Exercises for Course on Modeling and System Identification (MSI)
Albert-Ludwigs-Universität Freiburg – Winter Term 2018-2019

Exercise 2: Linear Least-Squares Introduction

(to be returned on Nov 16, 2018, 10:00 in SR 00-010/014, or before in building 102, 1st floor, 'Anbau')

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In this exercise, you will discover basic properties of the linear least-squares estimation method, as well as the covariance operator.

Exercise Tasks

1. ON PAPER: The covariance matrix of a vector-valued random variable X in $\mathbb{R}^n$ with mean $\mathbb{E}\{X\} = \mu_X$ is defined by

$$\text{cov}\,(X) := \mathbb{E}\left\{(X - \mu_X)\,(X - \mu_X)^\top\right\}.$$

Prove that the covariance matrix of a vector-valued variable $Y = AX + b$ with constant $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ is given by

$$\text{cov}\,(Y) = A\,\text{cov}\,(X)\,A^\top.$$

(2 points)

2. ON PAPER: Suppose we are measuring a constant $x_0 \in \mathbb{R}$ perturbed by random noise ϵ with mean $\mu_\epsilon = 0$ and variance $\sigma_\epsilon^2 > 0$, i.e. we have

$$x = x_0 + \epsilon.$$

   (a) State the mean μ_x and the variance σ_x^2 of the random variable x. (1 point)

   (b) Let $x(n) = (x_1, \dots, x_n)$ denote a sample of n observations of x. In the lecture, we have seen that the sample mean $\bar{x}(n) = \frac{1}{n}\sum_{i=1}^n x_i$ is an unbiased estimator of the mean μ_x. What is the variance of $\bar{x}(n)$? (1 point)

   (c) Prove that the LLS estimate for x_0 is the sample mean $\bar{x}(n)$. State the minimization problem explicitly. Is it convex? (2 points)

3. Let's consider atmospheric data taken by a radiosonde at the Freiburg airport. This data is available in its complete form [1], or lightly pre-processed on the course page.

   A radiosonde[2] is a weather-balloon equipped with a barometer to measure air pressure, a resistance thermistor to measure air temperature, and - on older radiosondes - a mechanical switch that connected the thermistor at predetermined intervals of the pressure. The altitude is calculated from the temperature and pressure measurements.

   In the MATLAB grader template, we already provide code loading the altitude z and pressure p data (measured in [m] and [Pa] respectively).

   (a) MATLAB: We would like to fit a 5-th order polynomial to the data using linear least-squares. Choose $y \in \mathbb{R}^N$ and $\mathbf{\Phi} \in \mathbb{R}^{N \times d}$ to correspond to the linear least-squares problem. Here, N denotes the number of data points in the given dataset and d denotes the number of coeffcients. As we would like to fit a polynomial of order 5, we have $d = 6$.

   Hint: You might want to have a look at the lecture notes, Section 4.2, Example 2.

   Which properties should the data fulfil for linear least squares to be an appropriate estimation method, i.e. state the model assumption and the assumed noise distribution. (2 points)

   (b) ON PAPER: In the MATLAB template, we provided four different ways of solving the given linear least-squares problem. Explain briefly (!) what each of them does and why MATLAB returns a warning (or not). Do the results coincide?

   Note: MATLAB grader does not report warnings, thus you have to run your script using your local MATLAB application or the online version of MATLAB. (2 points)

   (c) MATLAB: Plot the data (using x markers) together with your estimated polynomials (one for each of the four coefficient vectors). (1 point)

   (d) MATLAB: Rescale your fitting problem, using units of [10^5 Pa] for p and [km] for z. Again, try all four methods for solving the (rescaled) linear system.

   ON PAPER: Compare the resulting coefficients. Does scaling the measurements improve the performance of linear least-squares? Why? (1 point)

This sheet gives in total 12 points.

[1] `www1.ncdc.noaa.gov/pub/data/igra/data/data-por/GMXUAC00001-data.txt.zip`

[2] More information can be found at www.aos.wisc.edu/ hopkins/wx-inst/wxi-raob.htm