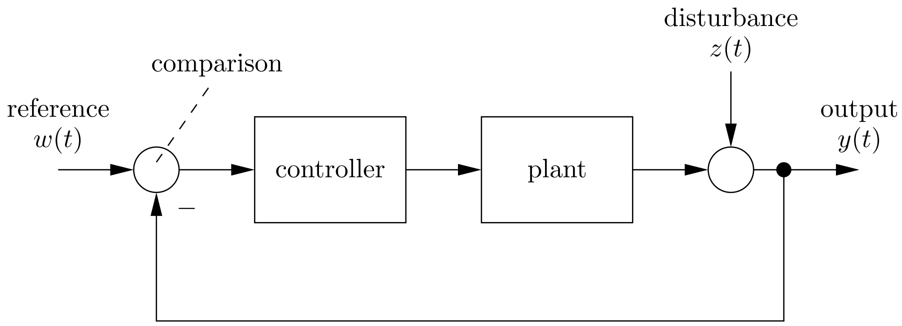
Chapter 3

State Feedback Control

3.1 Feedback control

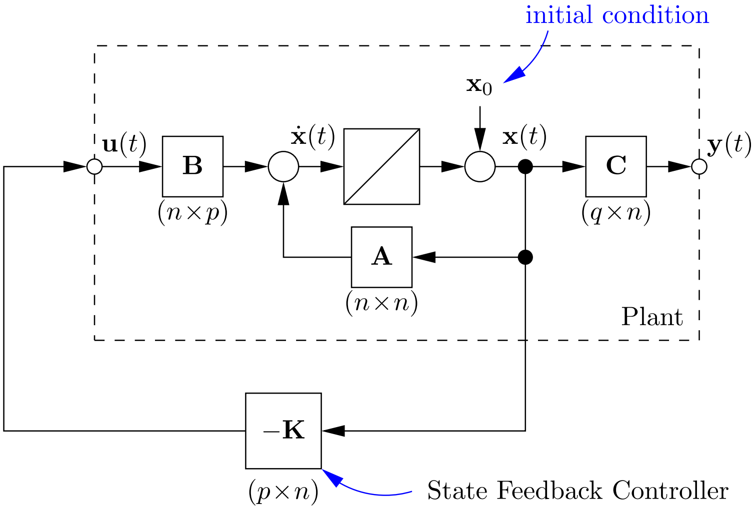
Before diving into the details of state feedback control, we remind ourselves of the ’classical’ *closed loop control*



The distinguishing feature is the feedback of the output, which is compared to the reference value and thereby enables the control loop to compensate for disturbances $z(t) \neq 0$. In this chapter, the implementation of feedback controllers for state space systems will be discussed. Note that in the subsequent sections, we will focus on state feedback, which is different to the output feedback of the ’classical’ control loop above. Firstly, this is done as equations become easier as for output feedback. Secondly and more importantly, state feedback can be implemented as there are methods to reconstruct the state from output measurements by observers, which will be discussed in detail when we study output feedback.

3.2 State feedback

For further considerations, we assume $\mathbf{D} = \mathbf{0}$ to simplify notation. Now, a feedback is added to our state space system as follows



The state feedback controller is defined by

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) \tag{3.1}$$

where $\mathbf{K}$ is a constant matrix.

Inserting this equation into the state space ODE $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ yields the following ODE for the feedback system

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) \tag{3.2}$$

This *closed loop system* can be considered as an autonomous system. Our interest is to choose $\mathbf{K}$ such that the state space control loop is stable.

$\Updownarrow$

For any initial value $\mathbf{x}_0 \neq \mathbf{0}$, $\mathbf{x}(t) \overset{t\to\infty}{\longrightarrow} \mathbf{0}$.

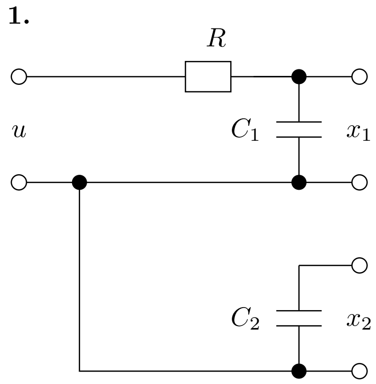
$\Updownarrow$

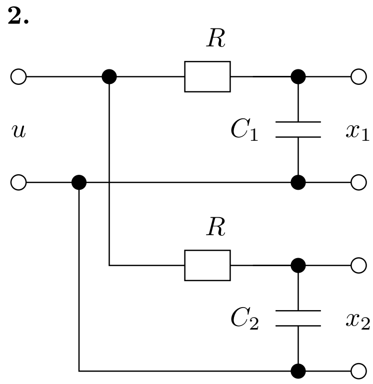
$(\mathbf{A} - \mathbf{B}\mathbf{K})$ is a stable matrix, i.e., all its eigenvalues have a negative real part.

3.3 Controllability

3.3.1 SISO systems

We consider controllability, i.e., control of the state $\mathbf{x} = [x_1, x_2]^\top$ by input u for the following introductory examples





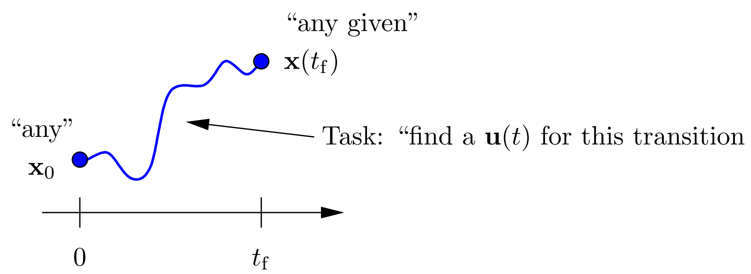
1. Is not controllable as x_2 is 'disconnected'.

2. Here we have to distinguish two cases. For $C_1 = C_2$ the subsystems would behave equally, hence the states can not be manipulated separately. System not controllable. For $C_1 \neq C_2$ any state can be generated by appropriate choice of $u(t)$, hence system controllable.

Controllability

A system is controllable, if in finite time t_f any initial state $\mathbf{x}(0)$ can be driven to any given final state $\mathbf{x}(t_\mathrm{f})$ by appropriate choice of the control signal $\mathbf{u}(t)$ for $0 \leq t \leq t_\mathrm{f}$.

This can be depicted as follows



By consideration of the solution of the state space ODE

$$\mathbf{x}(t_\mathrm{f}) = e^{\mathbf{A}t_\mathrm{f}}\mathbf{x}(0) + \int_0^{t_\mathrm{f}} e^{\mathbf{A}(t_\mathrm{f}-\tau)}\mathbf{B}\mathbf{u}(\tau)\,\mathrm{d}\tau \tag{3.3}$$

we get

$$\underbrace{\mathbf{x}(t_\mathrm{f}) - e^{\mathbf{A}t_\mathrm{f}}\mathbf{x}(0)}_{\triangleq -e^{\mathbf{A}t_\mathrm{f}}\mathbf{x}_\mathrm{i}} = \int_0^{t_\mathrm{f}} e^{\mathbf{A}(t_\mathrm{f}-\tau)}\mathbf{B}\mathbf{u}(\tau)\,\mathrm{d}\tau \tag{3.4}$$

The value $\mathbf{x}_\mathrm{i}$ is defined by setting the LHS equal to $-e^{\mathbf{A}t_\mathrm{f}}\mathbf{x}_\mathrm{i}$. As the equation has to be valid for any $\mathbf{x}(t_\mathrm{f})$ and any $\mathbf{x}(0)$, the following equation has to hold for all $\mathbf{x}_\mathrm{i} \in \mathrm{I\!R}^n$.

$$-e^{\mathbf{A}t_\mathrm{f}}\mathbf{x}_\mathrm{i} = \int_0^{t_\mathrm{f}} e^{\mathbf{A}(t_\mathrm{f}-\tau)}\mathbf{B}\mathbf{u}(\tau)\,\mathrm{d}\tau \tag{3.5}$$

The system is controllable, if for any $\mathbf{x}_\mathrm{i} \in \mathrm{I\!R}^n$, a finite t_f and a control input $\mathbf{u}(t)$ for $0 \leq t \leq t_\mathrm{f}$ can be found, such that (3.5) holds. In other words: by appropriate choice of $\mathbf{u}(t)$, the system can be driven from any initial state $\mathbf{x}_\mathrm{i}$ to the zero state in finite time t_f.

Controllability for SISO Systems

Criterion by Kalman (1960). Define controllability matrix

$$\mathscr{C} \triangleq \left[\mathbf{b}, \mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}, \ldots, \mathbf{A}^{n-1}\mathbf{b}\right] \tag{3.6}$$

The system $(\mathbf{A}, \mathbf{b})$ is controllable, if $\mathscr{C}$ is invertible, i.e. $\det(\mathscr{C}) \neq 0$.

Proof: consider

$$-e^{\mathbf{A}t_\mathrm{f}}\mathbf{x}_\mathrm{i} = \int_0^{t_\mathrm{f}} e^{\mathbf{A}(t_\mathrm{f}-\tau)}\mathbf{b}u(\tau)\,\mathrm{d}\tau \tag{3.7}$$

Hence

$$
\begin{aligned}
-\mathbf{x}_\mathrm{i} &= \int_0^{t_\mathrm{f}} e^{-\mathbf{A}\tau}\mathbf{b}u(\tau)\,\mathrm{d}\tau = \int_0^{t_\mathrm{f}} \left(\sum_{\nu=0}^{\infty} \frac{(-\mathbf{A})^\nu \tau^\nu}{\nu!}\right)\mathbf{b}u(\tau)\,\mathrm{d}\tau \\
&= \sum_{\nu=0}^{\infty} \mathbf{A}^\nu \mathbf{b} \underbrace{\int_0^{t_\mathrm{f}} \frac{(-1)^\nu \tau^\nu}{\nu!} u(\tau)\,\mathrm{d}\tau}_{u_\nu \triangleq} \qquad (3.8)
\end{aligned}
$$

Thus we get for $\mathbf{x}_\mathrm{i}$

$$
\mathbf{x}_\mathrm{i} = -\sum_{\nu=0}^{\infty} \mathbf{A}^\nu \mathbf{b} u_\nu \qquad (3.9)
$$

This equation has a solution for any $\mathbf{x}_\mathrm{i}$, if $\mathbf{A}^\nu\mathbf{b}$ span up the complete vector space, such that any $\mathbf{x}_\mathrm{i}$ can be composed by appropriate choice of the u_ν coefficients.

It remains to show that $\mathbf{A}^\nu\mathbf{b}$ with $\nu = 0, \dots, \infty$ span up the complete vector space if $\mathbf{b}, \mathbf{Ab}, \mathbf{A}^2\mathbf{b}, \dots, \mathbf{A}^{n-1}\mathbf{b}$ are linearly independent, i.e. $\mathcal{C}$ is non-singular. The argument is based on the theorem of Cayley-Hamilton for the characteristic polynomial $p(\mathbf{A}) = 0$, which states that $\mathbf{A}^n$ can be written as linear combination (LC) of $\mathbf{A}^0, \dots, \mathbf{A}^{n-1}$. Hence, $\mathbf{A}^{n+1} = \mathbf{A}\mathbf{A}^n$ can be written as LC of $\mathbf{A}^0, \dots, \mathbf{A}^n$ and recursively as LC of $\mathbf{A}^0, \dots, \mathbf{A}^{n-1}$. As a consequence, it is sufficient to consider $\mathbf{A}^0, \dots, \mathbf{A}^{n-1}$.

□

Example We consider the introductory example on page 14. The system ODE read

$$
\dot{\mathbf{x}}(t) = \begin{bmatrix} -\frac{1}{RC_1} & 0 \\ 0 & -\frac{1}{RC_2} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \frac{1}{RC_1} \\ \frac{1}{RC_2} \end{bmatrix} u(t) \qquad (3.10)
$$

The controllability matrix is then

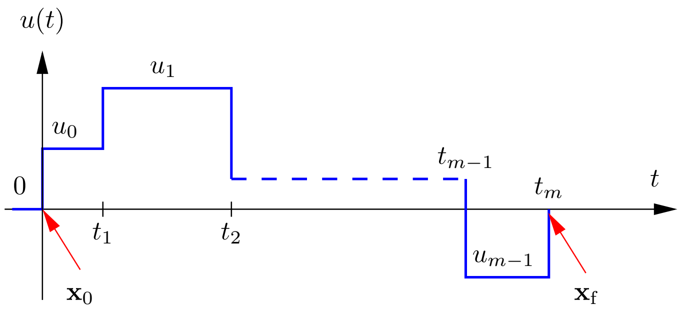
$$
\mathscr{C} = [\mathbf{b}, \mathbf{Ab}] = \begin{bmatrix} \frac{1}{RC_1} & -\frac{1}{(RC_1)^2} \\ \frac{1}{RC_2} & -\frac{1}{(RC_2)^2} \end{bmatrix} \qquad (3.11)
$$

Hence the system is controllable if

$$
\det(\mathscr{C}) = \frac{1}{(RC_1)(RC_2)} \left(-\frac{1}{RC_2} + \frac{1}{RC_1}\right) \neq 0 \qquad (3.12)
$$

which is equivalent to $C_1 \neq C_2$.

Control Input for State Transition The task is to control the state transition from $\mathbf{x}_0 \to \mathbf{x}_\mathrm{f}$ with a piece-wise constant control input given as follows



Using the solution of the ODE (3.3), we get

$$\mathbf{x}_\mathrm{f} = e^{\mathbf{A}t_m}\mathbf{x}_0 + \sum_{i=0}^{m-1}\left(\int_{t_i}^{t_{i+1}} e^{\mathbf{A}(t_m-\tau)}\mathbf{b}\,\mathrm{d}\tau\right)u_i \tag{3.13}$$

By defining

$$\mathbf{p}_i \triangleq \int_{t_i}^{t_{i+1}} e^{\mathbf{A}(t_m-\tau)}\mathbf{b}\,\mathrm{d}\tau \tag{3.14}$$

(3.13) can be written as

$$[\mathbf{p}_0, \ldots, \mathbf{p}_{m-1}]\begin{bmatrix} u_0 \\ \vdots \\ u_{m-1}\end{bmatrix} = \mathbf{x}_\mathrm{f} - e^{\mathbf{A}t_m}\mathbf{x}_0 \tag{3.15}$$

Hence the input amplitudes can be computed by

$$\begin{bmatrix} u_0 \\ \vdots \\ u_{m-1}\end{bmatrix} = [\mathbf{p}_0, \ldots, \mathbf{p}_{m-1}]^{-1}\left(\mathbf{x}_\mathrm{f} - e^{\mathbf{A}t_m}\mathbf{x}_0\right) \tag{3.16}$$

It should be remarked that due to the dimensions, (at least) n control steps are needed for an n-dimensional state vector. In addition, the times t_i have to be chosen such that the $\mathbf{p}_i$ are linearly independent.

3.3.2 Extension to MIMO systems

Having introduced controllability for SISO systems, we now sketch the criteria for MIMO systems.

Controllability for MIMO systems

The controllability matrix can now be defined as

$$\mathscr{C} \triangleq \left[\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, \ldots, \mathbf{A}^{n-1}\mathbf{B}\right] \tag{3.17}$$

The system $(\mathbf{A}, \mathbf{B})$ is controllable if $\mathrm{rank}(\mathscr{C}) = n$. (Note that $\mathscr{C}$ is a matrix of size $n \times (np)$.)

Proof (sketch): repeat basically the same as above by replacing $\mathbf{b}$ by $\mathbf{B}$ and $u(t)$ by $\mathbf{u}(t)$.

$$\hookrightarrow \cdots \hookrightarrow \mathbf{x}_0 = -\sum_{\nu=0}^{\infty}\mathbf{A}^\nu\mathbf{b}\mathbf{u}_\nu \tag{3.18}$$

Hence the columns of $\left[\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, \ldots, \mathbf{A}^{n-1}\mathbf{B}\right]$ have to span up the vector space $\mathbb{R}^n$. This condition is equal to $\mathrm{rank}(\mathscr{C}) = n$. Note that we can stop the sum at n due to the Cayley-Hamilton theorem.

□

Recap: Rank of a Matrix

$$\begin{aligned}\mathrm{rank}(\mathbf{M}) &= \text{number of linearly independent column vectors in } \mathbf{M} \\ &\qquad\text{(or alternatively)} \\ &= \text{number of linearly independent row vectors in } \mathbf{M}.\end{aligned}$$

Controllability for Discrete-time Linear Systems For discrete-time linear systems, the condition for controllability is also similar to the case for continuous-time linear systems: the system is controllable if and only if the controllability matrix as formulated in (3.17) has full rank n.

3.3.3 Stabilizability

Stabilizability is a weaker notion than controllability.

> **Stabilizability**
>
> The system $(\mathbf{A}, \mathbf{B})$ is stabilizable if there exist a matrix $\mathbf{K} \in \mathbb{R}^{p \times n}$ such that the matrix $\mathbf{A} - \mathbf{BK}$ is stable.

Recall that in the considered (continuous time) framework, a matrix $\mathbf{M}$ is stable if $\mathrm{Re}(\lambda_i) < 0$ for all eigenvalues λ_i of $\mathbf{M}$.

The idea of stabilizability is that all unstable modes of the system must be controllable, such that all eigenmodes of the matrix $\mathbf{A} - \mathbf{BK}$ can be made stable. That is formalized in the following theorem

> **Controllability and Stabilizability**
>
> If the system $(\mathbf{A}, \mathbf{B})$ is controllable, then it is stabilizable.

The converse is not true: as an example, a stable system with some uncontrollable modes is stabilizable (by choosing e.g. $\mathbf{K} = \mathbf{0}$) but not controllable.

3.3.4 Controllable canonical form

A controllable LTI system could be transformed into the *controllable canonical form*, which represent clearly the controllability of the system:

> **Controllable Canonical Form**
>
> $$\dot{\mathbf{x}}(t) = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & \vdots \\ \vdots & & \ddots & & \\ 0 & 0 & & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} u(t) \tag{3.19}$$

3.3.5 Transformation to the controllable canonical form

Given a controllable LTI system represented by $(\mathbf{A}, \mathbf{B})$, we want to find the transformation matrix $\mathbf{T}$ such that the transformed system obtained with the new state $\mathbf{z} = \mathbf{Tx}$ has the state-space representation $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$ in the controllable canonical form.

The values $a_1, \ldots, a_n$ of the matrix $\tilde{\mathbf{A}}$ if the controllable canonical form (3.19) are also coefficients of the characteristic polynomial:

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n\lambda \tag{3.20}$$

Note: A LTI system has the same characteristic polynomial, despite of the transformation. Hence, from the state-space representation $(\mathbf{A}, \mathbf{B})$ we can derive the characteristic polynomial, then pick the coefficients to construct the matrix $\tilde{\mathbf{A}}$.

The controllability matrix $\mathscr{C}$ associated with representation $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$ has a triangular structure (advantage: triangular structure simplifies linear algebra operations):

$$\tilde{\mathscr{C}} = \begin{bmatrix} 1 & -a_1 & a_1^2 - a_2 & \cdots & * \\ 0 & 1 & -a_1 & \dots & * \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{3.21}$$

With the transformation $\mathbf{z} = \mathbf{T}\mathbf{x}$, the relations between dynamics matrices and the control matrices of the original and the transformed systems are

$$\tilde{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}, \qquad \tilde{\mathbf{B}} = \mathbf{T}\mathbf{B} \tag{3.22}$$

The controllability matrix of the transformed system is

$$\tilde{\mathscr{C}} = [\ \tilde{\mathbf{B}} \quad \tilde{\mathbf{A}}\tilde{\mathbf{B}} \quad \dots \quad \tilde{\mathbf{A}}^{n-1}\tilde{\mathbf{B}}\ ]$$

in which each element can be expressed in terms of $\mathbf{A}, \mathbf{B}, \mathbf{T}$ as

$$\begin{aligned} \tilde{\mathbf{B}} &= \mathbf{T}\mathbf{B} \\ \tilde{\mathbf{A}}\tilde{\mathbf{B}} &= \mathbf{T}\mathbf{A}\mathbf{B} \\ &\dots \\ \tilde{\mathbf{A}}^{n-1}\tilde{\mathbf{B}} &= \mathbf{T}\mathbf{A}^{n-1}\mathbf{B} \end{aligned}$$

Hence, we get the relation between two controllability matrices:

$$\begin{aligned} \tilde{\mathscr{C}} &= [\ \mathbf{TB} \quad \mathbf{TAB} \quad \dots \quad \mathbf{TA}^{n-1}\mathbf{B}\ ] \\ &= \mathbf{T}[\ \mathbf{B} \quad \mathbf{AB} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}\ ] \\ \Rightarrow \tilde{\mathscr{C}} &= \mathbf{T}\mathscr{C} \end{aligned} \tag{3.23}$$

Notice that $\mathscr{C}$ is invertible (since the system is controllable), we can multiply $\mathscr{C}^{-1}$ to the right of two sides of (3.23), and get the formula for $\mathbf{T}$ that transform $(\mathbf{A}, \mathbf{B})$ to $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$:

$$\mathbf{T} = \tilde{\mathscr{C}}\mathscr{C}^{-1} \tag{3.24}$$