the domain ($\boldsymbol{x} = \boldsymbol{x}'$). This is especially problematic in numerical approximations of the integral. There's a discussion of how to regularize the Biot-Savart expression both before and after integration in sections 41.8 of [5].

There are a number of strategies for solving the Poisson equation for vorticity (12a) with its boundary conditions (12b). The first option is to solve the problem directly, typically with a Fast Poisson solver[13] The second option, is to recognize that ∇^2 is a linear operator. This means that elementary solutions to (12a) can be stacked in order to match the boundary conditions (12b), using a principle called *superposition*. Superposition is incredibly appealing if the appropriate elementary solutions are known, as the procedure leads to highly parallelizable solutions. Some relevant elementary solutions are described in section 13.

8 the Biot-Savart integral, or how to solve the Poisson equation

The vector potential can be found by solving the Poisson equation (12a) with the boundary conditions (12b). For an evaluation point $\boldsymbol{x} \in \mathbb{R}^n$ and the location of a source $\boldsymbol{x}' \in \mathbb{R}^n$, it is known that a Poisson partial differential

[13] See [11].

equation of form $\nabla^2 F(\boldsymbol{x}) = f(\boldsymbol{x})$, will have the solution:

$$F(\boldsymbol{x}) = \int G(\boldsymbol{x},\boldsymbol{x}')f(\boldsymbol{x}')d^3\boldsymbol{x}', \tag{13}$$

$$G(\boldsymbol{x},\boldsymbol{x}') = \begin{cases} -\frac{1}{2\pi}\log||\boldsymbol{x}-\boldsymbol{x}'||_2, & \text{for } \boldsymbol{x}\in\mathbb{R}^2, \\ -\frac{1}{4\pi}\frac{1}{||\boldsymbol{x}-\boldsymbol{x}'||_2}, & \text{for } \boldsymbol{x}\in\mathbb{R}^3. \end{cases} \tag{14}$$

Here, $G(\boldsymbol{x},\boldsymbol{x}')$ is the Green's function that solves the partial differential equation $\nabla^2 G(\boldsymbol{x},\boldsymbol{x}') = \delta(\boldsymbol{x}-\boldsymbol{x}')$. [21]

For the domain Ω, this gives a solution to the vector potential Poisson equation as:

$$\boldsymbol{\psi}(\boldsymbol{x}) = \frac{1}{4\pi}\int_\Omega \frac{\boldsymbol{\omega}(\boldsymbol{x}')}{||\boldsymbol{x}-\boldsymbol{x}'||_2} d^3\boldsymbol{x}' \tag{15}$$

In order to evaluate the Neumann boundary conditions (12b), we can take the curl of (15):

$$\boldsymbol{u}_{\mathrm{i}}(\boldsymbol{x}) = \nabla\times\boldsymbol{\psi}(\boldsymbol{x}) = \frac{1}{4\pi}\int_\Omega \nabla\times\frac{\boldsymbol{\omega}(\boldsymbol{x}')}{||\boldsymbol{x}-\boldsymbol{x}'||_2}\, d^3\boldsymbol{x}', \tag{16}$$

where $\boldsymbol{u}_{\mathrm{i}}$ is the *induced velocity*, the portion of the velocity field determined by the vorticity distrbution.

With some vector identies - $\boldsymbol{A}\times\boldsymbol{A}\times\boldsymbol{A} = \boldsymbol{0}$ and $\nabla\times(a\boldsymbol{b}) = a(\nabla\times\boldsymbol{b}) + (\nabla a)\times\boldsymbol{b}$ - and the recollection that the vorticity is the curl of the velocity, the curl of the vector potential, (16) can be simplified into the classical 3D Biot-Savart relationship:

$$\boldsymbol{u}_{\mathrm{i}}(\boldsymbol{x}) = -\frac{1}{4\pi}\int_\Omega \frac{(\boldsymbol{x}-\boldsymbol{x}')}{||\boldsymbol{x}-\boldsymbol{x}'||_2^3}\times\boldsymbol{\omega}(\boldsymbol{x}')\, d^3\boldsymbol{x}'. \tag{17}$$

Notice that the influence of the vorticity decreases asymptotically with radius in this solution - the requirement previously specified for the scalar potential $\phi(\boldsymbol{x})$ to be constant.

It remains to specify the boundary conditions that will determine $\boldsymbol{\omega}(\boldsymbol{x})$ for $\boldsymbol{x}\in\Omega$, applying (17) into (12b):

$$\hat{\boldsymbol{n}}(\boldsymbol{x})\cdot\left(\boldsymbol{u}_{\mathrm{sol}}(\boldsymbol{x}) - \frac{1}{4\pi}\int_\Omega \frac{(\boldsymbol{x}-\boldsymbol{x}')}{||\boldsymbol{x}-\boldsymbol{x}'||_2^3}\times\boldsymbol{\omega}(\boldsymbol{x}')\, d^3\boldsymbol{x}', \quad \text{for } \boldsymbol{x}\in\partial\Omega\right) = \boldsymbol{u}_{\mathrm{n}}(\boldsymbol{x}) \tag{18}$$

If you remember that the time evolution of $\boldsymbol{\omega}$ must satisfy (1), we see that solving (18) is not a trivial challenge. Luckily, there are some cases that help us to simplify the problem.

9 physical considerations that simplify the Biot-Savart integral, or the classical vorticity theorems

9.1 vorticity at separation: the Kutta condition

The first way that we might simplify the Biot-Savart expression concerns locations where the vorticity is known to be zero. This is true wherever the flow separates from a body, as is explained in this section.

At a separation point the boundary layer separates from the body. This occurs because the pressure gradient beyond the separation point is strong enough to force the flow to travel in the 'upstream' direction. What we see, then, is that flow before the separation point is moving 'downstream' close to the wall, where flow after the separation point is moving 'upstream' close to the wall.[14] Such a switch in flow direction close to the wall requires that there be a point where the wall-parallel velocity asymptotically approaches zero at the wall. By squeezing the 'upstream' and 'downstream' points closer together around the separation point, we see that this point of stationary near-wall, wall-parallel velocity must occur at the separation point itself.

[14] More specifically, this is true on average. Separated flow are characterized by strong turbulence, whose eddies make it impossible to say that any given flow element must necessarily be moving in any particular direction.

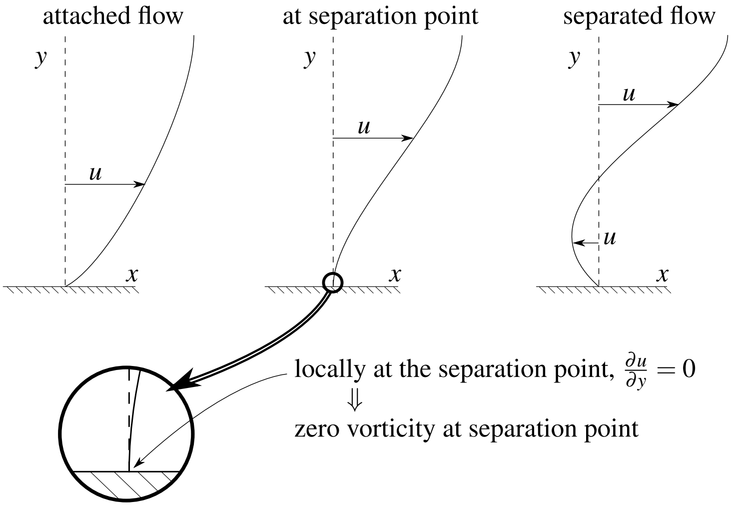

Figure 3: A sketch of wall-parallel flow velocity in the boundary layer at separation, as explanation of the Kutta condition. Here, u is the wall-parallel $\hat{\boldsymbol{x}}$ component of the velocity and v is the wall-normal $\hat{\boldsymbol{y}}$ component, such that $\boldsymbol{u} = u\,\hat{\boldsymbol{x}} + v\,\hat{\boldsymbol{y}}$.

Consequently, at the separation point:

$$\left.\frac{\partial u}{\partial y}\right|_{x_{\text{sep}}} = 0. \tag{19}$$

Among other practical effects, this means a solid boundary experiences no shear-stress at the separation point.

On another note, provided that the wall is impermeable, we see that the no-through condition requires that the wall-normal velocity be everywhere zero along the wall. As the separation point is on the wall, we know that :

$$\left.\frac{\partial u}{\partial y}\right|_{x_{\text{sep}}} = 0. \tag{20}$$

From these two statements, it must be true that the curl of the velocity is zero at the separation point. That is:

$$\boldsymbol{\omega}|_{x_{\text{sep}}} = \boldsymbol{0}. \tag{21}$$

This argument (sketched in Figure 3) can be extended to three dimensions when following a separation line.

The physical requirement that smooth separation implies zero vorticity at the separation location is called the *Kutta Condition*. You can convince yourself that fluids do - in fact - separate smoothly by admiring the beautiful photos of [17].

9.2 vortex transport rules: Helmholtz's first vorticity theorem

Another consideration to help simplify the problem above is to assume that the structures containing vorticity - the vortices defined nebulously in section 2 - convect with the flow. This is Helmholtz's first vorticity theorem, with a proof given in section 2.6.3 of [5].

Theorem 6 (Helmholtz's First Vorticity Theorem)**.** *For an inviscid and incompressible fluid experiencing only conservative body forces, vortices are material structures that convect with the flow.*

That is, if the location of a freely-convecting vortex is $\boldsymbol{x}_{\text{v}}(t)$, then we can express an ordinary differential equation describing the evolution of $\boldsymbol{x}_{\text{v}}$ in time:

$$\dot{\boldsymbol{x}}_{\text{v}}(t) = \boldsymbol{u}_{\text{sol}}(t) + \boldsymbol{u}_{\text{i}}(\boldsymbol{x}_{\text{v}}, t), \tag{22}$$

where $\boldsymbol{u}_{\text{i}}$ is the flow field determined by the Biot-Savart integral (17). This may not seem like much of a simplification, but it gives a necessary translation between the vorticity's material derivative expressed by (1) in terms of a vortex's location.

9.3 vorticity evolution in externally incompressible and inviscid flow with conservative forces: Kelvin's circulation theorem

The third consideration that may be useful concerns the flux of vorticity through a surface, such as a cross-section of Ω.

This flux of the vorticity - called circulation Γ - through a surface S bounded by the closed contour ∂S, is defined as:

$$\Gamma = \int_S \boldsymbol{\omega} \cdot \hat{\boldsymbol{n}}\, dS = \oint_{\partial S} \boldsymbol{u} \cdot d\boldsymbol{s}. \tag{23}$$

From this definition of circulation, we get[15] Kelvin's circulation theorem.

Theorem 7 (Kelvin's circulation theorem)**.** *For an inviscid and incompressible flow with conservative body forces, the material derivative of the circulation around a fluid curve is zero:*

$$\frac{D\Gamma}{Dt} = 0.$$

Kelvin's circulation theorem says that depending on the properties of the external flow, circulation is a conserved quantity from a Langrangian standpoint.

If we can find some domain containing an obstacle in an externally-potential flow, through which the vorticity flux is zero at some point in time, then, total circulation over that surface will remain zero as its boundary deforms with the flow. That is, if there ever was a previous time where the fluid within Ω was at rest, and therefore was vorticity-free with zero circulation around $\partial\Omega$, then the total circulation around the boundary must always be zero.

10 Kutta, Helmholtz and Kelvin walk into a bar, or vortex shedding from separation locations

In Sections 5 and 9.3 we've made two statements that would appear to contradict each other. That is:

[15] Kelvin's circulation theorem is derived here with the parameteric derivation given in section 4.9.1 of [6].

First, let's parameterize the closed space-curve $\partial S = \{\boldsymbol{s}(t) \mid t \in [0,1]\}$. Since ∂S is closed, we know that the boundaries $\boldsymbol{s}(0) = \boldsymbol{s}(1)$ and $\boldsymbol{u}(\boldsymbol{s}(0)) = \boldsymbol{u}(\boldsymbol{s}(1))$ are periodic.

This re-phrases the line integral of the circulation as:

$$\Gamma = \int_0^1 \boldsymbol{u} \cdot \frac{\partial \boldsymbol{s}}{\partial t} dt.$$

Then, we can take the time derivative of the circulation as:

$$\frac{D\Gamma}{Dt} = \int_0^1 \frac{D\boldsymbol{u}}{Dt} \cdot \frac{\partial \boldsymbol{s}}{\partial t} dt + \int_0^1 \boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial t} dt = \int_0^1 \frac{D\boldsymbol{u}}{Dt} \cdot \frac{\partial \boldsymbol{s}}{\partial t} dt + \frac{1}{2}\left[\boldsymbol{u} \cdot \boldsymbol{u}\right]_0^1.$$

Notice that the periodic conditions of $\boldsymbol{u}$ will make the second term vanish.

We can get the material derivative of the velocity from the incompressible and inviscid momentum equation:

$$\frac{D\boldsymbol{u}}{Dt} = \boldsymbol{k} - \frac{\nabla p}{\rho} = -\nabla K - \frac{\nabla p}{\rho},$$

since $\boldsymbol{k}$ has been specified to be a conservative force density, and is therefore the gradient of a potential K.

Applying the material derivative of the velocity back into the circulation derivative, we find:

$$\frac{D\Gamma}{Dt} = -\int_0^1 \left(\nabla K + \frac{\nabla p}{\rho}\right) \cdot \frac{\partial \boldsymbol{s}}{\partial t} dt$$

At this point, we remember the gradient theorem which tells us that:

$$f(\boldsymbol{s}(\alpha)) - f(\boldsymbol{s}(\beta)) = \int_\beta^\alpha \nabla f(\boldsymbol{s}(t)) \cdot \frac{\partial \boldsymbol{s}}{\partial t} dt$$

Consequently, we can apply periodicity to say that:

$$\frac{D\Gamma}{Dt} = -\left[K + \frac{p}{\rho}\right]_0^1 = 0.$$

- the lift on an airfoil - which is definitely not constant - is determined by the bound vorticity within $V \subset \Omega$, and
- that the flux of vorticity through any cross-section[16] of our full domain Ω is conserved over time for inviscid and incompressible flows.

How can we reconcile that the vorticity within a portion of Ω varies while there is a constant flux of vorticity through a cross-section of the entire domain?

The change in bound vorticity of V must be compensated by an equal-and-opposite change in the vorticity of $\Omega \setminus V$.

Further, this compensatory vorticity - called *shed vorticity* - that is created within $\Omega \setminus V$ should be created so that Kutta's condition from 9.1 is satisfied. The logical idea, then, would be to introduce the shed vorticity into $\Omega \setminus V$ from the separation locations[17] such that the vorticity at the separation locations always sums to $\mathbf{0}$.

After the shed vorticity has been released, it can be convected downstream by the flow as required by Helmholtz's first theorem.

In this way, the shed vorticity enforces conservation of angular momentum in the flow, even as the lift on the airfoil changes.

11 the lucky case of an externally-potential flow with a thin boundary layer

If we are particularly lucky, we will have both an externally-potential flow and a thin boundary layer.

The externally-potential flow assumption argues that the domain Ω can be divided into two parts: Ω_{BL} and Ω_{EF}, and that the contribution to the Biot-Savart integral of the entire external flow domain Ω_{EF} is negligible. This is shown in Figure 4, and gives a reduction in the relevant integration domain of the Biot-Savart integral.That is:

$$\boldsymbol{u}_{\mathrm{i}}(\boldsymbol{x}) = -\frac{1}{4\pi}\int_{\Omega} \frac{(\boldsymbol{x}-\boldsymbol{x}')}{||\boldsymbol{x}-\boldsymbol{x}'||_2^3} \times \boldsymbol{\omega}(\boldsymbol{x}')\, d^3\boldsymbol{x}' = -\frac{1}{4\pi}\int_{\Omega_{\mathrm{BL}}} \frac{(\boldsymbol{x}-\boldsymbol{x}')}{||\boldsymbol{x}-\boldsymbol{x}'||_2^3} \times \boldsymbol{\omega}(\boldsymbol{x}')\, d^3\boldsymbol{x}'. \tag{24}$$

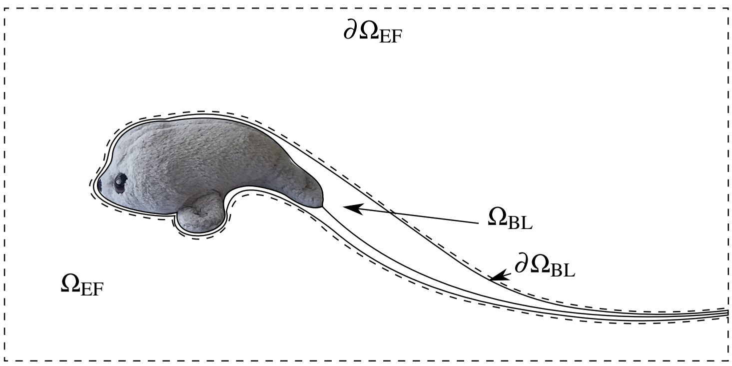

Figure 4: Separating the domain Ω around Fizzy into an external flow domain Ω_{EF} (dashed) which does not contribute to the Biot-Savart integral, and a boundary layer Ω_{BL} (solid) which does contribute to the integral.

If the boundary layer is thin, then the domain Ω_{BL} will be much smaller than Ω. Frequently, this thinness can be exaggerated such that $\Omega_{\mathrm{BL}} \approx \partial\Omega_{\mathrm{BL}}$, implying a decrease in the dimensionality of the required integration.

That is:

$$\boldsymbol{u}_{\mathrm{i}}(\boldsymbol{x}) = -\frac{1}{4\pi}\int_{\partial\Omega_{\mathrm{BL}}} \frac{(\boldsymbol{x}-\boldsymbol{x}')}{||\boldsymbol{x}-\boldsymbol{x}'||_2^3} \times \boldsymbol{\omega}(\boldsymbol{x}')\, d^2\boldsymbol{x}'. \tag{25}$$

[16] deforming with the fluid

[17] You can convince yourself that the shedding of vorticity from separation locations is physical, and not just convenient mathematical trickery, by looking at the experimental photos published - for example - at the back of [16], or by dragging a spoon through a pot of broth.

Typically, what we mean by 'thin' requires that the flow remain fully attached[18] to the body until it separates at the trailing edge. An example of the various domains and boundaries in the case of such a thin boundary layer, is shown in Figure 5.

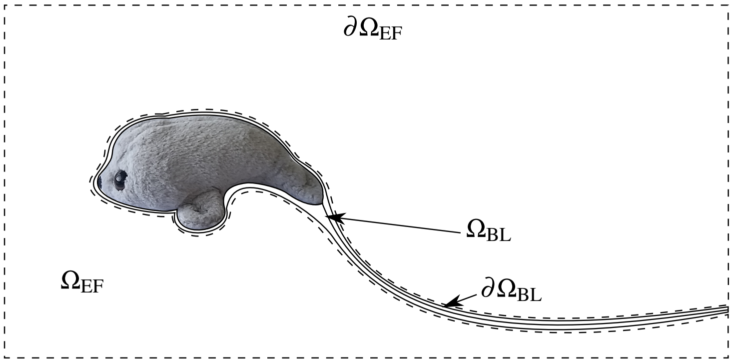

Figure 5: A depiction of the lucky case with an external potential flow in Ω_{EF} and boundary $\partial\Omega_{\mathrm{EF}}$ (dashed) - and a thin boundary layer in Ω_{BL} with boundary $\partial\Omega_{\mathrm{BL}}$ (solid), around Fizzy.

Such a combination of externally-potential flow and a thin boundary layer implies a significant savings of computational effort when solving the flow boundary problem (18).

12 general procedure of a Biot-Savart superposition solution, or how typical vortex methods work

As previously mentioned, one way to solve (18) is with the superposition of many elementary solutions to the (17) until the boundary conditions are satisfied. The elementary solution vortices must be chosen to match the expected geometry of the vorticity distribution within the domain.

The position of the vortices at any point in time is determined by solving an initial value problem, by integrating the derivative (22) from the v^{th} vortex's known location at some time t_v.

The vorticity distribution within the v^{th} vortex has a specific number of degrees of freedom depending on the type of elementary solution vortex. For vortices with constant vorticity distributions, there is only one degree of freedom that can be used to describe the vorticity distribution: a vortex strength γ_v. If the flow is externally-inviscid, γ_v remains constant over time; otherwise, the vortex strength will decrease[19] in time as viscous dissipation robs the vortex of energy .

In the case of an elementary solution with uniform vorticity distribution and externally potential flow, the v^{th} vortex is defined by four values: the three position coordinates $\boldsymbol{x}_v$ and the vortex strength γ_v defined at some time t_v. We need to determine these values for each vortex $v \in \mathscr{V}$ such that the boundary conditions (no-normal flow through the bound and shed boundaries) are satisfied.

This arranging problem is typically simplified by placing some of the vortices ($v \in \mathscr{B} \subset \mathscr{V}$) at the very inside of the solid object's boundary layer where the flow relative to the body is zero. These *bound vortices* are discretizations of the bound vorticity and have a simplified velocity ODE:

$$\dot{\boldsymbol{x}}_v(t) = \frac{\partial}{\partial t}\left(\partial\Omega(\boldsymbol{x}_v, t)\right) \tag{26}$$

Some subset (possibly the full subset) of the remaining vortices ($v \in \mathscr{S} \subset \mathscr{V}$ such that $\mathscr{S} \cap \mathscr{B} = \emptyset$) have their positions at time t_v chosen to ensure that the Kutta condition (21) is satisfied at all separation locations. The way this is accomplished, practically speaking, is to place the v^{th} vortex at the n^{th} separation location at time t_v: $\boldsymbol{x}_v(t_v) = \boldsymbol{x}_{\mathrm{sep},n}(t_v)$. This is done so that all $n \in \mathscr{N}$ separation locations are covered.

[18] In yet another example of potentially-confusing naming conventions, 'fully attached' means one flow attachement point, typically near the leading edge of an airfoil, and one flow separation point, almost always at the trailing edge of an airfoil.

[19] Numerically, this should satisfy whatever simplified version of (1) is desired.

The no-normal condition is satisfied over the shed boundary as a natural result of allowing the vortices of $\mathscr{S}$ to convect freely. That is, a converged solution will move the *shed vortices* with the flow, such that there cannot be any velocity difference between the the shed vortices and the immediately-surrounding fluid. The converged locations of the shed vortices will then end up marking the position of the shed boundary.

With a finite number of elementary vortices, we cannot satisfy the infinite boundary conditions given by a continuous boundary in a continuous time interval. As such, we chose some finite set of evaluation times, as well as positions along the bound boundary to place bound vortices. In the end, the number of unknowns for all $|\mathscr{V}|$ vortices should be equal to the number of boundary conditions enforceable - counting both the no-normal conditions as well as the Kutta conditions. This description gives the general procedure for many members of the family of vortex methods.

One common member of the vortex method family is the vortex lattice method, in which quadrilateral vortex rings of constant vortex strength are tiled over a wing surface as well as the infinitely-thin shed-boundary that describes the flow shed from the wing's trailing edge. A sketch of such a vortex placement is given in Figure 6.

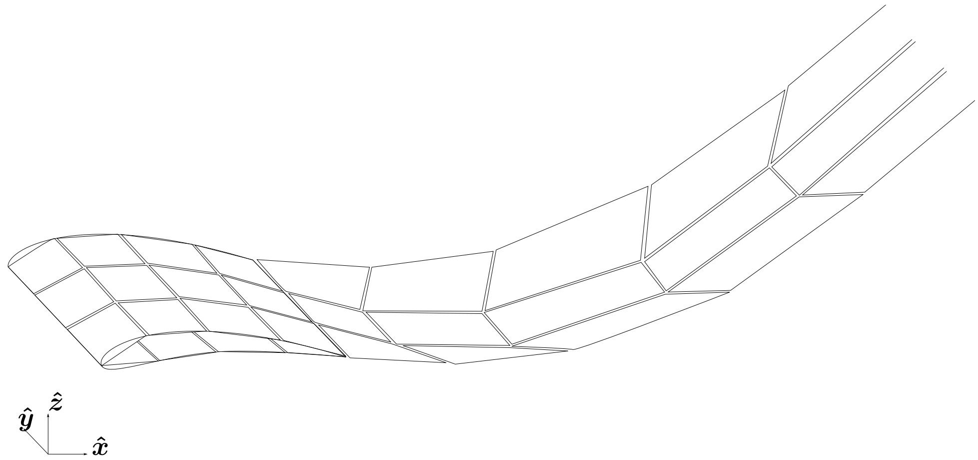

Figure 6: A sketch of a vortex lattice for a finite length wing. The angle of attack is exaggerated for visibility.

13 elementary solutions to the Biot-Savart integral

Due to the linearity of the ∇^2 operator, known solutions to the Poisson equation relating the vector potential and the vorticity (12a) can be superposed to find the solution which will satisfy the boundary conditions (12b) at any point in time. As a result, elementary solutions to the Biot-Savart integral (17) are very useful.

These elementary solutions are - by virtue of implying a concentration of vorticity - vortices. It is important to remember however that not all vortices are elementary solutions. Vortices are only elementary solutions if it is possible to analytically integrate the Biot-Savart integral over their geometry and vorticity distribution.

In the following section, some relevant elementary solutions are presented.

13.1 line-segment (filament) of constant vorticity distribution

In the case of a line-segment of constant vorticity distribution, the vortex has the parametric curve:

$$\boldsymbol{x}' = \boldsymbol{x}_0' + t(\boldsymbol{x}_1' - \boldsymbol{x}_0'), \quad \text{for } t \in [0,1], \tag{27}$$

such that the vorticity within the domain Ω will only be non-zero over the line-segment $\boldsymbol{x}'$.

Then, the vorticity field on that domain of the segment is parallel to the segment, with a magnitude ω. From the vorticity carried within the vortex and the segment length of the vortex, a combined *vortex strength* γ can be defined [20]

[20] Recall that we defined a quantity called circulation $\Gamma = \int_S \boldsymbol{\omega} \cdot \hat{\boldsymbol{n}}\, dS$ that is conserved in externally-inviscid and incompressible flows with conservative body forces.

$$\boldsymbol{\omega}(t) = \omega \frac{d\boldsymbol{x}'}{dt} \Big/ \left|\left|\frac{d\boldsymbol{x}'}{dt}\right|\right|_2 = \gamma \frac{d\boldsymbol{x}'}{dt} = \gamma(\boldsymbol{x}'_1 - \boldsymbol{x}'_0) \tag{28}$$

Notice that $\boldsymbol{\omega}$ is - by the definition of this scenario - constant in t. So that the cross product can be moved outside of the integral.

When (27) is substituted in the Biot-Savart integral (17), it gives the following integral:

$$\begin{aligned}
\boldsymbol{u}_{\mathrm{i}}(\boldsymbol{x}) &= -\frac{\gamma}{4\pi}\left(\int_0^1 \frac{(\boldsymbol{x}-\boldsymbol{x}'(t))}{||\boldsymbol{x}-\boldsymbol{x}'(t)||_2^3}\, dt\right) \times (\boldsymbol{x}'_1 - \boldsymbol{x}'_0) && (29)\\
&= -\frac{\gamma}{4\pi}\left((\boldsymbol{x}-\boldsymbol{x}'_0)\int_0^1 \frac{1}{||\boldsymbol{x}-\boldsymbol{x}'(t)||_2^3}\, dt - (\boldsymbol{x}'_1-\boldsymbol{x}'_0)\int_0^1 \frac{t}{||\boldsymbol{x}-\boldsymbol{x}'(t)||_2^3}\, dt\right) \times (\boldsymbol{x}'_1 - \boldsymbol{x}'_0) && (30)\\
&= -\frac{\gamma}{4\pi}\left(\int_0^1 \frac{1}{||\boldsymbol{x}-\boldsymbol{x}'(t)||_2^3}\, dt\right)\big((\boldsymbol{x}-\boldsymbol{x}'_0) \times (\boldsymbol{x}'_1 - \boldsymbol{x}'_0)\big) && (31)
\end{aligned}$$

Notice that as the cross product is distributive and $(\boldsymbol{a}-\boldsymbol{b})\times(\boldsymbol{a}-\boldsymbol{b}) = 0$, the cross product of the first and second legs of a triangle will be equivalent to the cross product of the first and third legs. That is, $(\boldsymbol{a}-\boldsymbol{b})\times(\boldsymbol{c}-\boldsymbol{b}) = (\boldsymbol{a}-\boldsymbol{b})\times(\boldsymbol{c}-\boldsymbol{a})$. Then, the above cross-product can be written in a more symmetrical fashion:

$$\boldsymbol{u}_{\mathrm{i}}(\boldsymbol{x}) = \frac{\gamma}{4\pi}\left(\int_0^1 \frac{1}{||\boldsymbol{x}-\boldsymbol{x}'(t)||_2^3} dt\right)\big((\boldsymbol{x}-\boldsymbol{x}'_0)\times(\boldsymbol{x}-\boldsymbol{x}'_1)\big). \tag{32}$$

The integral above can be evaluated as:

$$\begin{aligned}
\int_0^1 \frac{1}{||\boldsymbol{x}-\boldsymbol{x}'(t)||_2^3} dt = &\left[\frac{(\boldsymbol{x}'_0-\boldsymbol{x}'_1)\cdot(\boldsymbol{x}-\boldsymbol{x}'_0) + \big|\big|\boldsymbol{x}'_0-\boldsymbol{x}'_1\big|\big|_2^2\, t}{\left(\big|\big|\boldsymbol{x}-\boldsymbol{x}'_0\big|\big|_2^2 + 2\left((\boldsymbol{x}-\boldsymbol{x}'_1)\cdot(\boldsymbol{x}-\boldsymbol{x}'_0) - \big|\big|\boldsymbol{x}-\boldsymbol{x}'_0\big|\big|_2^2\right)\, t + \big|\big|\boldsymbol{x}'_0-\boldsymbol{x}'_1\big|\big|_2^2\, t^2\right)^{\frac{1}{2}}} \cdots \right.\\
&\left.\frac{1}{\big|\big|(\boldsymbol{x}-\boldsymbol{x}'_0)\times(\boldsymbol{x}-\boldsymbol{x}'_1)\big)\big|\big|_2^2}\right]_{t=0}^{t=1} \qquad (33)
\end{aligned}$$

When evaluated and separated, this gives:

$$\begin{aligned}
\int_0^1 \frac{1}{||\boldsymbol{x}-\boldsymbol{x}'(t)||_2^3} dt &= \frac{(\boldsymbol{x}'_1-\boldsymbol{x}'_0)\cdot\left(\frac{\boldsymbol{x}-\boldsymbol{x}'_0}{||\boldsymbol{x}-\boldsymbol{x}'_0||_2} - \frac{\boldsymbol{x}-\boldsymbol{x}'_1}{||\boldsymbol{x}-\boldsymbol{x}'_1||_2}\right)}{\big|\big|(\boldsymbol{x}-\boldsymbol{x}'_0)\times(\boldsymbol{x}-\boldsymbol{x}'_1)\big)\big|\big|_2^2} && (34)\\
&= \frac{\left(\big|\big|\boldsymbol{x}-\boldsymbol{x}'_0\big|\big|_2 + ||\boldsymbol{x}-\boldsymbol{x}'_1||_2\right)}{\big|\big|\boldsymbol{x}-\boldsymbol{x}'_0\big|\big|_2\,\big|\big|\boldsymbol{x}-\boldsymbol{x}'_1\big|\big|_2\left(\big|\big|\boldsymbol{x}-\boldsymbol{x}'_0\big|\big|_2\,\big|\big|\boldsymbol{x}-\boldsymbol{x}'_1\big|\big|_2 + (\boldsymbol{x}-\boldsymbol{x}'_0)\cdot(\boldsymbol{x}-\boldsymbol{x}'_1)\right)}. && (35)
\end{aligned}$$

after some manipulation[21].

Let's take a plane S that is perpendicular to the vortex filament's orientation, and find the corresponding circulation. Since the normal vector is parallel to the vortice's orientation, and consequently the vorticity; and since the vorticity is uniform within the vortex, we see that:

$$\Gamma = ||\boldsymbol{\omega}||_2 A_{\mathrm{v}} = \gamma\big|\big|\boldsymbol{x}'_1 - \boldsymbol{x}'_0\big|\big|_2 A_{\mathrm{v}},$$

where A_{v} is the cross-sectional area of the vortex filament.
We see then that if a vortex filament stretches over time (that is, $\big|\big|\boldsymbol{x}'_1 - \boldsymbol{x}'_0\big|\big|_2$ increases) while Γ and γ remain constant, that the cross-sectional area of the filament must decrease. That is, the so-called 'core' radius of a vortex must be inversely proportional to the square root of the filament length.

[21] Starting from (34), we can rearrange and expand the dot product. This gives:

$$\int_0^1 \frac{1}{||\boldsymbol{x}-\boldsymbol{x}'(t)||_2^3} dt = \frac{\big((\boldsymbol{x}-\boldsymbol{x}'_0)-(\boldsymbol{x}-\boldsymbol{x}'_1)\big)\cdot\left(\frac{\boldsymbol{x}-\boldsymbol{x}'_0}{||\boldsymbol{x}-\boldsymbol{x}'_0||_2} - \frac{\boldsymbol{x}-\boldsymbol{x}'_1}{||\boldsymbol{x}-\boldsymbol{x}'_1||_2}\right)}{\big|\big|(\boldsymbol{x}-\boldsymbol{x}'_0)\times(\boldsymbol{x}-\boldsymbol{x}'_1)\big)\big|\big|_2^2} = \frac{\left(\frac{||\boldsymbol{x}-\boldsymbol{x}'_0||_2^2 - (\boldsymbol{x}-\boldsymbol{x}'_1)\cdot(\boldsymbol{x}-\boldsymbol{x}'_0)}{||\boldsymbol{x}-\boldsymbol{x}'_0||_2} + \frac{||\boldsymbol{x}-\boldsymbol{x}'_1||_2^2 - (\boldsymbol{x}-\boldsymbol{x}'_0)\cdot(\boldsymbol{x}-\boldsymbol{x}'_1)}{||\boldsymbol{x}-\boldsymbol{x}'_1||_2}\right)}{\big|\big|(\boldsymbol{x}-\boldsymbol{x}'_0)\times(\boldsymbol{x}-\boldsymbol{x}'_1)\big)\big|\big|_2^2}$$

When these pieces are re-assembled, we find the induced velocity field due to the vortex filament to be:

$$\boldsymbol{u}_{\mathrm{i}}(\boldsymbol{x}) \quad = \quad \frac{\gamma}{4\pi}\left(\frac{\left(\left|\left|\boldsymbol{x}-\boldsymbol{x}_0'\right|\right|_2+\left|\left|\boldsymbol{x}-\boldsymbol{x}_1'\right|\right|_2\right)\left(\left(\boldsymbol{x}-\boldsymbol{x}_0'\right)\times\left(\boldsymbol{x}-\boldsymbol{x}_1'\right)\right)}{\left(\left|\left|\boldsymbol{x}-\boldsymbol{x}_0'\right|\right|_2^2\left|\left|\boldsymbol{x}-\boldsymbol{x}_1'\right|\right|_2^2+\left|\left|\boldsymbol{x}-\boldsymbol{x}_0'\right|\right|_2\left|\left|\boldsymbol{x}-\boldsymbol{x}_1'\right|\right|_2\left(\boldsymbol{x}-\boldsymbol{x}_0'\right)\cdot\left(\boldsymbol{x}-\boldsymbol{x}_1'\right)\right)}\right). \tag{36}$$

This derivation method is chosen to provide some framework to integrations of the Biot-Savart over other geometries, because the solution - found using a different derivation mainly based on angle identities - is widely published in descriptions of vortex methods such as [10].

We can expand this with Lagrange's identity ($||\boldsymbol{a}\times\boldsymbol{b}||_2^2 = ||\boldsymbol{a}||_2^2\,||\boldsymbol{b}||_2^2-(\boldsymbol{a}\cdot\boldsymbol{b})^2$) and multiply the denominator to give:

$$\int_0^1 \frac{1}{||\boldsymbol{x}-\boldsymbol{x}'(t)||_2^3}dt = \frac{\left(||\boldsymbol{x}-\boldsymbol{x}_1'||_2\,\frac{||\boldsymbol{x}-\boldsymbol{x}_0'||_2^2-(\boldsymbol{x}-\boldsymbol{x}_1')\cdot(\boldsymbol{x}-\boldsymbol{x}_0')}{||\boldsymbol{x}-\boldsymbol{x}_0'||_2||\boldsymbol{x}-\boldsymbol{x}_1'||_2}+||\boldsymbol{x}-\boldsymbol{x}_0'||_2\,\frac{||\boldsymbol{x}-\boldsymbol{x}_1'||_2^2-(\boldsymbol{x}-\boldsymbol{x}_0')\cdot(\boldsymbol{x}-\boldsymbol{x}_1')}{||\boldsymbol{x}-\boldsymbol{x}_0'||_2||\boldsymbol{x}-\boldsymbol{x}_1'||_2}\right)}{||(\boldsymbol{x}-\boldsymbol{x}_0')||_2^2\,||(\boldsymbol{x}-\boldsymbol{x}_1')||_2^2-\left((\boldsymbol{x}-\boldsymbol{x}_0')\cdot(\boldsymbol{x}-\boldsymbol{x}_1')\right)^2}$$

With some factoring:

$$\int_0^1 \frac{1}{||\boldsymbol{x}-\boldsymbol{x}'(t)||_2^3}dt = \frac{\left(||\boldsymbol{x}-\boldsymbol{x}_0'||_2+||\boldsymbol{x}-\boldsymbol{x}_1'||_2\right)\left(||\boldsymbol{x}-\boldsymbol{x}_0'||_2\,||\boldsymbol{x}-\boldsymbol{x}_1'||_2-(\boldsymbol{x}-\boldsymbol{x}_0')\cdot(\boldsymbol{x}-\boldsymbol{x}_1')\right)}{||\boldsymbol{x}-\boldsymbol{x}_0'||_2\,||\boldsymbol{x}-\boldsymbol{x}_1'||_2\left(||(\boldsymbol{x}-\boldsymbol{x}_0')||_2^2\,||(\boldsymbol{x}-\boldsymbol{x}_1')||_2^2-\left((\boldsymbol{x}-\boldsymbol{x}_0')\cdot(\boldsymbol{x}-\boldsymbol{x}_1')\right)^2\right)}$$

Finally, notice that $(a^2-b^2)=(a+b)(a-b)$.

References

[1] I. H. Abbott and A. E. V. Doenhoff. Summary of airfoil data. (824), 1945.

[2] J. D. Anderson. Ludwig Prandtl's Boundary Layer. Physics Today, 58(12):42–48, dec 2005.

[3] W. H. Beyer. CRC standard mathematical tables and formulae. CRC Press, Boca Raton, Florida, USA, 29 edition, 1991.

[4] H. Bhatia, G. Norgard, V. Pascucci, and P.-T. Bremer. The Helmholtz-Hodge DecompositionA Survey. IEEE Transactions on Visualization and Computer Graphics, 19(8):1386–1404, aug 2013.

[5] E. Branlard. Wind turbine aerodynamics and vorticity-based methods : fundamentals and recent applications. Springer, Roskilde, Denmark, 2017.

[6] P. Fife. A Gentle Introduction to the Physics and Mathematics of Incompressible Flow. University of Utah, Salt Lake City, Utah, USA, 2000.

[7] W. P. Graebel. Advanced fluid mechanics. Academic Press, Burlington, MA, USA, 2007.

[8] Y. F. Gui and W.-B. Dou. A Rigorous and completed statement on Helmholtz theorem. Progress In Electromagnetics Research, 69:287–304, 2007.

[9] J. C. Hardin. The velocity field induced by a helical vortex filament. Physics of Fluids, 25(11):1949, 1982.

[10] J. Katz and A. Plotkin. Low-speed aerodynamics. Cambridge University Press, Cambridge, UK, 2 edition, 2010.

[11] R. Peyret. Vorticity-Streamfunction Equations. In Spectral Methods for Incompressible Viscous Flow, chapter 6, pages 167–206. Springer Verlag, New York, New York, USA, 2002.

[12] L. Prandtl. Tragflügeltheorie. I. Mitteilung. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, 1918.

[13] L. Prandtl. Motion of fluids with very little viscosity. (Trans: Ueber Flussigkeitbewegung bei sehr kleiner Reibung). Technical report, National Advisory Committee for Aeronautics. NACA-TM-452. Trans. by Dwight M. Miner., Washington, DC, USA, 1928.

[14] T. Sebastian and M. Lackner. Development of a free vortex wake method code for offshore floating wind turbines. Renewable Energy, 46:269–275, oct 2012.

[15] A. Shapiro. Vorticity, 1961.

[16] O. G. Tietjens, L. Prandtl, and L. Rosenhead. Fundamentals of hydro- and aeromechanics. Dover Publications, 1957.

[17] M. van Dyke. An Album of Fluid Motion. Parabolic Press, Stanford, CA, USA, 1982.

[18] W. van Hoydonck, M. Gerritsma, and M. van Tooren. On Core and Curvature Corrections used in Straight-Line Vortex Filament Methods. page 34, apr 2012.

[19] G. H. Vatistas, V. Kozel, and W. C. Mih. A simpler model for concentrated vortices. Experiments in Fluids, 11(1):73–76, apr 1991.

[20] G. N. Watson. A treatise on the theory of Bessel functions. Cambridge University Press, Cambridge, UK, 2 edition, 1966.

[21] E. W. Weisstein. Green's Function–Poisson's Equation.

[22] G. Winckelmans and A. Leonard. Contributions to Vortex Particle Methods for the Computation of Three-Dimensional Incompressible Unsteady Flows. Journal of Computational Physics, 109(2):247–273, dec 1993.

[23] J.-Z. Wu, H.-Y. Ma, and M.-D. Zhou. Vorticity and Vortex Dynamics. Springer Science & Business Media, Berlin, Germany, 2007.