Numerical Optimal Control – Exam

Prof. Dr. Moritz Diehl, IMTEK, Universität Freiburg, September 25th, 2019, 9:00-12:00, Freiburg,
Georges-Koehler-Allee 101 SR 01 016

Page	0	1	2	3	4	5	6	7	8	9	Sum
Points on page (max)	3	10	11	11	9	8	10	8	10	0	80
Points obtained											
Intermediate sum											
Intermediate sum (max)	3	13	24	35	44	52	62	70	80	80	

Mark: Exam inspected on: Signature of examiner:

Surname: First Name: Matriculation number:

Subject: Programme: Bachelor ☐ Master ☐ Lehramt ☐ others ☐ Signature:

Please fill in your name above. For the multiple choice questions, which give exactly one point, tick exactly one box for the right answer. For the text questions, give a short formula or text answer just below the question in the space provided and, if necessary, write on the **backpage of the same sheet** where the question appears, and add a comment “see backpage”. Do not add extra pages (for fast correction, all pages will be separated for parallelization). The exam is a closed book exam, i.e. no books or other material are allowed besides 2 sheets (with 4 pages) of notes and a non-programmable calculator. Some legal comments are found in a footnote[1].

1. Which of the following functions $f(x)$, $f : \mathbb{R}^n \to \mathbb{R}$, is NOT convex ($c \in \mathbb{R}^n$)?

(a) ☐ $\exp(-c^\top x)$	(b) ☐ $\exp(x^\top x)$	(c) ☐ $\exp(-x^\top x)$	(d) ☐ $\exp(c^\top x)$

1

2. Which of the following sets is NOT convex ($x, c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$)?

(a) ☐ $\{x \mid \|Ax\|_2^2 \leq 1\}$	(b) ☐ $\{x \mid x^\top x \geq 1\}$	(c) ☐ $\{x \mid \|x\|_1 \leq 1\}$	(d) ☐ $\{x \mid c^\top x \geq 0\}$

1

3. Which of the following functions $f(x)$, $f : \mathbb{R}^n \to \mathbb{R}$, is NOT globally Lipschitz continuous ($A \in \mathbb{R}^{m\times n}$)?

(a) ☐ $\cos(\|x\|_2)$	(b) ☐ $\sqrt{x^\top x}$	(c) ☐ $\sqrt{\|x\|_2}$	(d) ☐ $\|Ax\|_2$

1

[1]WITHDRAWING FROM AN EXAMINATION: In case of illness, you must supply proof of your illness by submitting a medical report to the Examinations Office. Please note that the medical examination must be done at the latest on the same day of the missed exam. In case of illness while writing the exam please contact the supervisory staff, inform them about your illness and immediately see your doctor. The medical certificate must be submitted latest 3 days after the medical examination. More information’s: http://www.tf.uni-freiburg.de/studies/exams/withdrawing_exam.html
CHEATING/DISTURBING IN EXAMINATIONS: A student who disrupts the orderly proceedings of an examination will be excluded from the remainder of the exam by the respective examiners or invigilators. In such a case, the written exam of the student in question will be graded as ’nicht bestanden’ (5.0, fail) on the grounds of cheating. In severe cases, the Board of Examiners will exclude the student from further examinations.

4. Consider a primal-dual interior point method. What can be said about the iterates computed by the algorithm?

(a) ☐ they converge sublinearly to a local minimum	(b) ☐ the Hessian is always positive definite
(c) ☐ they do not require globalization	(d) ☐ they can be infeasible with respect to constraints

1	

5. How does CasADi compute derivatives?

(a) ☐ Algorithmic Differentiation	(b) ☐ Finite differences
(c) ☐ Symbolic Differentiation	(d) ☐ Imaginary trick

1	

6. How many optimization variables does the NLP arising in the direct **multiple shooting** method have, if the system has n_x states, n_u controls, the initial value is fixed, and the time horizon is divided into N control intervals (piecewise-constant)?

(a) ☐ $Nn_x^3 + Nn_u^2$	(b) ☐ Nn_u	(c) ☐ $\frac{1}{3}N^3n_u^3$	(d) ☐ $(N+1)n_x + Nn_u$

1	

7. How many optimization variables does the NLP arising in the direct **single shooting** method have, if the system has n_x states, n_u controls, the initial value is fixed, and the time horizon is divided into N control intervals (piecewise-constant)?

(a) ☐ $\frac{1}{3}N^3n_u^3$	(b) ☐ Nn_u	(c) ☐ $(N+1)n_x + Nn_u$	(d) ☐ $Nn_x^3 + Nn_u^2$

1	

8. Newton's Method: Consider the function $f(x) = 1 + x^5$. We want to solve the root finding problem $f(x) = 0$ with (exact) Newton's Method. Given the current iterate x_k, write down – as explicitly as possible – the formula that determines the next iterate x_{k+1}.

2	

9. Algorithmic Differentiation: Give one advantage and one disadvantage of the backward mode of AD compared to the forward mode when evaluating the gradient of a scalar function $f : \mathbb{R}^n \to \mathbb{R}$.

4	

Points on page (max. 10)	

10. Equality constrained optimization: Regard the numerical solution of the constrained nonlinear least-squares problem

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|F(x)\|_2^2 \quad \text{subject to} \quad g(x) = 0$$

by the constrained Gauss-Newton (CGN) Method.

(a) Write down the Gauss-Newton Hessian approximation $B(x)$.

2	

(b) State the Quadratic Program (QP) which CGN is solving at every iteration. Denote the current iterate by x_k.

2	

(c) From the KKT conditions of the QP, derive the equation that determines the next iterate $z_{k+1} = (x_{k+1}, \lambda_{k+1})$ from the current iterate x_k, where λ denotes the Lagrange multipliers.

4	

(d) Name the local convergence rate of the CGN method and the two conditions under which it converges faster.

3	

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11. Forward Algorithmic Differentiation: regard the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by the code

```
function f = myfunction(x1,x2,x3)
   v1 = x1 * x2;
   v2 = exp(v1);
   f = v2 / x3;
end
```

We now write an algorithm that – given values for $x \in \mathbb{R}^3$ and a seed $\dot{x} \in \mathbb{R}^3$ – computes the directional derivative $\nabla f(x)^\top \dot{x}$ via the forward mode of AD. Add the missing three lines to the following template. Use the variable names `v1dot`, `v2dot`, `fdot` in the intermediate lines.

```
function [f, fdot] = forwardAD(x1,x2,x3,x1dot,x2dot,x3dot)
   v1 = x1 * x2;

   ...
   v2 = exp(v1);

   ...
   f = v2 / x3;

   ...
end
```

5	

12. Numerical Integration: Consider the scalar system $\dot{x}(t) = -\lambda x(t)$ with $\lambda \in \mathbb{R}, \lambda > 0$. We want to integrate it using explicit Euler with timestep $h > 0$.

(a) Give the formula for computing x_{k+1} from x_k doing one step of explicit Euler for this system.

2	

(b) Given an initial value x_0, derive an *explicit* formula for state x_k after k integration steps.

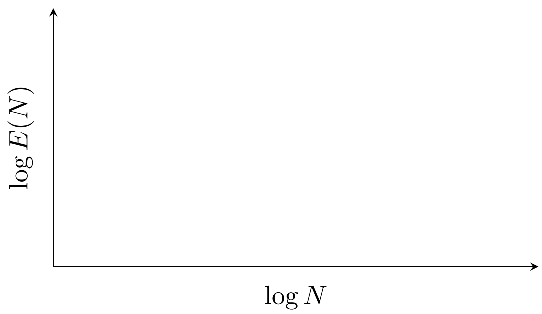
2	

(c) Derive for which values of h (depending on λ) this integrator is stable.

2	

Points on page (max. 11)	

13. Numerical Integration: Consider the numerical integration of the ODE $\dot{x} = f(x)$ over fixed time interval T, starting at initial value $x(0) = \bar{x}_0$. We compare explicit Euler and explicit Runge-Kutta of 4th order (RK4). For both methods, draw a sketch of the relation between the number of integration steps N and the integration error $E(N) = \|\tilde{x}(N) - x(T)\|$, where $x(T)$ is the true value at time T, and $\tilde{x}(N)$ denotes the result of numerical integration using N steps for the time interval $[0, T]$. Both axes are logarithmically scaled.



3

14. Dynamic Programming: We want to solve the following discrete time optimal control problem (OCP) given by

$$\begin{aligned}
\underset{x_0, u_0, \ldots, u_{N-1}, x_N}{\text{minimize}} \quad & \sum_{k=0}^{N-1} L(x_k, u_k) \quad + \quad E(x_N) \\
\text{subject to} \quad & \\
x_0 &= \bar{x} \\
x_{k+1} &= f(x_k, u_k), \quad k = 0, 1, \ldots, N-1, \\
x_{\min} \leq x_k &\leq x_{\max}, \quad k = 0, 1, \ldots, N, \\
u_{\min} \leq u_k &\leq u_{\max}, \quad k = 0, 1, \ldots, N-1.
\end{aligned}$$

where $x_i \in \mathbb{R}^{n_x}$, $u_i \in \mathbb{R}^{n_u}$.

(a) Briefly describe how this problem can be solved with Dynamic Programming.

4

(b) Give one advantage and one disadvantage of Dynamic Programming compared to Newton Type Optimization.

2

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15. Linear Quadratic Dynamic Programming: Use dynamic programming to compute the parametric optimal solution $u_0^*(\bar{x}_0)$ and the associated optimal cost $J^*(\bar{x}_0)$ to the following simple discrete time OCP with scalar state and scalar control where $x_0, x_1, x_2 \in \mathbb{R}$ and $u_0, u_1 \in \mathbb{R}$. Compute the intermediate cost-to-go and control policy for every step of the recursion. *Hint: notice that the cost is quadratic and convex and the system is linear.*

$$\begin{aligned}
\min_{x_0,x_1,x_2,u_0,u_1} \quad & \sum_{k=0}^{1} u_k^2 \ + \ x_2^2 \\
\text{s.t.} \quad & x_0 = \bar{x}_0 \\
& x_{k+1} = x_k + u_k, \quad k = 0, 1.
\end{aligned}$$

8	

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16. Pontryagin's Minimum Principle.

(a) Consider the following problem formulation:

$$\begin{aligned} \min_{x(\cdot),u(\cdot)} \quad & \int_0^T L(x(t),u(t))\mathrm{d}t + E(x(T)) \\ \text{s.t.} \quad & x_0 - \bar{x}_0 = 0 \\ & \dot{x}(t) - f(x(t),u(t)) = 0, \quad t \in [0,\, T]. \end{aligned}$$

State the boundary value problem (BVP) resulting from Pontryagin's Minimum Principle for the problem above.

4	

(b) Using Pontryagin's Minimum Principle, compute the explicit solution to the following optimal control problem:

$$\begin{aligned} \min_{x,u} \quad & \int_0^1 (x+u^2)\mathrm{d}t \\ \text{s.t.} \quad & \dot{x} = u \\ & x(0) = 0. \end{aligned}$$

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Points on page (max. 10)	

17. Direct Multiple Shooting: Consider the following dynamical system with constraints on the initial and the final state:

$$\dot{x}(t) = f(x(t), u(t)) \text{ for } t \in [0, T], \quad x(0) = \bar{x}_0, \quad x(T) = \bar{x}_\mathrm{F}$$

with state $x(t) \in \mathbb{R}^{n_x}$ and controls $u(t) \in \mathbb{R}^{n_u}$. The objective function is

$$\int_0^T g(x(t), u(t))\,\mathrm{d}t + M(x(T)).$$

This problem will be discretized using direct multiple shooting, with piecewise constant controls over N equidistant intervals. You don't have to explicitly write out an integration scheme, you can just write something like:

$$s_{k+1} \quad = \quad F(s_k, u_k).$$

(a) Write down the NLP for this problem, specifying the optimization variables, the objective function and the constraints. Explain how the NLP functions are obtained from the original OCP. Be sure to write down the dimensions of all optimization variables and to give correct ranges for all indices.

8	

Points on page (max. 8)

(b) For the obtained NLP, derive the first order necessary optimality conditions. Make sure to explicitly state the equations in a stage-wise fashion.

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(c) Now assume $N = 3$, $n_x = 2$, $n_u = 1$. Sketch the sparsity pattern a) of the Jacobian of the equality constraints and b) the Hessian of the Lagrangian.

5	

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