State-Space Control Systems – Exam

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Points on page (max)	0	12	14	8	10	10	6
Points obtained							
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Mark: Exam inspected on: Signature of examiner:

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Please fill in your name above. For each question, give a short formula or text answer just below the question in the space provided, and, if necessary, write on the **backpage of the same sheet** where the question appears, and add a comment "see backpage". Do not add extra pages (for fast correction, all pages will be separated for parallelization). The exam is a closed book exam, i.e. no books or other material are allowed besides 2 sheets (total 4 pages) of notes and a non-programmable calculator. Some legal comments are found in a footnote.[1]

Note: Data given in the questions are enough to figure the answers. Engineers often face situations with 'vague' data, it's indeed additional freedom for them to choose reasonable parameters.

1. **Control of DC motor speed**

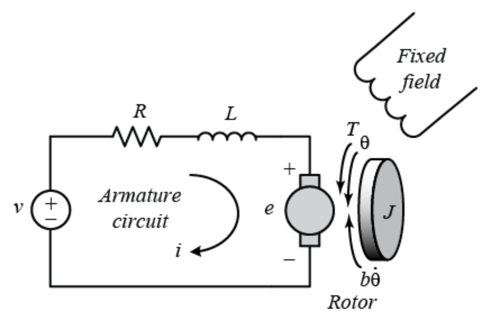

Figure 1: Schematical depiction of a DC motor.
Courtesy Control Tutorials for MATLAB and Simulink.

Given a DC motor with the following dynamical equations:

$$\ddot{\theta}(t) + 10\dot{\theta}(t) = i(t) \tag{1}$$

$$\frac{di}{dt} + 2i(t) = V(t) - 0.02\dot{\theta}(t) \tag{2}$$

[1]WITHDRAWING FROM AN EXAMINATION: In case of illness, you must supply proof of your illness by submitting a medical report to the Examinations Office. Please note that the medical examination must be done at the latest on the same day of the missed exam. In case of illness while writing the exam please contact the supervisory staff, inform them about your illness and immidiatelly see your doctor. The medical certificate must be submitted latest 3 days after the medical examination. More informations: http://www.tf.uni-freiburg.de/studies/exams/withdrawing_exam.html

CHEATING/DISTRUBING IN EXAMINATIONS: A student who disrupts the orderly proceedings of an examination will be excluded from the remainder of the exam by the respective examiners or invigilators. In such a case, the written exam of the student in question will be graded as 'nicht bestanden' (5.0, fail) on the grounds of cheating. In severe cases, the Board of Examiners will exclude the student from further examinations.

(a) Write the state-space representation of this system, taking $V(t)$ as the control input, $\dot{\theta}(t)$ as the output.

2	

(b) Calculate the controllability matrix of this system. Is the system fully controllable? Why?

2	

(c) Calculate the observability matrix of this system. Is the system fully observable? Why?

2	

(d) We want to design a full-state feedback controller $\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t)$, so that the closed-loop system has two eigenvalues at locations: $-5 \pm j$, where j denotes the imaginary unit. Calculate the suitable matrix $\mathbf{K}$ to achieve that.

4	

(e) We want the output $\mathbf{y}$ of the controlled system to track a reference $\mathbf{r}$, using a prefilter $\mathbf{K}_f$ such that $\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{K}_f\mathbf{r}$. Write the formula for computing $\mathbf{K}_f$ based on $\mathbf{K}$ computed from step (1d).

2	

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2. **LQR**

Given a state-space continuous system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

with the following parameters:

$$\mathbf{A} = \begin{bmatrix} -10 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

(a) Determine from the state-space system: which state is controllable, which state is observable.

2	

(b) Calculate the eigenvalues of the closed-loop system using a state-feedback controller $\mathbf{u} = -\bar{\mathbf{K}}\mathbf{x}$ with $\bar{\mathbf{K}} = \begin{bmatrix} 0 & 2 \end{bmatrix}$.

2	

(c) Explain why the output signal of the closed-loop system always exhibits a stable behaviour, regardless of the state-feedback gain $\mathbf{K}$ that is used in the controller $\mathbf{u} = -\mathbf{K}\mathbf{x}$.

2	

(d) Write the algebraic Riccati equation to be solved, in order to design a state-feedback controller using linear quadratic regulator (LQR) with the cost function:

$$J(\mathbf{x}, \mathbf{u}) = \int_0^\infty \left(\mathbf{x}^\top(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^\top(t)\mathbf{R}\mathbf{u}(t) \right) \, \mathrm{d}t \tag{3}$$

2	

(e) **(*)** Lily the Queen of Regulation claims that $\bar{\mathbf{K}} = \begin{bmatrix} 0 & 2 \end{bmatrix}$ is an optimal controller in some sense. Your task is to trace back her claim, by finding which weighting matrices $\mathbf{Q}, \mathbf{R}$ used in the LQR formulation, such that $\bar{\mathbf{K}}$ is the solution.

6	

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(f) Later the Queen found that her 'optimal' state-feedback controller with $\bar{\mathbf{K}} = \begin{bmatrix} 0 & 2 \end{bmatrix}$ is indeed impossible to be implemented in practice. Point out and explain the trouble she would face when implementing this controller.

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3. **Ball on beam**

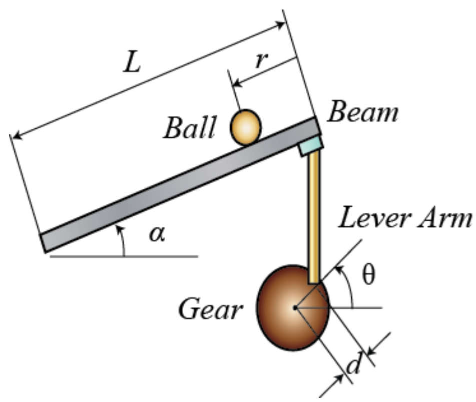

Figure 2: Schematical depiction of a ball on beam.
Courtesy Control Tutorials for MATLAB and Simulink.

Consider a ball on a controlled beam under some approximation, having the following differential equation:

$$0 = \left(\frac{J}{R^2} + m \right) \ddot{r}(t) + mg \sin \alpha(t) \tag{4}$$

and the relation $\alpha(t) = \frac{d}{L}\theta(t)$, in which J, R, m, g, d, L are positive constants.

(a) Let θ be the control input, r be the output. Write the nonlinear state-space system, using the state vector $\mathbf{x} = \begin{bmatrix} r \\ \dot{r} \end{bmatrix}$.

2	

(b) Find the equilibrium points of this system.

2	

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(c) Linearize the original system around the equilibrium $\mathbf{x}_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{u}_e = 0$. You can simplify the exposition by denoting $H = \frac{mgd}{L\left(\frac{J}{R^2}+m\right)}$.

2	

(d) Suppose we want to control the system around $(\mathbf{x}_e, \mathbf{u}_e)$, using the linearized model in the following form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{w}(t) \tag{5}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{v}(t) \tag{6}$$

where $\mathbf{w}$ and $\mathbf{v}$ are independent zero-mean, Gaussian white noises, with covariances:

$$\mathbb{E}(\mathbf{w}\mathbf{w}^T) = \mathbf{Q}_\mathbf{w}, \quad \mathbb{E}(\mathbf{v}\mathbf{v}^T) = \mathbf{R}_\mathbf{v}. \tag{7}$$

We want to design a Kalman filter for estimating states of this system.

Write the formula that would be used to obtain the state estimation, and how to find the gain of the filter.

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(e) We want to use Linear Quadratic Gaussian method to control this linearized system, with an optimal state-feedback gain $\mathbf{K}$ and an optimal observer gain $\mathbf{L}$.

Draw the block diagram of the LQG control system.

2	

(f) Provided that the closed-loop system is stable using K and L from the LQG design in step (3e). Explain why we don't need to redesign $\mathbf{K}$ for every time we use a different state estimator (i.e. $\mathbf{L}$ changes), as long as the new observer guarantees that the estimated state converges asymptotically to the real state.

4	

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(g) Uno the King of Filtering has an idea to first discretize the nonlinear continuous dynamics to obtain a discrete-time system in the form:

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{w}_k \quad (8)$$
$$\mathbf{y}_k = h(\mathbf{x}_k) + \mathbf{v}_k, \quad (9)$$

and then linearize it around $(\mathbf{x}_e, \mathbf{u}_e)$ to have the discrete-time linear system:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{w}_k \quad (10)$$
$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{v}_k, \quad (11)$$

then he uses discrete LQR to design the state-feedback controller, and a discrete nonlinear observer for state estimation.
Propose to Uno one type of nonlinear observer that he can use, describe the steps in the iteration of such observer, with general formula.

4	

4. **MPC for discrete-time system**

Consider a discrete linear time-invariant system:

$$x_{k+1} = \mathbf{A}x_k + \mathbf{B}u_k$$
$$y_k = \mathbf{C}x_k$$

with $\mathbf{A} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Given an arbitrary initial state $\bar{x}_0 = \begin{bmatrix} \bar{x}_0^1 \\ \bar{x}_0^2 \end{bmatrix}$ in the region: $\bar{x}_0^1 \in [-2; 2], \bar{x}_0^2 \in [-1; 1]$. We want to use state-feedback Model Predictive Control (MPC) to regulate the system, i.e. we aim to drive the states to zero. During control, we want following conditions to be satisfied:

$$-3 \leq y \leq 3 \quad (12)$$
$$-1 \leq u \leq 1 \quad (13)$$

(a) Formulate the optimization problem to be solved at each sampling time, using quadratic cost MPC with horizon N and zero terminal state constraint.

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(b) **(*)** Suppose we choose a short horizon: $N = 2$. Prove that there is some value of x_0 in $[-2; 2] \times [-1; 1]$ such that the MPC problem formulated at step (4a) cannot be solved (it is infeasible).

4	

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(c) **(*)** We want to formulate a less restrictive MPC problem, where we replace the zero terminal state constraint by:

- a terminal cost term $x_N^T \mathbf{P} x_N$, where $\mathbf{P}$ is the solution of the discrete algebraic Riccati equation:

$$\mathbf{P} = \mathbf{Q} + \mathbf{A}^T\mathbf{P}\mathbf{A} - \mathbf{A}^T\mathbf{P}\mathbf{B}\left(\mathbf{R} + \mathbf{B}^T\mathbf{P}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{P}\mathbf{A} \tag{14}$$

  in which $\mathbf{Q}$ and $\mathbf{R}$ are respectively the weighting matrices for state and control input in the stage cost; and

- a terminal constaint $x_N \in X_f$, where X_f is an invariant set for the closed-loop system using the state-feedback controller $\mathbf{u} = -\mathbf{K}\mathbf{x}$, with

$$\mathbf{K} = \left(\mathbf{R} + \mathbf{B}^T\mathbf{P}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{P}\mathbf{A}, \tag{15}$$

  'invariant' means if we have a state $\bar{x} \in X_f$, then we will have $(\mathbf{A} - \mathbf{B}\mathbf{K})\bar{x} \in X_f$.

Prove that such MPC closed-loop system is stable, using Lyapunov stability theory.

6	

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