1.5 Discrete-time state-space systems

A discrete-time system represents the evolution of a system at discrete instants of time, taken at time points kT_s with $k \in \mathbb{Z}$ and T_s a short amount of time, called *sampling time.*

Discrete-time systems can be expressed as difference equations:

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{f}(\mathbf{x}[k], \mathbf{u}[k]) \\ \mathbf{y}[k] &= \mathbf{g}(\mathbf{x}[k], \mathbf{u}[k]) \end{aligned}$$

Transformation between continuous-time System and discrete-time system Consider linear continuous-time dynamics:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{1.24}$$

We can express the relation of $\mathbf{x}[k+1] \triangleq \mathbf{x}\big((k+1)T_s\big)$ and $\mathbf{x}[k] \triangleq \mathbf{x}\big(kT_s\big)$, using the solution (1.15) for continuous-time ODE:

$$\mathbf{x}\big((k+1)T_s\big) = e^{\mathbf{A}T_s}\mathbf{x}\big(kT_s\big) + \int_0^{T_s} e^{\mathbf{A}(T_s-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \tag{1.25}$$

Assume that the control signal is constant over each sampling interval, i.e. for an interval from k^{th} to $(k+1)^{\text{th}}$ samples: $\mathbf{u}(t) = \mathbf{u}[k], t \in [kT_s, (k+1)T_s)$ (this is called *zero order hold* sampling).

By defining:

$$\mathbf{\Phi} = e^{\mathbf{A}T_s}, \quad \Gamma = \left(\int_0^{T_s} e^{\mathbf{A}s}\mathrm{d}s\right)\mathbf{B} \tag{1.26}$$

we have the discrete-time system that is sampled from (1.24):

$$\mathbf{x}[k+1] = \mathbf{\Phi}\mathbf{x}[k] + \Gamma\mathbf{u}[k] \tag{1.27}$$

Reversely, with given matrices for the discrete-time system, we can also find the matrices of the continuous-time system using relations:

$$\mathbf{A} = \frac{1}{T_s}\log\mathbf{\Phi}, \quad \mathbf{B} = \left(\int_0^{T_s} e^{\mathbf{A}s}\mathrm{d}s\right)^{-1}\Gamma \tag{1.28}$$

If $\mathbf{A}$ is invertible, we have the relation:

$$\Gamma = \mathbf{A}^{-1}\big(e^{\mathbf{A}T_s} - I\big)\mathbf{B} \tag{1.29}$$

MATLAB command for converting continuous-time and discrete-time linear systems:
sysd = c2d(sysc, Ts); sysc = d2c(sysd).

Discrete-time System as an Approximation of continuous-time system For nonlinear continuous-time systems, we don't have analytic solution of the nonlinear ODE, hence it is difficult to find the exact transformation from continuous-time to discrete-time like in the linear case. Instead, we use numerical approximation to compute the discretized values.

Consider Taylor expansion series of function $\mathbf{x}(t)$ around the point $t_0 = kT_s$:

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \nabla\mathbf{x}(t_0)\Delta_t + \mathcal{O}\big(\Delta_t^2\big) \tag{1.30}$$

with $\Delta_t = \mathbf{x}(t) - \mathbf{x}(t_0)$. The notation $\nabla\mathbf{x}(t_0)$ is the derivative of the multi-dimensional function $\mathbf{x}$ with respect to time t.

We omit the last term which stands for the contribution of high order derivatives, and get linear approximation for $t = (k+1)T_s$:

$$\mathbf{x}[k+1] \approx \mathbf{x}[k] + \nabla \mathbf{x}(t_0) \tag{1.31}$$

In order to implement the computation following (1.31), at each sampling time kT_s, we first compute the derivative of $\mathbf{x}$ at that time (we may substitute $\mathbf{x}[k], \mathbf{u}[k]$ into the nonlinear ODE to get that derivative).

The formula (1.31) is called *Forward Euler* approximation. There are other ways that are widely used: *Backward Euler*, *Runge Kutta* methods.

1.6 Linearization of nonlinear systems

Given a nonlinear dynamical system, we may want to examine the behavior of the system around a reference or steady-state point by linearization of the ODE. The following procedure is applied:

1. Find equilibrium points.
2. Linearize system around the equilibrium point that we are interested.

Equilibrium of a dynamical system Recall the ODE describing a system:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y} &= \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \end{aligned}$$

Equilibrium points are the points $(\mathbf{x}_e, \mathbf{u}_e)$ so that $\mathbf{f}(\mathbf{x}_e, \mathbf{u}_e) = 0$, hence $\dot{\mathbf{x}}|_{(\mathbf{x}_e, \mathbf{u}_e)} = 0$, the state does not change and remains at $\mathbf{x}_e$.

Then, we define a new set of variables: $\mathbf{z}, \mathbf{v}, \mathbf{w}$ are new state, input, and output variables, respectively:

$$\mathbf{z} = \mathbf{x} - \mathbf{x}_e, \quad \mathbf{v} = \mathbf{u} - \mathbf{u}_e, \quad \mathbf{w} = \mathbf{y} - \mathbf{g}(\mathbf{x}_e, \mathbf{u}_e) \tag{1.32}$$

Linearization around an equilibrium point The linearized system has the form:

$$\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{v} \tag{1.33}$$

$$\mathbf{w} = \mathbf{C}\mathbf{z} + \mathbf{D}\mathbf{v} \tag{1.34}$$

where:

$$\mathbf{A} = \left.\frac{\mathrm{D}\mathbf{f}}{\mathrm{D}\mathbf{x}}\right|_{(\mathbf{x}_e, \mathbf{u}_e)}, \quad \mathbf{B} = \left.\frac{\mathrm{D}\mathbf{f}}{\mathrm{D}\mathbf{u}}\right|_{(\mathbf{x}_e, \mathbf{u}_e)}, \quad \mathbf{C} = \left.\frac{\mathrm{D}\mathbf{g}}{\mathrm{D}\mathbf{x}}\right|_{(\mathbf{x}_e, \mathbf{u}_e)}, \quad \mathbf{D} = \left.\frac{\mathrm{D}\mathbf{g}}{\mathrm{D}\mathbf{u}}\right|_{(\mathbf{x}_e, \mathbf{u}_e)} \tag{1.35}$$

The notation $\frac{\mathrm{D}\mathbf{f}}{\mathrm{D}\mathbf{x}}$ means the partial Jacobian matrix of the multi-dimensional function $\mathbf{f} \in \mathbb{R}^m$ with respect to variables $\mathbf{x} \in \mathbb{R}^n$, with formula:

$$\frac{\mathrm{D}\mathbf{f}}{\mathrm{D}\mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \tag{1.36}$$