## 4.5 Kalman Filter

Kalman filter is an optimal state estimator, which is widely used in many disciplines. In this section, we study the derivation of the discrete Kalman filter, and see that it is a dual concept to LQR. We also provide results on continuous Kalman filter, without derivation.

### 4.5.1 Discrete Kalman Filter

Consider the discrete-time LTI system with the disturbance vector $\mathbf{w}$ and the noise vector $\mathbf{v}$:

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k] + \mathbf{w}[k] && (4.20) \\ \mathbf{y}[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{v}[k] && (4.21) \end{aligned}$$

in which $\mathbf{w}$ is called *process noise* or *disturbance*, $\mathbf{v}$ is called *measurement noise*, these signals are uncertainties (we don't know the values, even after they have appeared).

Due to the uncertainties, the output that we measured, denoted $\hat{\mathbf{y}}$ is not the true output $\mathbf{y}$, hence the job of state observation - computing the estimated state $\hat{\mathbf{x}}$ - depends also on the characteristic of $\mathbf{w}$ and $\mathbf{v}$.

We assume that $\mathbf{w}$ and $\mathbf{v}$ are independent zero-mean, Gaussian white noise, with covariances: $\mathbb{E}(\mathbf{w}[k]\mathbf{w}[k]^T) = \mathbf{Q_w}[k]$, $\mathbb{E}(\mathbf{v}[k]\mathbf{v}[k]^T) = \mathbf{R_v}[k]$. In case of stationary noises, $\mathbf{Q_w}$ and $\mathbf{R_v}$ are constants.

We use the Luenberger observer for estimating states:

$$\hat{\mathbf{x}}[k+1] = \mathbf{A}\hat{\mathbf{x}}[k] + \mathbf{B}\mathbf{u}[k] + \mathbf{L}[k](\mathbf{y}[k] - \mathbf{C}\hat{\mathbf{x}}[k]) \tag{4.22}$$

The estimation error $\mathbf{e}[k] = \mathbf{x}[k] - \hat{\mathbf{x}}[k]$ has the dynamics

$$\mathbf{e}[k+1] = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}[k] + \mathbf{w}[k] - \mathbf{L}\mathbf{v}[k] \tag{4.23}$$

In 1961, Kalman has published his work on state estimation, which is then named *Kalman filter*, the target is to minimize the mean square error:

$$J[k] = \mathbb{E}(e_1[k]^2 + \cdots + e_n[k]^2) = \sum_{j=1}^{n} \mathbb{E}(e_j[k]^2) \tag{4.24}$$

This corresponds to the sum of diagonal entries of the covariance matrix:

$$\mathbf{P}[k] = \mathbb{E}(\mathbf{e}[k]\mathbf{e}[k]^T) = \mathbb{E}\left((\mathbf{x}[k] - \hat{\mathbf{x}}[k])(\mathbf{x}[k] - \hat{\mathbf{x}}[k])^T\right)$$

Note that the sum of diagonal entries of a square matrix is called the *trace* of the matrix, and it is a linear operator on the matrices, i.e. following properties hold:

$$\begin{aligned} \mathrm{tr}(\mathbf{A} + \mathbf{B}) &= \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B}) \\ \mathrm{tr}(c\mathbf{A}) &= c\,\mathrm{tr}(\mathbf{A}) \end{aligned}$$

Some other properties of the *trace* operator:

$$\begin{aligned} \mathrm{tr}(\mathbf{A}) &= \mathrm{tr}(\mathbf{A}^T) \\ \mathrm{tr}(\mathbf{A}) &= \sum_{i=1}^{n} \lambda_i, \end{aligned}$$

where $\lambda_i$ are eigenvalues of $\mathbf{A} \in \mathbb{R}^{n\times n}$.

In summary, the observer needs to minimize $\mathrm{tr}(\mathbf{P}[k])$ for each time step $k$.

We want to find $\mathbf{P}$ in a recursive manner, given $\mathbf{P}[0] = \mathbb{E}\left((\mathbf{x}[0] - \hat{\mathbf{x}}_0)(\mathbf{x}[0] - \hat{\mathbf{x}}_0)^T\right)$.

With $\mathbf{P}[k]$ available, we want to find the relation between $\mathbf{P}[k+1]$ and $\mathbf{P}[k]$:

$$
\begin{aligned}
\mathbf{P}[k+1] = \mathbf{e}[k+1]\mathbf{e}[k+1]^T &= \mathbb{E}\left((\mathbf{A}-\mathbf{LC})\mathbf{e}[k]+\mathbf{w}[k]-\mathbf{Lv}[k])(\mathbf{A}-\mathbf{LC})\mathbf{e}[k]+\mathbf{w}[k]-\mathbf{Lv}[k])^T\right) \\
&= (\mathbf{A}-\mathbf{LC})\mathbb{E}(\mathbf{e}[k]\mathbf{e}[k]^T)(\mathbf{A}-\mathbf{LC})^T + \mathbb{E}(\mathbf{w}[k]\mathbf{w}[k]^T) + \mathbf{L}\mathbb{E}(\mathbf{v}[k]\mathbf{v}[k]^T)\mathbf{L}^T \\
&= (\mathbf{A}-\mathbf{LC})\mathbf{P}[k](\mathbf{A}-\mathbf{LC})^T + \mathbf{Q_w}[k] + \mathbf{LR_v}[k]\mathbf{L}^T \\
&= \mathbf{AP}[k]\mathbf{A}^T + \mathbf{Q_w}[k] - \mathbf{AP}[k]\mathbf{C}^T\mathbf{L}^T - \mathbf{LCP}[k]\mathbf{A}^T + \mathbf{L}(\mathbf{R_v}[k]+\mathbf{CP}[k]\mathbf{C}^T)\mathbf{L}^T
\end{aligned}
$$

Note that $\mathbf{w}$, $\mathbf{v}$ and $\mathbf{e}$ are uncorrelated, hence the covariances of cross-terms are zeros.

Denote $\mathbf{R}_\epsilon[k] = (\mathbf{R_v}[k] + \mathbf{CP}[k]\mathbf{C}^T)$, we obtain further:

$$
\begin{aligned}
\mathbf{P}[k+1] &= \mathbf{AP}[k]\mathbf{A}^T + \mathbf{Q_w}[k] - \mathbf{AP}[k]\mathbf{C}^T\mathbf{L}^T - \mathbf{LCP}[k]\mathbf{A}^T + \mathbf{LR}_\epsilon[k]\mathbf{L}^T \\
&= \mathbf{AP}[k]\mathbf{A}^T + \mathbf{Q_w}[k] + (\mathbf{L} - \mathbf{AP}[k]\mathbf{C}^T\mathbf{R}_\epsilon^{-1})\mathbf{R}_\epsilon(\mathbf{L}-\mathbf{AP}[k]\mathbf{C}^T)^T - \mathbf{AP}[k]\mathbf{C}^T\mathbf{R}_\epsilon[k]^{-1}\mathbf{CP}[k]\mathbf{A}^T
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\mathrm{tr}\mathbf{P}[k+1] = {} & \mathrm{tr}\left(\mathbf{AP}[k]\mathbf{A}^T\right) + \mathrm{tr}\left(\mathbf{Q_w}[k]\right) + \mathrm{tr}\left(\mathbf{L} - \mathbf{AP}[k]\mathbf{C}^T\mathbf{R}_\epsilon[k]^{-1})\mathbf{R}_\epsilon[k](\mathbf{L}-\mathbf{AP}[k]\mathbf{C}^T)^T\right) \\
& - \mathrm{tr}\left(\mathbf{AP}[k]\mathbf{C}^T\mathbf{R}_\epsilon[k]^{-1}\mathbf{CP}[k]\mathbf{A}^T\right) \qquad (4.25)
\end{aligned}
$$

Since the matrix in the third term is positive semidefinite, its mininum trace would be 0 when we choose $\mathbf{L} = \mathbf{AP}[k]\mathbf{C}^T\mathbf{R}_\epsilon[k]^{-1}$. This leads to the recursive Kalman filter:

**Recursive discrete Kalman filter**

The optimal gain at time step $k$ is

$$
\mathbf{L}[k] = \mathbf{AP}[k]\mathbf{C}^T(\mathbf{R_v} + \mathbf{CP}[k]\mathbf{C}^T)^{-1} \qquad (4.26)
$$

where

$$
\begin{aligned}
\mathbf{P}[k+1] &= (\mathbf{A}-\mathbf{LC}[k])\mathbf{P}[k](\mathbf{A}-\mathbf{LC}[k])^T + \mathbf{Q_w}[k] + \mathbf{LR_v}[k]\mathbf{L}^T, \qquad (4.27) \\
\mathbf{P}[0] &= \mathbb{E}\left((\mathbf{x}[0]-\mathbf{x}_0)(\mathbf{x}[0]-\mathbf{x}_0)^T\right) \qquad (4.28)
\end{aligned}
$$

When the system is invariant, the noises are stationary, and if $\mathbf{P}[k]$ converges, the observer gain is constant:

$$
\mathbf{L} = \mathbf{APC}^T(\mathbf{R_v} + \mathbf{CPC}^T)^{-1}, \qquad (4.29)
$$

where $\mathbf{P}$ satisfies:

$$
\mathbf{P} = \mathbf{APA}^T + \mathbf{Q_w} - \mathbf{APC}^T(\mathbf{R_v} + \mathbf{CPC}^T)^{-1}\mathbf{CPA}^T \qquad (4.30)
$$

Equation (4.30) is the *discrete algebraic Riccati equation* (DARE), see (3.62). In MATLAB, the command `dlqe` will solve this DARE and also computes the gain $\mathbf{L}$ as in (4.29).

### 4.5.2 Continuous Kalman Filter

A similar result for Kalman filter is also available for the continuous setting. Consider the LTI system with disturbance and measurement noise:

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{Ax}(t) + \mathbf{Bu}(t) + \mathbf{w}(t) \qquad (4.31) \\
\mathbf{y}(t) &= \mathbf{Cx}(t) + \mathbf{v}(t) \qquad (4.32)
\end{aligned}
$$

where $\mathbf{w}$ and $\mathbf{v}$ are independent zero-mean, Gaussian white noise, with covariances: $\mathbb{E}(\mathbf{ww}^T) = \mathbf{Q_w}$, $\mathbb{E}(\mathbf{vv}^T) = \mathbf{R_v}$.

Below is the recursive Kalman filter for this continuous system.

**Recursive continuous Kalman filter**

The optimal estimator has the form of a linear observer:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}), \quad \hat{\mathbf{x}}(0) = \mathbb{E}(\mathbf{x}(0))$$

with the gain

$$\mathbf{L} = \mathbf{P}\mathbf{C}^T\mathbf{R}_{\mathbf{v}}^{-1}, \text{where } \mathbf{P} = \mathbb{E}\left((\mathbf{x} - \hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}})^T\right)$$

and the covariance $\mathbf{P}$ satisfies

$$\dot{\mathbf{P}} = \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T - \mathbf{P}\mathbf{C}^T\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{C}\mathbf{P} + \mathbf{Q}_{\mathbf{w}}, \tag{4.33}$$

$$\mathbf{P}(0) = \mathbb{E}\left((\mathbf{x}(0) - \mathbf{x}_0)(\mathbf{x}(0) - \mathbf{x}_0)^T\right) \tag{4.34}$$

All matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{R}_{\mathbf{v}}$, $\mathbf{R}_{\mathbf{w}}$, $\mathbf{P}$ and $\mathbf{L}$ can be time varying. The essential condition is that the Riccati equation (4.33) has a unique positive solution.

When the system is invariant, and if $\mathbf{P}(t)$ converges, we obtain the constact observer gain

$$\mathbf{L} = \mathbf{P}\mathbf{C}^T\mathbf{R}_{\mathbf{v}}^{-1}, \tag{4.35}$$

and $\mathbf{P} = \mathbf{P}^T \succ 0$ is the unique solution to the *algebraic Riccati equation*:

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T - \mathbf{P}\mathbf{C}^T\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{C}\mathbf{P} + \mathbf{Q}_{\mathbf{w}} = 0 \tag{4.36}$$

In MATLAB, the command `lqe` will solve this ARE and also computes the gain $\mathbf{L}$ as in (4.35).

### 4.5.3 Two-step Kalman filter

In practice, usually we prefer to use the latest measurement $\mathbf{y}[k+1]$ instead of $\mathbf{y}[k]$ to correct the predicted state in (4.22), so that the new estimated state $\mathbf{x}[k+1]$ uses the latest available information. This leads to the presentation of Kalman filter with two steps. Below we provide the formula for this Kalman filter for the discrete case (there would be no difference for the continuous case), the derivation of this representation is left as an exercise.

**Two-step recursive discrete Kalman filter**

Given: initial mean state $\bar{\mathbf{x}}_0$, covariances $\mathbf{Q}[k], \mathbf{R}[k]$, with $k = 0, 1, \ldots$.
Iterate for each $k = 0, 1, \ldots$:

1. *Prediction step*: project the state and covariance to the next sampling time:

$$\hat{\mathbf{x}}[k+1]^- = \mathbf{A}\hat{\mathbf{x}}[k] \tag{4.37}$$

$$\mathbf{P}[k+1]^- = \mathbf{A}\mathbf{P}[k]\mathbf{A}^T + \mathbf{Q}[k] \tag{4.38}$$

$$\hat{\mathbf{x}}[0]^- = \bar{\mathbf{x}}_0, \quad \mathbf{P}[0]^- = \mathbf{Q}[0] \tag{4.39}$$

   Note that we use superscript $^-$ to represent the prediction using the information from previous sampling time.

2. Compute Kalman gain for the next sampling time:

$$\mathbf{L}[k+1] = \mathbf{P}[k+1]^-\mathbf{C}^T(\mathbf{C}\mathbf{P}[k+1]^-\mathbf{C}^T + \mathbf{R}[k+1])^{-1} \tag{4.40}$$

3. *Innovation step*: After obtaining the new measurement $\mathbf{y}[k+1]$, perform an update on estimated state and covariance:

$$\begin{aligned}\hat{\mathbf{x}}[k+1] &= \hat{\mathbf{x}}[k+1]^- + \mathbf{L}[k+1](\mathbf{y}[k+1] - \hat{\mathbf{y}}[k+1]^-) \\ &= \hat{\mathbf{x}}[k+1]^- + \mathbf{L}[k+1](\mathbf{y}[k+1] - \mathbf{C}\hat{\mathbf{x}}[k+1]^-) \qquad (4.41)\end{aligned}$$

$$\mathbf{P}[k+1] = \mathbf{P}[k+1]^- - \mathbf{L}[k+1]\mathbf{C}\mathbf{P}[k+1]^- \tag{4.42}$$