Exercises for Course on State-Space Control Systems (SSC)
Albert-Ludwigs-Universität Freiburg – Summer Term 2019

Exercise 7: Kalman Filter

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Exercises

1. *(Other setting for Kalman filter)* Consider a discrete LTI system with state-space matrices $(A, B, C, 0)$, there are disturbance w_k and measurement noise v_k, both are independent zero-mean, Gaussian white noises. Suppose that instead of using the Luenberger observer:

$$\hat{x}[k+1] = A\hat{x}[k] + Bu[k] + L_k(y[k] - C\hat{x}[k])$$

we use the modified observer with the formula:

$$\hat{x}[k+1] = \hat{x}^-[k+1] + L_{k+1}(y[k+1] - C\hat{x}^-[k+1])$$

where

$$\hat{x}^-[k+1] = A\hat{x}[k] + Bu[k]$$

Derive the recursive Kalman filter in this case.

2. *(Matlab simulation)* Consider an inverted cart system that is linearized around the downward position (stable equilibrium), characterized by the matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.184 & 2.822 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.48 & -32.93 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1.84 \\ 0 \\ -4.8 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad D = 0$$

Taking into account the zero-mean, stationary Gaussian white processes: disturbance $w \sim \mathcal{N}(0, V_d)$, measurement noise $v \sim \mathcal{N}(0, V_n)$ with $V_d = 0.1 * I_4, V_n = 1$.

   (a) Use the Matlab command `lqe` to calculate the gain of the Kalman filter.
   (b) Construct the combined state-space representation of the system with control vector $[u, w, v]^T$.
   (c) Generate random series for the disturbance and the measurement noise, apply impulse input for u, simulate the combined system (with Matlab command `lsim`), and compare the true states against the estimated states using Kalman filter.
   (d) If the system is linearized around the upward position (unstable equilibrium), can you use Kalman filter with impulse response?

3. *(Discrete-time random walk)* Suppose that we wish to estimate the position of a particle that is undergoing a random walk in one dimension (i.e., along a line). We model the position of the particle as

$$x[k+1] = x[k] + u[k],$$

where x is the position of the particle and u is a white noise processes with $E(u[i]) = 0$ and

$$E(u[i]u[j]) = R_u \delta_{ij}, \quad \delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

We assume that we can measure x subject to additive, zero-mean, Gaussian white noise with covariance 1.

(a) Compute the expected value and covariance of the particle as a function of k.

(b) Construct a Kalman filter to estimate the position of the particle given the noisy measurements of its position. Compute the steady-state expected value and covariance of the error of your estimate.

(c) Suppose that $E(u[0]) = \mu \neq 0$ but is otherwise unchanged. How would your answers to parts (a) and (b) change?

4. *(Kalman filter for scalar ODE)* Consider a scalar control system

$$\frac{dx}{dt} = \lambda x + u + \sigma_w w, \quad y = x + \sigma_v v$$

where w and v are zero-mean, Gaussian white noise processes with covariance 1, and $\sigma_w, \sigma_v > 0$. Assume that the initial value of x is modeled as a Gaussian with mean x_0 and variance $\sigma_{x_0}^2$.

(a) Write down the Kalman filter for the optimal estimate of the state x and compute the steady-state value(s) of the mean and covariance of the estimation error.

(b) Assume that we initialize our filter such that the initial covariance starts near a steady-state value p^*. Give conditions on λ such that error covariance is locally stable around this solution.