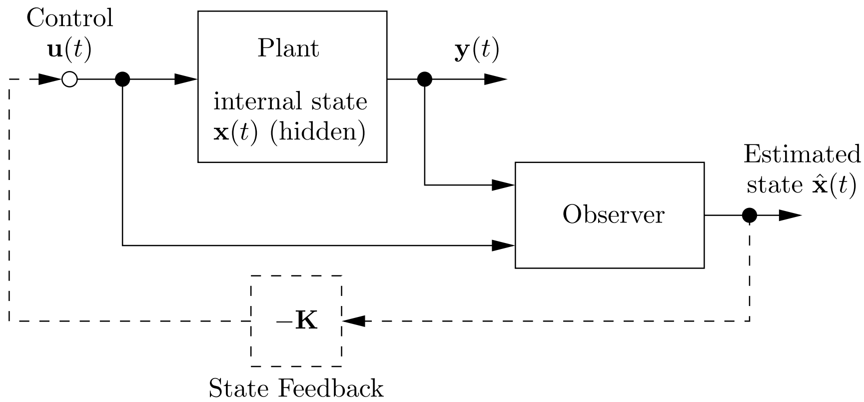
Chapter 4

Output Feedback Control

For many systems, we do not directly have the information on all the states, instead only outputs are measured. In order to implement a state feedback controller, the states of the systems need to be estimated (observed).

The task of an *observer* (also known as *state estimator*) is to reconstruct the (hidden) state vector of an system, which is based on the knowledge of the system dynamics and recorded inputs and outputs over time. The output feedback control diagram using an observer can be depicted as follows



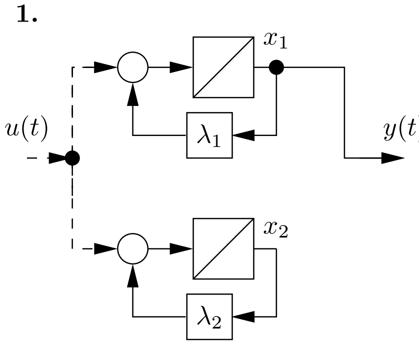
In this chapter, we study how to design an observer for a linear time-invariant system, and how to couple an observer and a state feedback controller to obtain an output feedback controller.

In the following section, we address the first question: is a linear system observable?

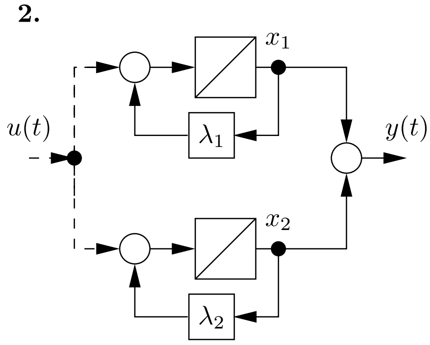
4.1 Observability for linear systems

Introductory examples: can the state $\mathbf{x}(t_0)$ be determined from $y(t)$?

1.



2.

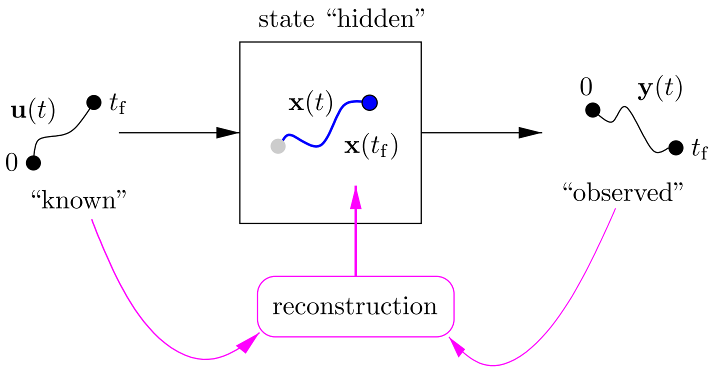


1. is not observable, as state variable x_2 is not connected to the output.
2. is observable if $\lambda_1 \neq \lambda_2$. Note that the output has to be observed for an interval of finite duration in order to discriminate the values x_1 and x_2.

Observability

A linear system is observable, if for every $t_\mathrm{f} > 0$, it is possible to determine the state of the system $\mathbf{x}(t_\mathrm{f})$ from the knowledge of the control input $u(t)$ and the output $y(t)$ over a finite time interval $[0, t_\mathrm{f}]$.

Illustration:



4.1.1 Checking observability

The observability of LTI systems can be checked using *observability matrix*, as follow:

Observability for MIMO systems

The observability matrix $\mathcal{O}$ is defined as

$$\mathcal{O} \triangleq \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \tag{4.1}$$

The system $(\mathbf{A}, \mathbf{C})$ is observable if and only if $\mathcal{O}$ is full rank (for $\mathbf{x}$ of size n, $\mathbf{y}$ of size q, then $\mathcal{O}$ is a matrix of size $(nq) \times n$, it is full rank if $\mathrm{rank}(\mathcal{O}) = n$.

In MATLAB, the command `obsv(A,C)` will compute the observability matrix with the given $\mathbf{A}, \mathbf{C}$ matrices.

4.1.2 Detectability

Detectability stands to observability similarly to how stabilizability stands to controllability. Namely, detectability is a weaker notion than observability.

Detectability

The system $(\mathbf{A}, \mathbf{C})$ is detectable if there exist a matrix $\mathbf{L} \in \mathbb{R}^{n \times q}$ such that the matrix $\mathbf{A} - \mathbf{LC}$ is stable.

The idea of detectability is that all unstable modes of the system must be observable, such that all modes of the system $(\mathbf{A} - \mathbf{LC}, \mathbf{C})$ can be made stable.

Observability and Detectability

If the system $(\mathbf{A}, \mathbf{C})$ is observable, then it is detectable.

The converse is not true: a stable system with some unobservable modes is detectable but might not be observable.

4.1.3 Observable canonical form

An observable SISO system could be transformed into the *observable canonical form*:

Observable Canonical Form

$$\dot{\mathbf{z}}(t) = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \ddots & \vdots \\ -a_3 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & & 0 & 1 \\ -a_n & 0 & \cdots & & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix} u(t) \tag{4.2}$$

$$y(t) = [1, 0, \ldots, 0]\, \mathbf{z}(t) \tag{4.3}$$

When a system is represented in the observable canonical form (4.2)-(4.3), the observability matrix has the triangular form:

$$\bar{\mathcal{O}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -a_1 & 1 & 0 & \ldots & 0 \\ a_1^2 - a_2 & -a_1 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ * & * & \cdots & * & 1 \end{bmatrix} \tag{4.4}$$

and it can be easily verified that the system is observable.
The values $a_1, \ldots, a_n$ are also coefficients of the characteristic polynomial of the system:

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n\lambda$$

It can be shown that the inverse of the observability matrix has a simple form given by

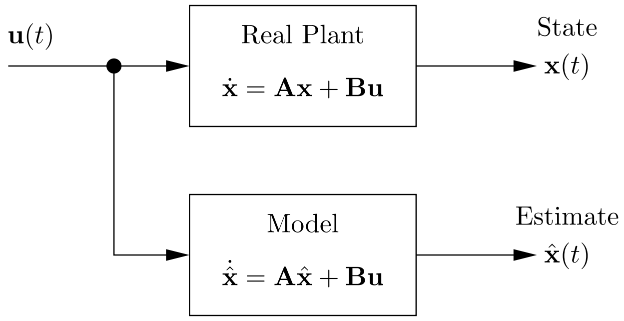
$$\bar{\mathcal{O}}^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \ldots & 0 \\ a_2 & a_1 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & 1 \end{bmatrix}$$

Note that the observability matrix of the observable canonical form is the transpose of the controllability matrix for the controllable canonical form of the same system (same characteristic polynomial), i.e. $\bar{\mathcal{O}} = \mathscr{C}^T$. Using the similar derivation like in section 3.3.5, we can show the relation between the observability matrices of the original and the observable canonical form: $\bar{\mathcal{O}} = \mathcal{O}\mathbf{T}$ where $\mathbf{T}$ is the transformation matrix for $\mathbf{z} = \mathbf{T}\mathbf{x}$, and hence we can compute $\mathbf{T}$ using $\mathbf{T} = \mathcal{O}^{-1}\bar{\mathcal{O}}$.

4.2 Luenberger observer

Luenberger observer is a popular observer type, where the state is estimated using a *predictor* that copies the dynamics of the real system, and a *corrector* that gives feedback on the error or the output.

In principle state estimation could be accomplished by the following prediction scheme

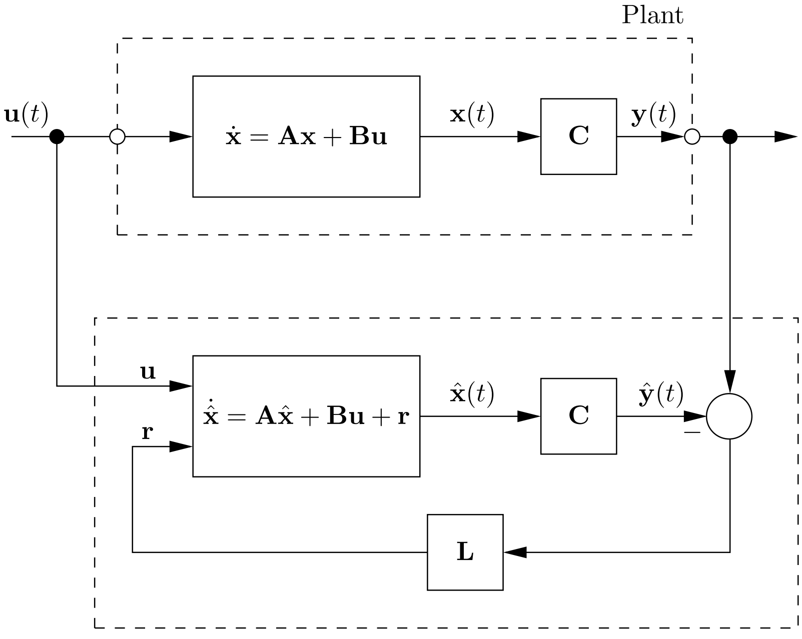


There are some prerequisites that $\hat{\mathbf{x}}(t)$ becomes a 'good' estimate for the state vector $\mathbf{x}(t)$.

- The system has to be *stable*.
- Absence of significant disturbances.
- Model should be accurate.

In order to obtain a better estimate or make the estimation feasible for unstable plants, a *feedback* is introduced, which is the correction part.

This leads to the **Luenberger observer** depicted in the following



where $\mathbf{L}$ is the feedback matrix.

The ODE for the observer reads

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{r}(t) \tag{4.5}$$

Insertion of

$$\mathbf{r}(t) = \mathbf{L}(\mathbf{y}(t) - \hat{\mathbf{y}}(t)) = \mathbf{L}\mathbf{y}(t) - \mathbf{L}\mathbf{C}\hat{\mathbf{x}}(t) \tag{4.6}$$

yields

$$\dot{\hat{x}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}\mathbf{y}(t) \tag{4.7}$$

Considering the ODE for the estimation error, defined by

$$\mathbf{e}(t) \triangleq \mathbf{x}(t) - \hat{\mathbf{x}}(t) \tag{4.8}$$

gives with $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$

$$\begin{aligned} \dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) - (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(t) - \mathbf{B}\mathbf{u}(t) - \mathbf{L}\mathbf{y}(t) \\ &= (\mathbf{A} - \mathbf{L}\mathbf{C})\left(\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right) \end{aligned} \tag{4.9}$$

Hence the dynamics is described by the state equation

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t) \tag{4.10}$$

In order to obtain a reasonable estimate, we demand for the following

- The observer must be *stable*, i.e. $\mathbf{e}(t) \to 0$ for $t \to \infty$.
- As a consequence, the real parts of the eigenvalues of $(\mathbf{A} - \mathbf{L}\mathbf{C})$ must be negative $\mathrm{Re}(\lambda_i) < 0$ for $i = 1, \ldots, n$.
- The speed of the observer is determined by the locations of the eigenvalues of $(\mathbf{A} - \mathbf{L}\mathbf{C})$.

Eigenvalue assignment for Luenberger observer We tune the gain $\mathbf{L}$ to design the Luenberger observer, so that the eigenvalues of $(\mathbf{A} - \mathbf{L}\mathbf{C})$ are at desired locations. The process to assign eigenvalues of the observer is similar to the eigenvalue assignment of the state feedback controller. The eigenvalue assignment for the observer can be derived using the observable canonical form $(\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}, \bar{\mathbf{D}})$ as a bridge.

Eigenvalue assignment for Observable canonical form (SISO)

For a system given in observable canonical form (4.2)-(4.3), the characteristic polynomial

$$p(\lambda) = \lambda^n + l_1\lambda^{n-1} + \cdots + l_{n-1}\lambda + l_n \tag{4.11}$$

is implemented by the feedback

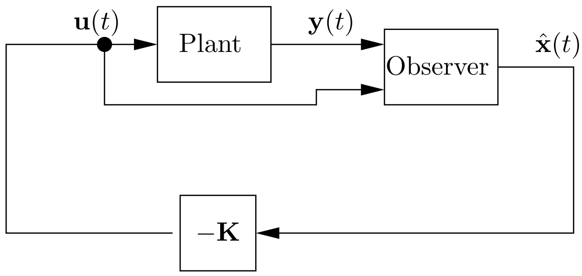
$$\bar{\mathbf{L}} = \begin{bmatrix} (l_1 - a_1) \\ \vdots \\ (l_n - a_n) \end{bmatrix} \tag{4.12}$$

The gain of the observer for the system is original state space $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is obtained by:

$$\mathbf{L} = \mathbf{T}\bar{\mathbf{L}} = \mathcal{O}^{-1}\bar{\mathcal{O}} \begin{bmatrix} (l_1 - a_1) \\ \vdots \\ (l_n - a_n) \end{bmatrix} \tag{4.13}$$

4.3 Control loop with state feedback and observer

In this section, a state feedback of the *estimated* state vector will be considered as follows



Hence the feedback is given by

$$\mathbf{u}(t) = -\mathbf{K}\hat{\mathbf{x}}(t) \tag{4.14}$$

Combining the plant system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{B}\mathbf{k}_r\mathbf{r}(t) \tag{4.15}$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{4.16}$$

and the state equation of the error (4.10):

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t) \tag{4.17}$$

into a set of ODE for the combined system yields

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{K}) & \mathbf{B}\mathbf{K} \\ \mathbf{0} & (\mathbf{A} - \mathbf{L}\mathbf{C}) \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}\mathbf{k}_r \\ \mathbf{0} \end{bmatrix} \mathbf{r} \tag{4.18}$$

The eigenvalues of the combined system are the roots of the characteristic polynomial, which is the determinant of the matrix $\mathcal{A}$ and can be decomposed due to the block-diagonal structure:

$$\begin{aligned} 0 = p(\lambda) &= \det\left(\begin{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{K}) & \mathbf{B}\mathbf{K} \\ \mathbf{0} & (\mathbf{A} - \mathbf{L}\mathbf{C}) \end{bmatrix}\right) \\ &= \underbrace{\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K}))}_{\text{state feedback}} \underbrace{\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{L}\mathbf{C}))}_{\text{observer}} \end{aligned} \tag{4.19}$$

Consequently, we could see that the eigenvalues of the closed-loop system are the union of state feedback eigenvalues and observer eigenvalues, this is called **separation theorem**. Based on this, the state feedback design can be carried out independently from the observer.

On the choice of eigenvalues for the observer, the following could be stated

- The eigenvalues should be placed to the left of the closed loop eigenvalues, otherwise the reaction of the system to disturbances, which cause differences between the state of the plant and the estimate, would be too slow.
- Theoretically, the observer could be made arbitrarily fast. As the algorithm involves differentiation, this is critical w.r.t. noise in measurements. Hence, the observer should be made faster than the state feedback, but not significantly faster.

4.4 Kalman decomposition

Kalman and his collaborators have shown that the two properties of controllability and observability can be used to classify the dynamics of a system. The key result is Kalman's decomposition theorem, which says that a linear system can be divided into four subsystems: S_{co} which is controllable and observable, $S_{\mathrm{c\bar{o}}}$ which is controllable but not observable, $S_{\mathrm{\bar{c}o}}$ which is not controllable but is observable, and $S_{\mathrm{\bar{c}\bar{o}}}$ which is neither controllable nor observable.

For the special case of systems with one input and one output, and where the matrix $\mathbf{A}$ has distinct eigenvalues, we can find a set of coordinates such that the $\mathbf{A}$ matrix is diagonal and, with some additional reordering of the states, the system can be written as:

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \mathbf{A}_{\mathrm{co}} & 0 & 0 & 0 \\ 0 & \mathbf{A}_{\mathrm{c\bar{o}}} & 0 & 0 \\ 0 & 0 & \mathbf{A}_{\mathrm{\bar{c}o}} & 0 \\ 0 & 0 & 0 & \mathbf{A}_{\mathrm{\bar{c}\bar{o}}} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{B}_{\mathrm{co}} \\ \mathbf{B}_{\mathrm{c\bar{o}}} \\ 0 \\ 0 \end{bmatrix} \mathbf{u},$$
$$\mathbf{y} = \begin{bmatrix} \mathbf{C}_{\mathrm{co}} & 0 & \mathbf{C}_{\mathrm{\bar{c}o}} & 0 \end{bmatrix} \mathbf{x} + \mathbf{D}\mathbf{u}$$

All states x_k such that $B_k \neq 0$ are controllable, and all states such that $C_k \neq 0$ are observable. If we set the initial states to zero, the states given by $\mathbf{x}_{\bar{c}o}$ and $\mathbf{x}_{\bar{c}\bar{o}}$ will be zero, and $\mathbf{x}_c\bar{o}$ does not affect the output. Hence the output $\mathbf{y}$ can be determined from the system:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}_{\mathrm{co}}\mathbf{x}_{\mathrm{co}} + \mathbf{B}_{\mathrm{co}}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}_{\mathrm{co}}\mathbf{x}_{\mathrm{co}} + \mathbf{D}\mathbf{u}$$

Thus from the input/output point of view, it is **only the controllable and observable dynamics that matter**.

The general case of the Kalman decomposition is more complicated. The key result is that the state space can still be decomposed into four parts, but there will be additional coupling so that the equations have the form

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \mathbf{A}_{\mathrm{co}} & 0 & * & 0 \\ * & \mathbf{A}_{\mathrm{c\bar{o}}} & * & * \\ 0 & 0 & \mathbf{A}_{\mathrm{\bar{c}o}} & 0 \\ 0 & 0 & * & \mathbf{A}_{\mathrm{\bar{c}\bar{o}}} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{B}_{\mathrm{co}} \\ \mathbf{B}_{\mathrm{c\bar{o}}} \\ 0 \\ 0 \end{bmatrix} \mathbf{u},$$
$$\mathbf{y} = \begin{bmatrix} \mathbf{C}_{\mathrm{co}} & 0 & \mathbf{C}_{\mathrm{\bar{c}o}} & 0 \end{bmatrix} \mathbf{x} + \mathbf{D}\mathbf{u}$$

where $*$ denotes coupling blocks of appropriate dimensions.