Lecture Notes on Wind Energy Systems

Summer Semester 2020

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Edition I

Contents

Preface

This manuscript is based on lecture notes of the Wind Energy Systems (WES) master course given by Moritz Diehl in the summer semesters 2018 and 2020 at the University of Freiburg. This course was supported by Rachel Leuthold and Nick Harder who gave the exercise sessions, but also contributed significantly to the organization and contents of this text, for which I want to very warmly thank them. The initial latex typesetting and drawing of most of the figures in this manuscript was performed by Hsin Chen under a job student contract, whom I also want to thank. Finally, I want to also thank Paul Daum for proofreading of the manuscript. I hope that these lecture notes prove to be an additional useful resource for self study, in addition to the video recordings and the exercise sheets which are also released on the WES course webpage.

https://www.syscop.de/teaching/ss2020/wind-energy-systems

Moritz Diehl,
Freiburg, July 2020

Chapter 1

Introduction

1.1 Motivation and lecture overview

See slides: (click here for slides: https://tinyurl.com/yb8xskhn)

1.2 Energy content of the wind

How much power is in the wind?

Consider a cylindrical volume of air flowing through a "window" of area, $A[\mathrm{m}^2]$, with length, $L[\mathrm{m}]$ and air velocity, $V[\mathrm{m\,s^{-1}}]$. The mass of the air in this volume, m, can be found by $m = \rho \cdot L \cdot A$ with density of air, ρ, taken to be $1.2\,\mathrm{kg/m^3}$.

Kinetic energy in the volume of air is found by $E = \frac{1}{2}mv^2 = \frac{1}{2} \cdot \rho LA \cdot v^2$. Power, $P[\mathrm{W}]$, is given by $P = \frac{E}{t}$ with $t[\mathrm{s}]$ being the time it takes to move the volume through the window (as shown in Figure 1.1), given by $t = \frac{L}{V}$. Thus:

$$P = \frac{\frac{1}{2}\rho LAV^2}{L/V} = \frac{1}{2}\rho AV^3 \tag{1.1}$$

Note: P has a cubic relationship with wind velocity, V.

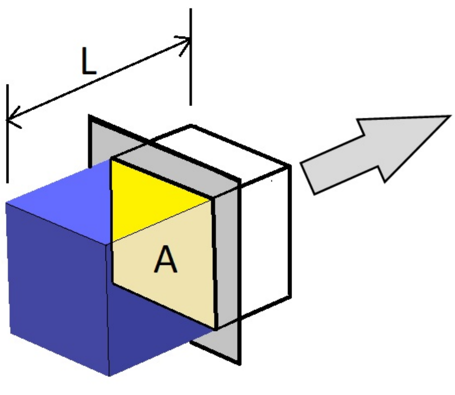


Figure 1.1 Power flowing through the window

Power density is "power per cross-sectional area" and given by

$$\frac{P}{A} = \frac{1}{2}\rho V^3 \tag{1.2}$$

SI-Unit of this expression is

$$\frac{\mathrm{kg}}{\mathrm{s}^3} = \underbrace{(\mathrm{kg}\cdot\frac{\mathrm{m}}{\mathrm{s}^2})}_{\mathrm{N}}\cdot(\frac{1}{\mathrm{m}\cdot\mathrm{s}}) = \underbrace{(\mathrm{N}\cdot\mathrm{m})}_{\mathrm{J}}\cdot(\frac{1}{\mathrm{m}^2\cdot\mathrm{s}}) = \underbrace{(\frac{\mathrm{J}}{s})}_{\mathrm{W}}\cdot(\frac{1}{\mathrm{m}^2}) = \frac{\mathrm{W}}{\mathrm{m}^2}$$

For $V = 10\,\mathrm{m/s}$ we get:

$$\frac{P}{A} = \frac{1}{2}\cdot 1.2\cdot 10^3\,\frac{\mathrm{W}}{\mathrm{m}^2} = 600\,\frac{\mathrm{W}}{\mathrm{m}^2} \tag{1.3}$$

At $V = 20\,\mathrm{m/s}$, a good strong wind, we have $\frac{P}{A} = 4.8\,\mathrm{kW\,m^{-2}}$.

Compare this with the average European's power need of 5 kW:

$\boxed{2\,\mathrm{m}^2}$ of cross-sectional area in very strong wind, or $\boxed{16\,\mathrm{m}^2}$ of area in good wind (of $V = 10\,\mathrm{m\,s^{-1}}$) or $\boxed{128\,\mathrm{m}^2}$ of area in weak wind (of $V = 5\,\mathrm{m\,s^{-1}}$), contain about 5 kW. (Not all of this can be harvested, due to the so called"Betz-Limit", which we will derive & discuss in chapter 3).

Strong winds constitute a fairly concentrated form of sustainable energy of a similar power density as solar power. Note that the cross-sectional area, A (shown in Figure 1.2), of wind turbines is given by the whole disc over which the rotor blades sweep.

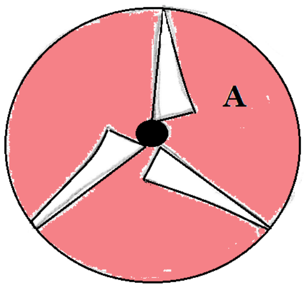


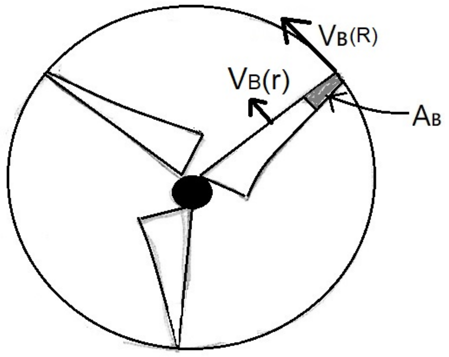
Figure 1.2 Rotor Blades

Thus, wind turbines can harvest from the entire area with relatively little blade area; this is one explanation of why wind power is comparably cheap and competitive.

For example: $V = 20\,\mathrm{m/s}$, Power density $= 4.8\,\mathrm{kW/m^2}$, $R = 35\,\mathrm{m}$, $A = \pi R^2 = 3850\,\mathrm{m}^2$, $P = 4.8\times 10^3\cdot 3850\mathrm{W} = 18.5\,\mathrm{MW}$, a large amount of power is accessible to the wind turbine.

1.3 Power density and blade area

Let us try to estimate how much power can be captured by a given blade area, A_B [m^2]. We regard only the outer part of a rotor blade (close to the wing-tips) which moves with a speed, V_{B}, in cross-wind direction.



Note that the inner part of the blade moves slower, but they are not our focus for now.

Figure 1.3

We simplify further by assuming that the blade-tip moves straight (not a circular path), the motion of the blade tip can now be compared to a sailing boat moving "half-wind" or "cross-wind." And it can be depicted from the top view as shown by Figure 1.4:

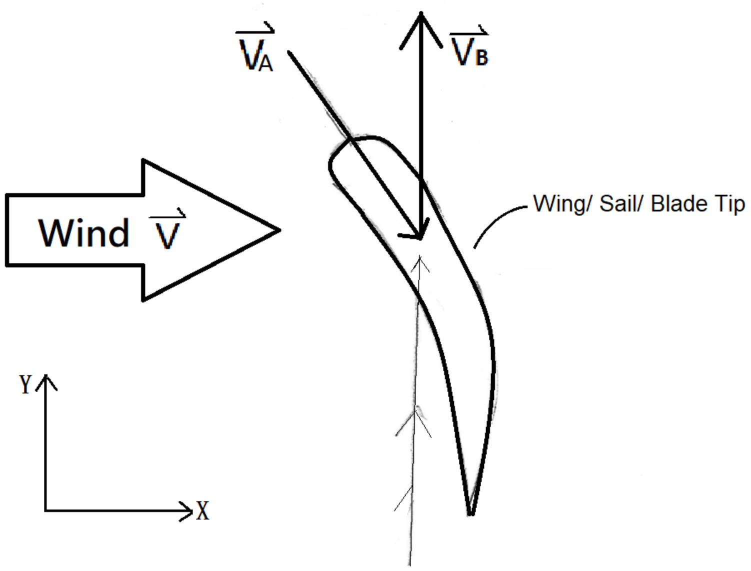


Figure 1.4

The apparent wind $\overrightarrow{V_{\mathrm{A}}}$ is given by $\overrightarrow{V_{\mathrm{A}}} = \overrightarrow{V} - \overrightarrow{V_{\mathrm{B}}}$ and therefore:

$$\overrightarrow{V_{\mathrm{A}}} = \begin{bmatrix} V \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ V_{\mathrm{B}} \end{bmatrix} = \begin{bmatrix} V \\ -V_{\mathrm{B}} \end{bmatrix} \tag{1.4}$$

The magnitude of the apparent wind is given by:

$$|\overrightarrow{V_{\mathrm{A}}}| = \sqrt{V_{\mathrm{B}}^2 + V^2} := V_{\mathrm{A}} \tag{1.5}$$

To determine the forces on the "wing" (we use this word now for the blade-tip of area A_{B}), we need one basic fact from aerodynamics: the force on a body in a moving fluid is proportional to the dynamic pressure $\frac{1}{2}\rho \cdot V_A^2$ and the area A_B. The force can be decomposed into "lift" and "drag", where lift force is perpendicular to $\overrightarrow{V_A}$ and drag force is aligned with it.

Lift and Drag

With lift-coefficient C_L and drag-coefficient C_D we have:

$$F_L = \frac{1}{2} C_L \cdot \rho A_B V_A^2 \tag{1.6}$$

$$F_D = \frac{1}{2} C_D \cdot \rho A_B V_A^2 \tag{1.7}$$

C_L & C_D depend upon:

- Angle of attack (orientation)
- Reynolds number (ratio of inertial forces to viscous forces)

Good wings have small drag and high lift, e.g. $C_L = 1.5$ and $C_D = 0.05$.

The $\boxed{\text{lift-over-drag ratio } \frac{C_L}{C_D}}$ has a nice interpretation for sailplanes: it determines how far a sailplane can go, depending on the initial altitude (see Figure 1.5). $\frac{C_L}{C_D}$ is therefore also called "gliding number".

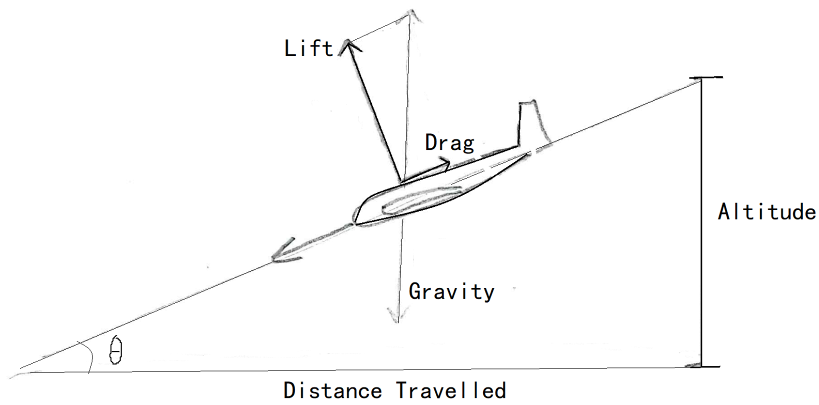


Figure 1.5 For a sailplane, distance travelled = $\frac{C_L}{C_D} \cdot$ altitude

For our rotor-blade we get the following picture (Figure 1.6):

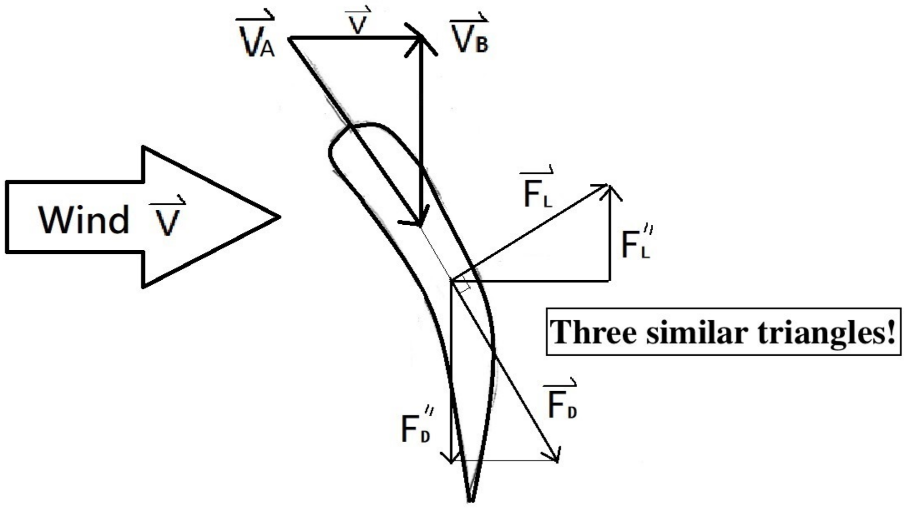


Figure 1.6

For rotation of the wind turbine, we are first only interested in the force component in the direction of motion of the wing, $F^{\parallel}$, as its product with V_{B} gives the mechanical power production:

$$P_{\mathrm{B}} = F^{\parallel} \cdot (-V_{\mathrm{B}}) \tag{1.8}$$

With $F^{\parallel} = F_L^{\parallel} + F_D^{\parallel} = F_L \cdot \frac{V}{V_{\mathrm{A}}} - F_D \cdot \frac{V_{\mathrm{B}}}{V_{\mathrm{A}}}$, where $F_L^{\parallel}$ and $F_D^{\parallel}$ are components of lift and drag which are parallel to the blade movement direction, bringing them all together gives:

$$P_{\mathrm{B}} = \frac{1}{2}\rho A_{\mathrm{B}} \cdot V_{\mathrm{A}}^2 \cdot V_{\mathrm{B}} \cdot \frac{1}{V_{\mathrm{A}}}(C_{\mathrm{L}} \cdot V - C_{\mathrm{D}} \cdot V_{\mathrm{B}}) \tag{1.9}$$

To simplify further, we introduce the $\boxed{\text{tip speed ratio } \lambda = \frac{V_{\mathrm{B}}(R)}{V}}$, such that $V_{\mathrm{B}} = \lambda V$ and $V_{\mathrm{A}} = \sqrt{1+\lambda^2} \cdot V = \sqrt{1+\frac{1}{\lambda^2}} \cdot \lambda V$. So the expression further simplifies to:

$$P_{\mathrm{B}} = \frac{1}{2}\rho A_{\mathrm{B}} \cdot V^3 \underbrace{\lambda^2 \sqrt{1+\frac{1}{\lambda^2}}(C_{\mathrm{L}} - C_{\mathrm{D}} \cdot \lambda)}_{:=\ \zeta \text{ (Power Harvesting Factor)}} \tag{1.10}$$

Note that at $\lambda = \frac{C_{\mathrm{L}}}{C_{\mathrm{D}}}$, no power is generated. ($\frac{C_{\mathrm{L}}}{C_{\mathrm{D}}}$ is the maximum possible tip speed ration. It is realized if the generator is switched off, which means there is no torque.)

A typical value for λ is $\boxed{\lambda = 7}$. And if $C_{\mathrm{L}} = 1.5$ and $C_{\mathrm{D}} = 0.05$, we can calculate the power harvesting factor:

$$\zeta = \lambda^2 \sqrt{1+\frac{1}{\lambda^2}}(C_{\mathrm{L}} - C_{\mathrm{D}}\lambda) \approx 49 \cdot 1 \cdot (1.5 - 0.05 \times 7) \approx 57 \tag{1.11}$$

(For $\lambda = 20$ we would even get $\zeta \approx 400 \cdot 0.5 = 200$.)

This is a remarkably high number. ζ shows how many times more power a blade area can harvest compared to the energy in the wind which would pass through the "window" of the same size as the blade area. Compared to the energy in the air for $\zeta = 50$ and $V = 10\,\mathrm{m/s}$, we thus get a power density of $\frac{P}{A_\mathrm{B}} = 50 \cdot 600\ \frac{\mathrm{W}}{\mathrm{m}^2} = 30\ \frac{\mathrm{kW}}{\mathrm{m}^2}$.

For the inner parts of the blade we can calculate $\boxed{\text{local speed ratio } \lambda_\mathrm{r} = \frac{V_\mathrm{B}(r)}{V}}$. As the inner parts of the blade move slower, their λ_r is smaller and therefore also their harvesting factors. This is one major reason why blades become thicker toward the center, as shown by Figure 1.7:

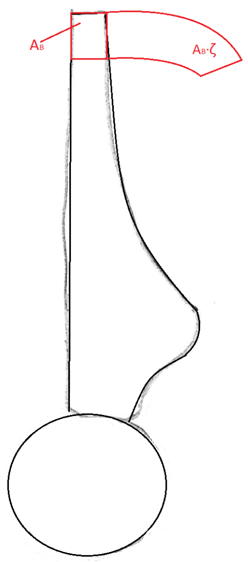


Figure 1.7

1.4 Components of a modern wind turbine

With its five joints (yaw, rotor, 3×pitch), a wind turbine can be regarded as a gigantic robot-arm, comparable to the six-joint robot arms in car manufacturing. However, it is an"energy-harvesting robot."

For an illustration of the components of a modern wind turbine, refer to figures 1.8, 1.9 and 1.10.

1.5 Blade & airfoil nomenclature

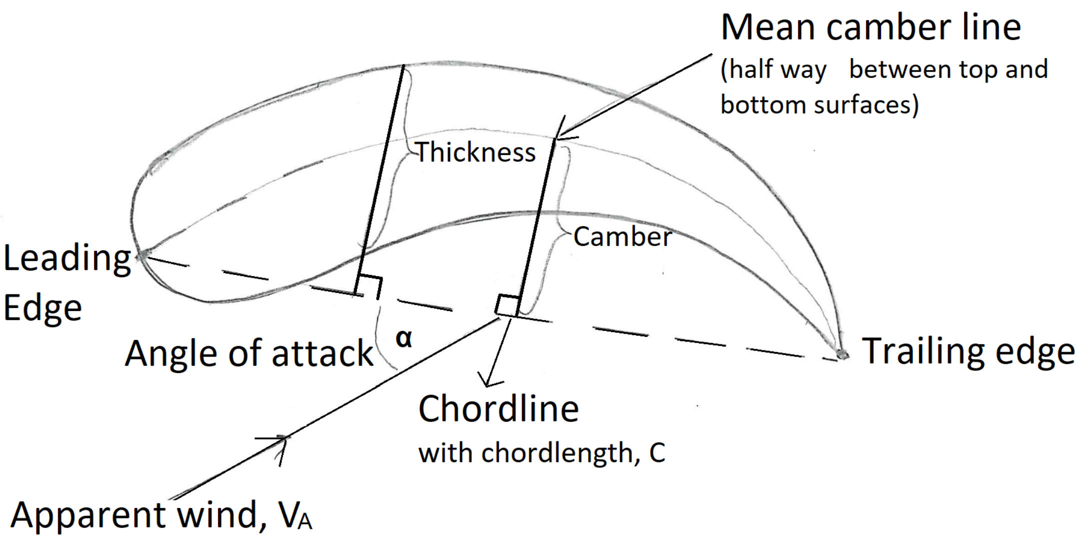


Figure 1.11 Airfoil

Note: Chordwise direction is along the chord line. Spanwise direction is orthogonal, along the radial direction of the turbine.

Surface area of a blade element, dA, by definition, is chord ($c(r)$) × span (dr) (see Figure. 1.12), therefore the whole blade area, A can be found by:

$$A = \int_0^R c(r)\,\mathrm{d}r \tag{1.12}$$

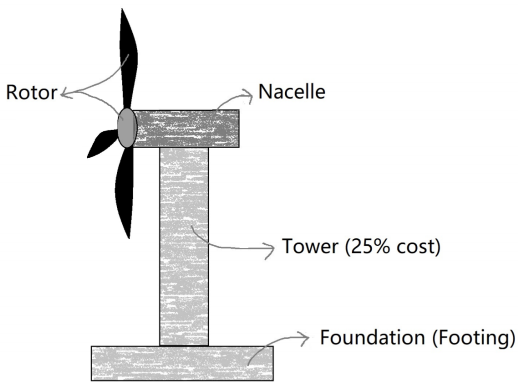


Figure 1.8 Wind turbine components

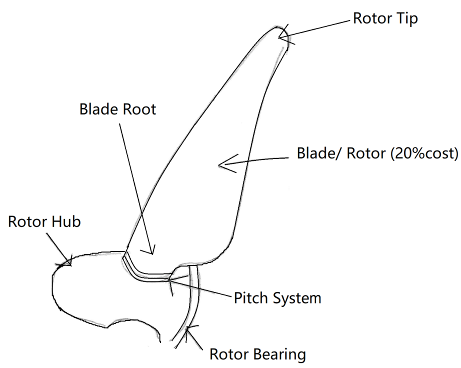


Figure 1.9 Rotor details

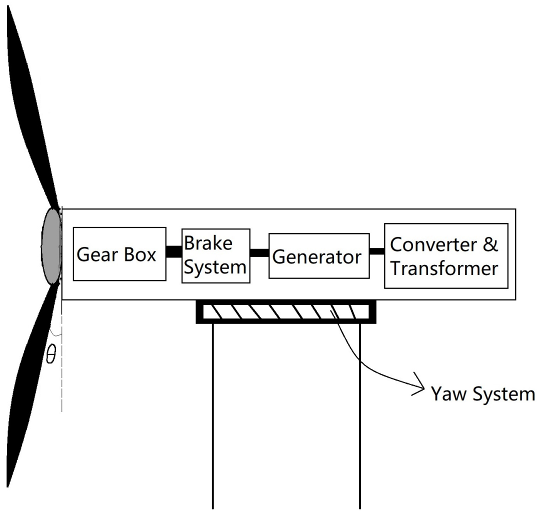


Figure 1.10 Rotor inner details

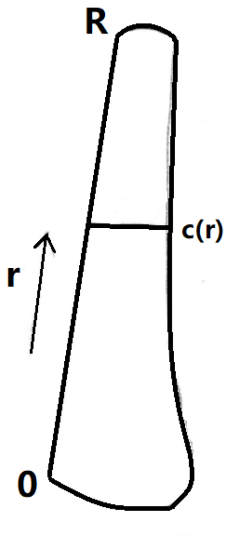


Figure 1.12 Surface Blade

Chapter 2

The Wind Resource

2.1 Origins

- Air heated up (by the sun, direct or indirect).
- Air density drops.
- Air rises and creates low pressure region.
- Other air fills the gap: "wind".

Heat capacity [1] of land is not as high as water. During the a sunny day, air over land is heated up as the temperature of the ground rises quickly and rises up. The temperature of water rises slowly and warm air is cooled by the ocean and sinks back down. Refer to Figure 2.1

During the night, the opposite happens, where air over land is cooled down and air over water is heated. Refer to Figure 2.2.

[1]heat capacity: the amount of energy it takes to increase the temperature of 1 kg of a substance by 1 degree Kelvin

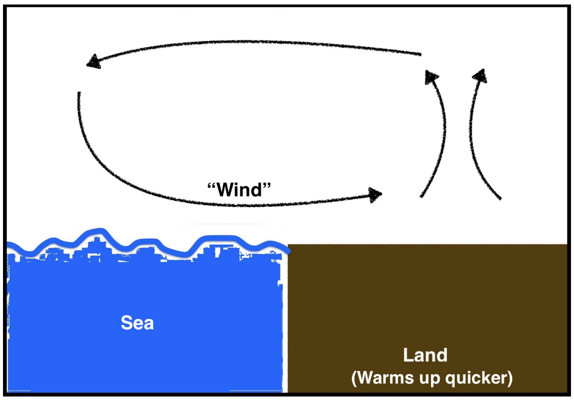


Figure 2.1 Sunny day at coast

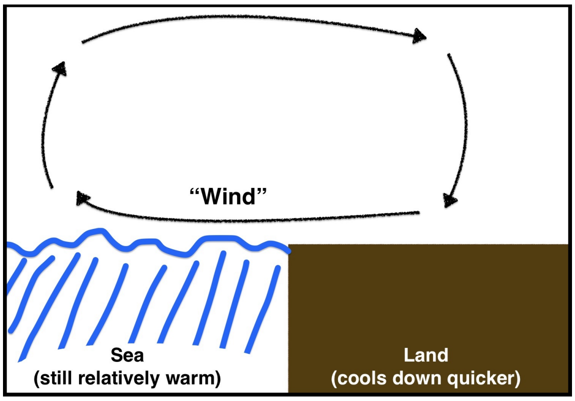


Figure 2.2 Clear night at coast

2.2 Global patterns

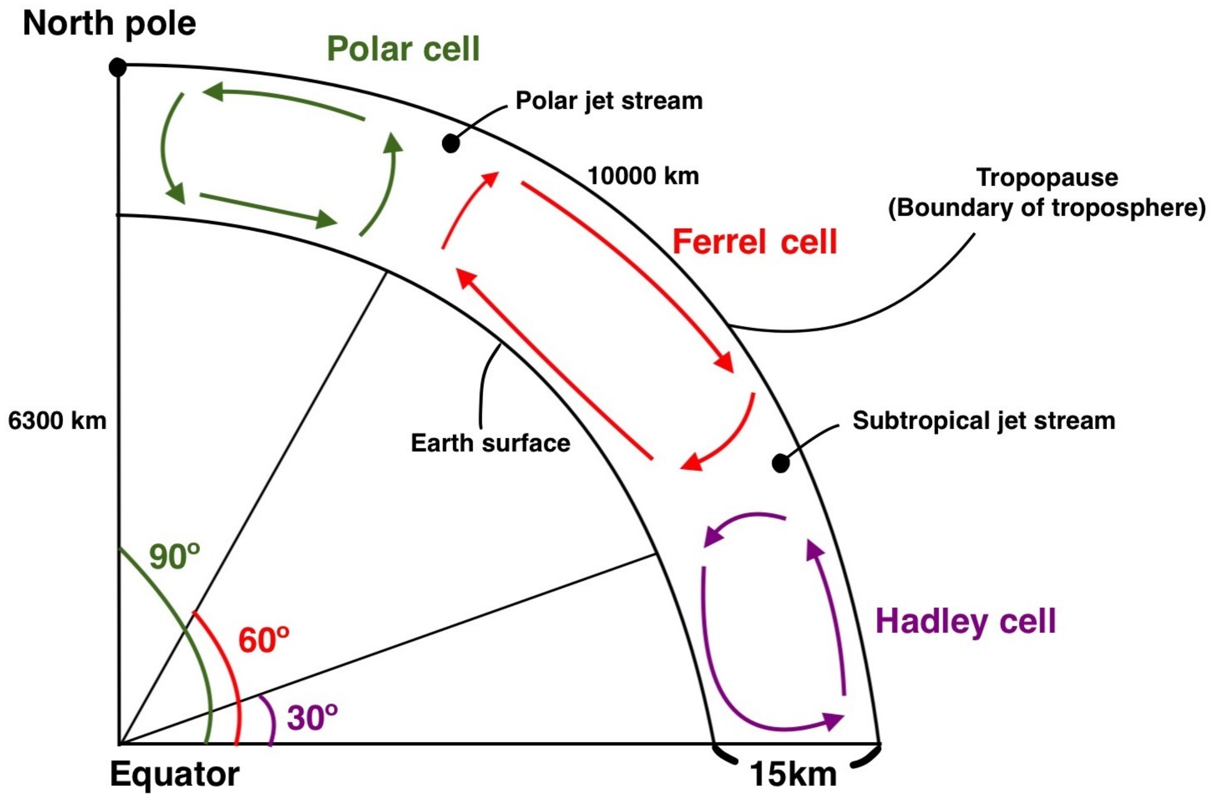


Figure 2.3 Air moves within troposphere (5-15 km altitude). Three big "cells" per hemisphere.

Note 1: The Ferrel cell is indirectly driven by the Hadley cell and the Polar cell.

Note 2: The distance along the surface of the Earth between the North Pole and the equator is about 10 000 km. The thickness of troposphere is only 5-15 km.

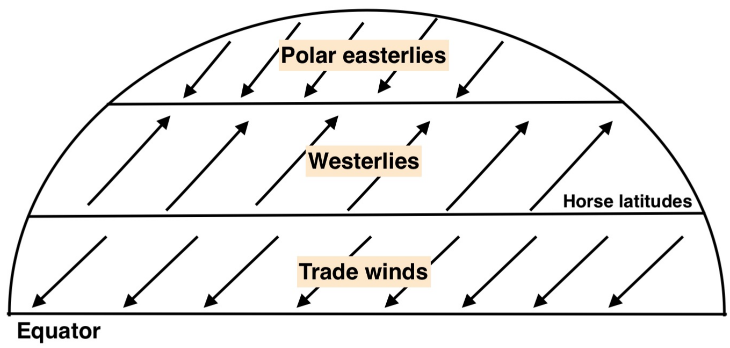


Figure 2.4 Due to the Coriolis force, winds get diverted to the right hand side on the northern hemisphere (relative to the direction of travel).

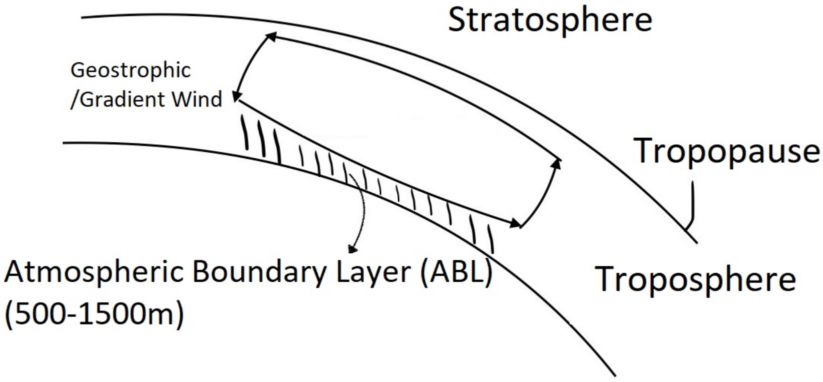


Figure 2.5 Strong wind shear in the Atmospheric Boundary Layer (ABL), magnitude and direction change with altitude. Ground friction is significant.

2.3 Mechanics of wind

Four main influences:

a) Pressure difference

b) Coriolis force

c) Centrifugal force

d) Friction

a) Pressure gradient

Regard a cylinder with length L and area A:

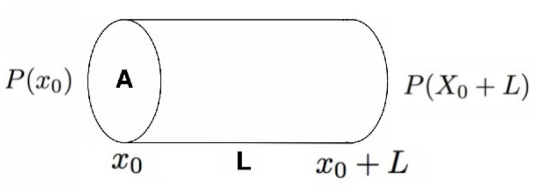


Figure 2.6

$$\left[\begin{array}{l} \text{Volume} : L \cdot A \\ \\ \text{Mass} : \rho \cdot L \cdot A = m \end{array}\right.$$

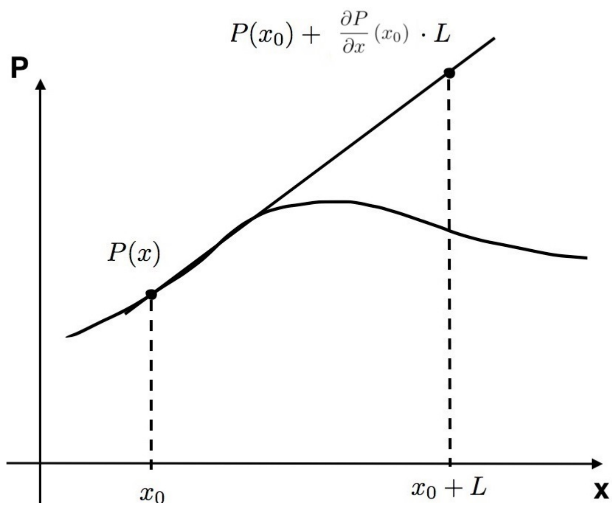


Figure 2.7

Pressure varies in space and time: $P(x,t)$

$P(x,t)$ unit: Pascal [Pa] $= 1\,\mathrm{N/m^2}$ (1 millibar = 1 hectopascal = 100 Pa)

Standard atmosphere pressure (sea level): $\boxed{101.325\,\mathrm{kPa}}$

Hence, the pressure gradient causes a net force on the air mass, $F =$ (Force on the left side)$-$(Force on the right side):

$$F = A \cdot P(x_0) - A \cdot P(x_0 + L) \tag{2.1a}$$

$$\approx A \cdot P(x_0) - A \cdot P(x_0) - A \cdot \frac{\partial P}{\partial x} \cdot L \tag{2.1b}$$

$$= -A \frac{\partial P}{\partial x}(x_0) \cdot L \tag{2.1c}$$

Acceleration a due to pressure gradient:

$$a = \frac{F}{m} = \frac{-A \frac{\partial P}{\partial x}(x_0) \cdot L}{\rho \cdot L \cdot A} = \frac{-\frac{\partial P}{\mathrm{d}x}(x_0)}{\rho} \; (\mathrm{m/s^2}) \tag{2.2}$$

b) Coriolis force (Due to rotation of Earth)

Consider a point on the surface of Earth, in Freiburg. This point is moving towards the east. Consider another point near the North Pole, it is also moving to the east, but because it is closer to the rotational axis of the Earth, it is moving slower to the east compared to Freiburg.

Now imagine wind moving from the North Pole towards the south. As it moves further south, the ground is moving faster and faster towards the east, causing the ground to "slide" away from wind. When viewed from the perspective of the ground, it appears that the wind is bending or accelerating to the right, see Figure 2.8. This is called the Coriolis Effect. This right-ward acceleration applies to wind blowing in all horizontal directions. However, in the Southern Hemisphere, the wind would accelerate to the left instead.

The Coriolis effect can be regarded as either a virtual force or an acceleration. On the north pole it is given by:

$$F = 2 \cdot m \cdot \omega_0 \cdot V_{\text{GEO}} \tag{2.3}$$

$$a = 2 \cdot \omega_0 \cdot V_{\text{GEO}} \tag{2.4}$$

where V_{GEO} is the geostrophic wind velocity, ω_0 is the rotation velocity of the earth.

The Coriolis effect depends on the latitude ϕ, which means there is no Coriolis force on the equator. So we have:

$$a = 2 \cdot \omega_0 \cdot \sin\phi \cdot V_{\text{GEO}} = \frac{-\frac{\partial P}{\partial x}}{\rho} \tag{2.5}$$

$$\boxed{V_{\text{GEO}} = \tfrac{1}{2\rho \cdot \omega_0 \sin\phi}(-\tfrac{\partial P}{\partial x})} \tag{2.6}$$

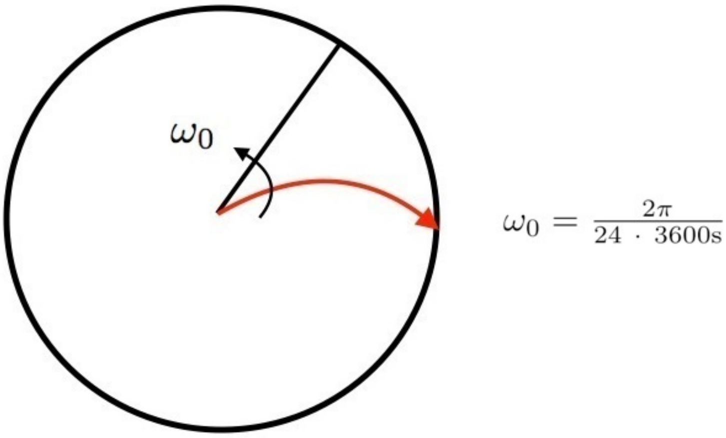


Figure 2.8 As viewed from above the North Pole, with the Earth rotation, ω_0, an air current traveling to the south would curve to the right.

Effect of pressure gradient and Coriolis force:

Geostrophic wind is a balance of pressure gradient and the Coriolis effect. In a simple case of straight isobars, e.g. in east-west direction, as shown in Figure 2.9, while the pressure gradient pushes the wind northwards, the Coriolis force pushes the wind southwards. The result is that the wind travels in parallel to the isobars, where the accelerations due to pressure gradient and the Coriolis effect are balanced:

$$\overbrace{\frac{-\frac{\partial P}{\partial x}}{\rho}}^{\text{pressure gradient}} = \underbrace{2\sin\phi \cdot \omega_0 \cdot V_{\text{GEO}}}_{\text{Coriolis effect}} \tag{2.7}$$

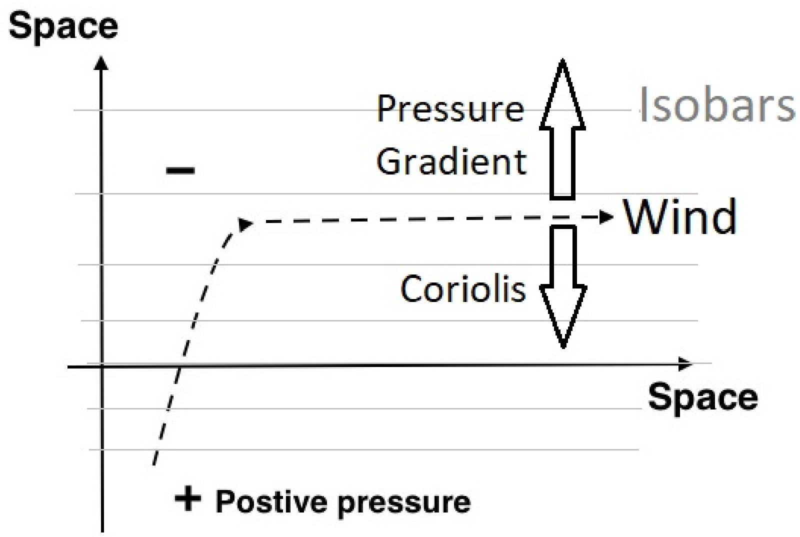


Figure 2.9 Geostrophic wind flows parallel to isobars.

Note: "Geostrophic Wind", V_{GEO}, is proportional to pressure gradient but parallel to isobars!

Weather Maps:

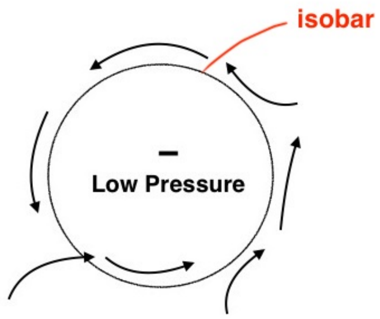


Figure 2.10

c) Centrifugal acceleration

Geostrophic wind considers the pressure gradient and Coriolis force, however when the isobars are curved, which is almost always the case, there is a third force which affects the wind, the centrifugal force, which we all know arises from travelling in a circular path.

A refinement of Geostrophic wind, V_{GEO}, is the Gradient wind, V_{G}.

Figure 2.11 shows a situation where there is a circular isobar and the wind is travelling along the isobar.

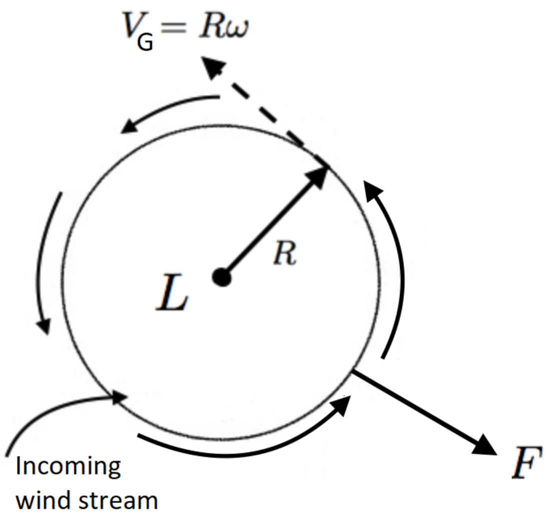


Figure 2.11

$a = \frac{{V_G}^2}{R} = \omega^2 R = \frac{\omega^2 R^2}{R}$

R = Radius of curvature of isobar

V_{G} = Gradient wind
(speed of wind along isobar)

Note: $V_{\mathrm{G}} \neq V_{\mathrm{GEO}}$
but still parallel to isobars.

Hence, around a low pressure region in the northern hemisphere, an extra centrifugal term is added that acts in the same direction as the Coriolis force:

$$a = \frac{-\frac{\partial P}{\partial x}}{\rho} = \overbrace{2\sin\phi \cdot \omega_0 \cdot V_{\mathrm{G}}}^{\text{Coriolis}} + \overbrace{\frac{V_G^2}{R}}^{\text{Centrifugal}} \tag{2.8a}$$

$$V_G^2 + (2R\,\omega_0 \sin\phi) \cdot V_{\mathrm{G}} + \frac{-\frac{\partial P}{\partial x} \cdot R}{\rho} = 0 \tag{2.8b}$$

$$V_{\mathrm{G}} = -R\omega_0 \sin\phi \pm \sqrt{R^2\omega_0^2 \sin^2\phi - \frac{\frac{\partial P}{\partial x} \cdot R}{\rho}} \tag{2.8c}$$

Note: $V_{\mathrm{G}} < V_{\mathrm{GEO}}$

To assess the relevance of the centrifugal force, compare **Coriolis** $\boxed{2\sin\phi \cdot \omega_0 V_{\mathrm{G}}}$ with **centrifugal** $\boxed{\frac{V_{\mathrm{G}}^2}{R}}$, we can compute the ratio between the two:

$$\frac{\text{Coriolis}}{\text{Centrifugal}} = \frac{2\omega_0 \sin\phi \cdot R}{V_{\mathrm{G}}} \tag{2.9}$$

Therefore, if:

$$\begin{pmatrix} \phi = 50 \to \sin\phi \approx 0.75 \\ \\ V_{\mathrm{G}} \approx 50\,\mathrm{km/h} \\ \\ R \approx 500\,\mathrm{km} \end{pmatrix}$$

$\frac{\text{Coriolis}}{\text{Centrifugal}} \approx \frac{2 \cdot 0.75 \cdot 2\pi \cdot 500\mathrm{km}}{24 \cdot 50\mathrm{km}}$

≈ 4

The centrifugal term adds about 1/4 to the Coriolis force in these strong wind conditions, and can therefore not be neglected for strong circular winds.

d) Friction

Friction is complex and depends on surface properties, but it generally slows down the air (only in the ABL[2]). This also decreases Coriolis & centrifugal forces. Therefore, very low altitude winds tend more towards the direction of negative pressure gradients. At the earth's surface, the wind speed is zero.

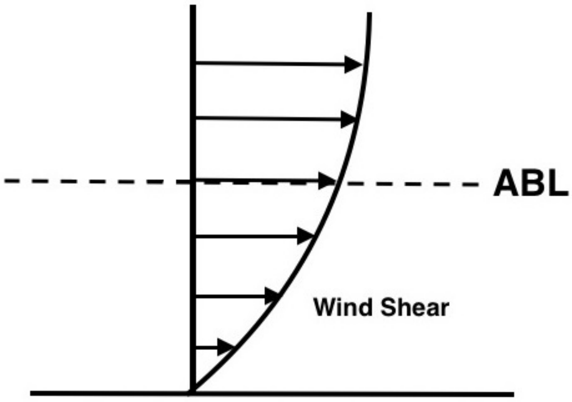


Figure 2.12

The change in wind speed over a change in altitude is called wind shear. A common description of the long-term time-averaged wind shear is the logarithmic profile.

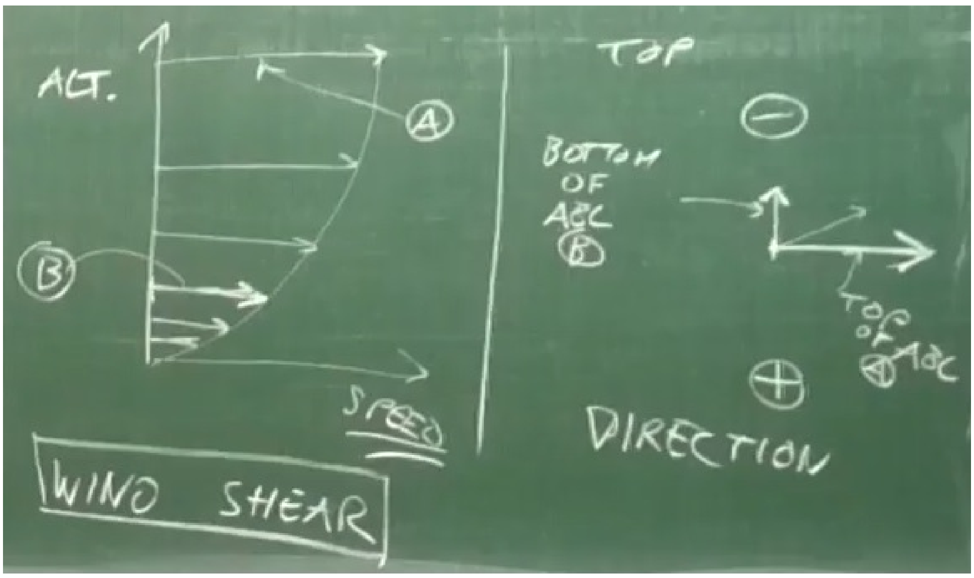


Figure 2.13

[2]ABL - Atmospheric Boundary Region - the thin part of the atmosphere closest to the ground, where friction effects are considered to be "significant".

$$V(z) = \frac{V_0 \cdot \log(\frac{Z}{Z_r})}{\log(\frac{Z_0}{Z_r})} \tag{2.10}$$

$V_0 =$ speed at altitude $Z = Z_0$,
$Z_r =$ "Roughness length" (a few millimeters for flat ground).

2.4 Stable and unstable atmospheric stratification

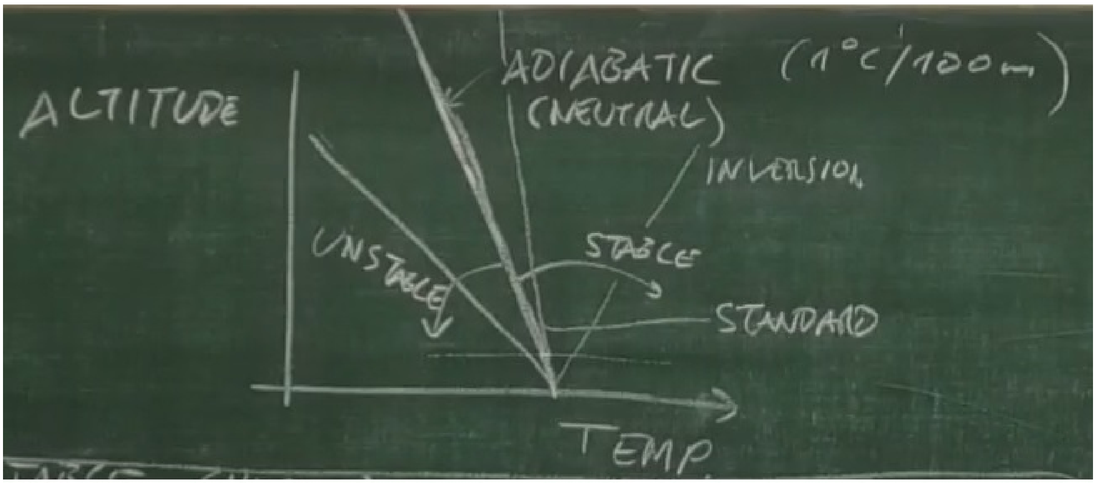


Figure 2.14

A hot piece of air becomes relatively lighter so it rises, but rising air expands and therefore gets cooler. The "dry adiabatic lapse rate" is about 1 °C/100 m, i.e. rising air cools down 1 °C per 100 m rise in altitude due to its own expanse. If the ambient air gets cooler slower than 1 °C/100 m, it means that the atmosphere is stable. If it gets cooler faster, it is unstable.

The standard atmospheric lapse rate is $\boxed{0.66\,°\mathrm{C}/100\,\mathrm{m.}}$ This corresponds to a **stable stratification**. Even more stable is an "inversion" (if air becomes hotter with height).

Generally, the wind shear is stronger ("strong" meaning a larger change in wind-speed over a change in altitude, and possibly a thinner ABL) for stable conditions, because less mixing between layers occur. Thus, for a given high-altitude wind speed, there will be less momentum transfered within the flow under stable conditions than under neutral (i.e. atmospheric lapse rate equals adiabatic lapse rate) conditions.

2.5 Statistics of wind

At a given site, wind speed and direction vary with time. If only speed is regarded, one can plot time series data similar to the following Figure 2.15. One can compute e.g. mean $\overline{U}$ and variance σ_u^2 with the hourly average wind speed over a year:

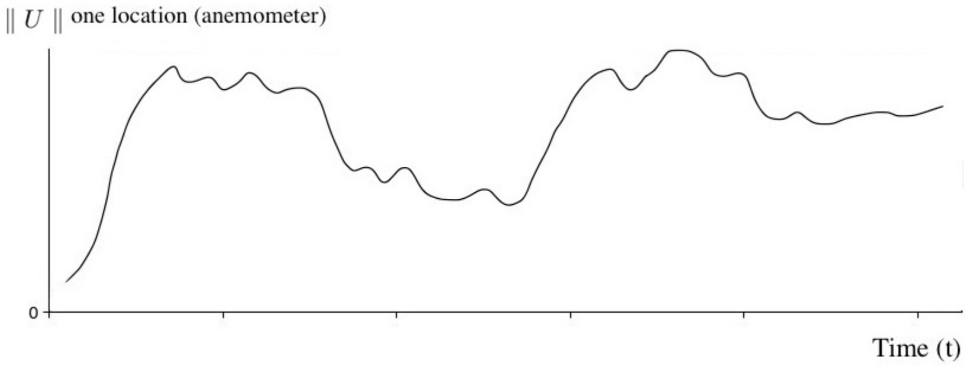


Figure 2.15 Hourly averages over one year

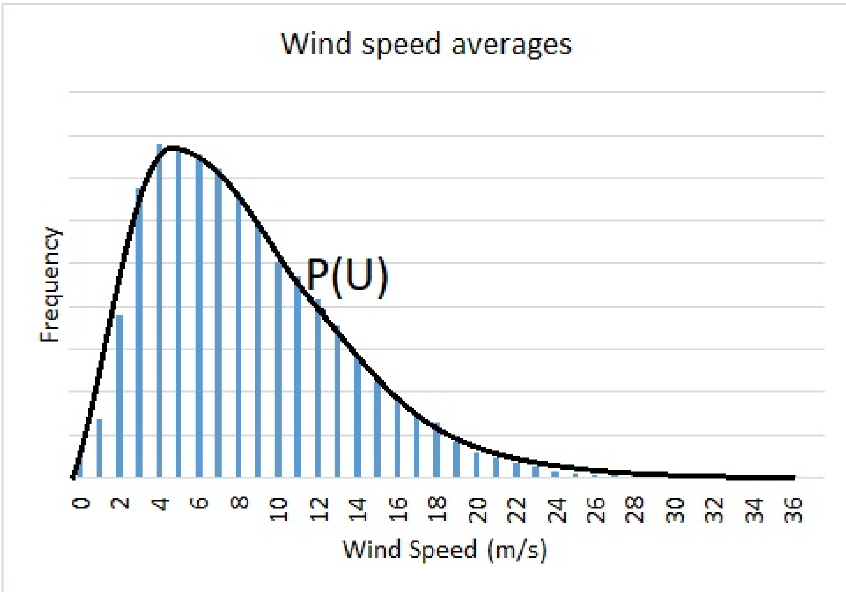


Figure 2.16 Histogram

Different distributions can be used to describe $P(U)$, the probability density function of wind speeds (PDF). Strongly related is its integral $F(U)$, which is called the cumulative distribution function (CDF).

$$\int_0^\infty P(U)\,\mathrm{d}U = 1 \tag{2.11}$$

$$F(U) = \int_0^U P(U)\,\mathrm{d}U \tag{2.12}$$

$$\boxed{P(U) = F(U)'} \tag{2.13}$$

The mean $\overline{U}$ and variance σ_u^2 of the probability density function $P(U)$ are defined as follows:

$$\overline{U} := \int_0^\infty U P(U)\,\mathrm{d}U \tag{2.14a}$$

$$\sigma_u^2 := \int_0^\infty (U - \overline{U})^2\, P(U)\,\mathrm{d}U \tag{2.14b}$$

$$= \int_0^\infty U^2 P(U)\,\mathrm{d}U - \overline{U}^2 \tag{2.14c}$$

Examples of different distributions:

- **Gaussian (Normal) Distribution**

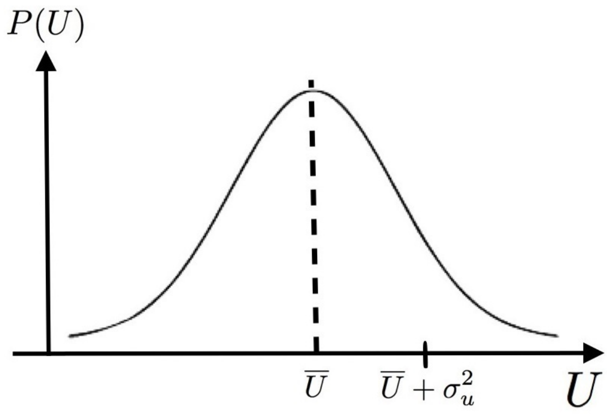


Figure 2.17

$$P(U) = \frac{1}{\sqrt{2\pi\sigma_u^2}} \exp\left(-\frac{\left(U - \overline{U}\right)^2}{2\sigma_u^2}\right) \tag{2.15}$$

- **Weibull Distribution**

  Wind velocity probabilities are described by a Weibull distribution, with "scale parameter" c and "shape parameter" k,

$$F(U) = 1 - \exp\left(-\left(\frac{U}{c}\right)^k\right) \tag{2.16}$$

$$P(U) = F(U)' = \left(\frac{k}{c}\right)\left(\frac{U}{c}\right)^{k-1} \exp\left(-\left(\frac{U}{c}\right)^k\right) \tag{2.17}$$

One can show that $\overline{U}$ and σ_u^2 can be computed from c & k using the "Gamma Function" as follows:

$$\Gamma(x) \coloneqq \int_0^\infty e^{-t}\ t^{x-1} \mathrm{d}t \tag{2.18}$$

$$(\Gamma(n) = (n-1)!,\ \Gamma(1) = 1,\ \Gamma(2) = 1, \dots)$$

$$\overline{U} = c \cdot \Gamma\left(1 + \frac{1}{k}\right) \tag{2.19}$$

$$\sigma_u^2 = c^2\ \Gamma\left(1 + \frac{2}{k}\right) - c^2 \left(\Gamma\left(1 + \frac{1}{k}\right)\right)^2 \tag{2.20a}$$

$$\left(= \int_0^\infty U^2 P(U) \mathrm{d}U - \overline{U}^2 \right) \tag{2.20b}$$

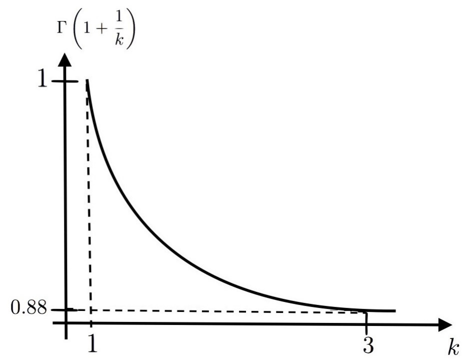


Figure 2.18 Gamma Function

- **Rayleigh Distribution**

A special case of Weibull distribution is the Rayleigh distribution with $k = 2$. Here, $\Gamma(1 + \frac{1}{2}) = \sqrt{\frac{\pi}{4}}$, i.e. $c = \frac{\overline{U}}{\sqrt{\frac{\pi}{4}}}$.

$$F(U) = 1 - \exp\left(-\left(\frac{U}{c}\right)^2\right) \tag{2.21}$$

$$P(U) = \frac{2}{c^2} U \cdot \exp\left(-\left(\frac{U}{c}\right)^2\right) \tag{2.22}$$

Note: The Rayleigh distribution corresponds to the PDF of the vector magnitude of a 2-dimensional Gaussian distribution.

Question: What is the average power per year?

Given power curve and wind speed distribution:

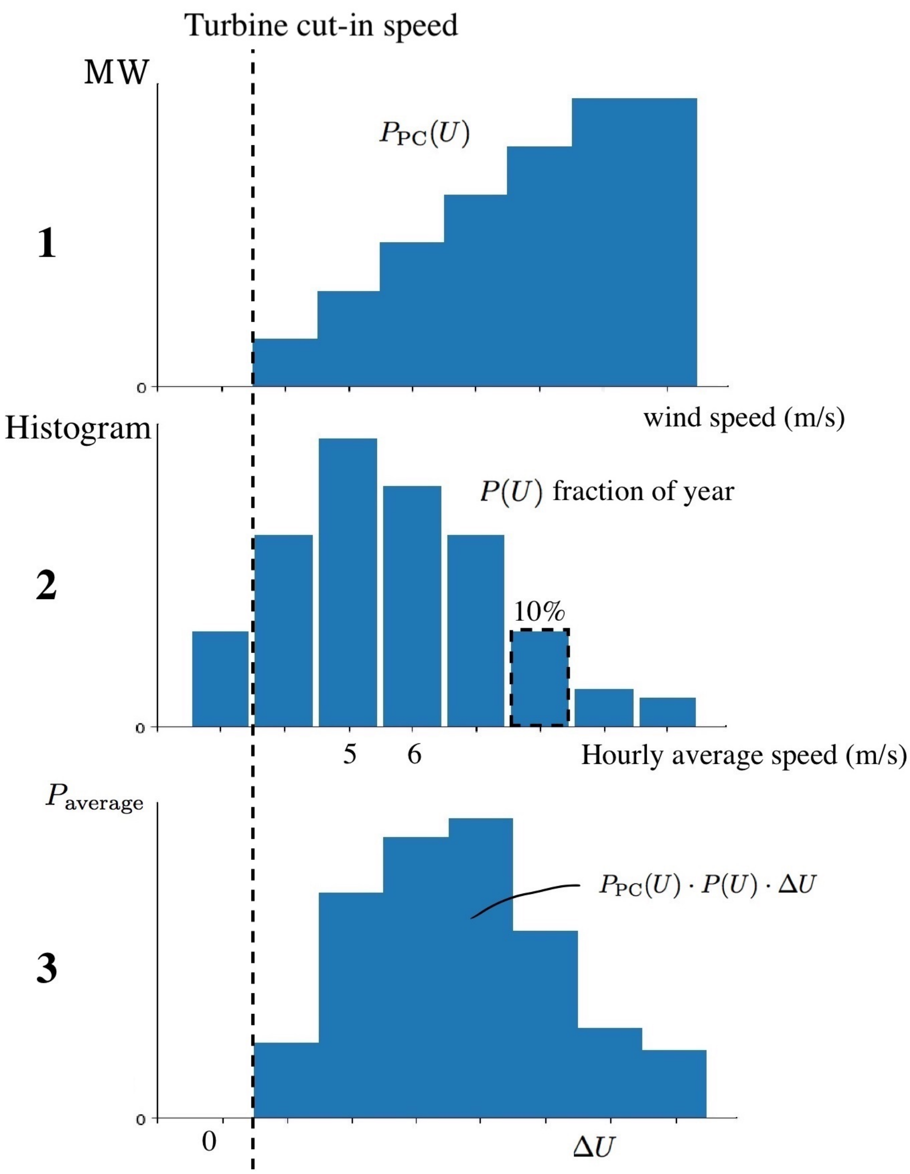


Figure 2.19

Answer:
Therefore the average power per year:

$$\boxed{P_{average} = \int_0^\infty P(U) P_{PC}(U) \, dU}$$

2.6 Spectral Properties of Wind

Autocorrelation & Power Spectral Density

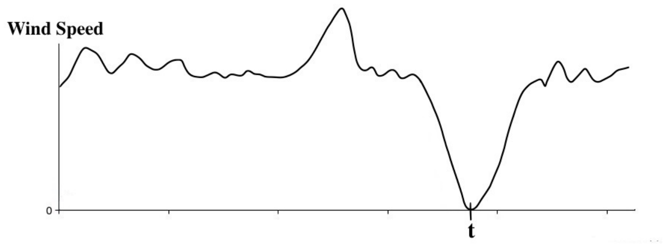


Figure 2.20

If a Fourier series is taken, the power spectral density $S(f)$ is obtained. It often looks as follows:

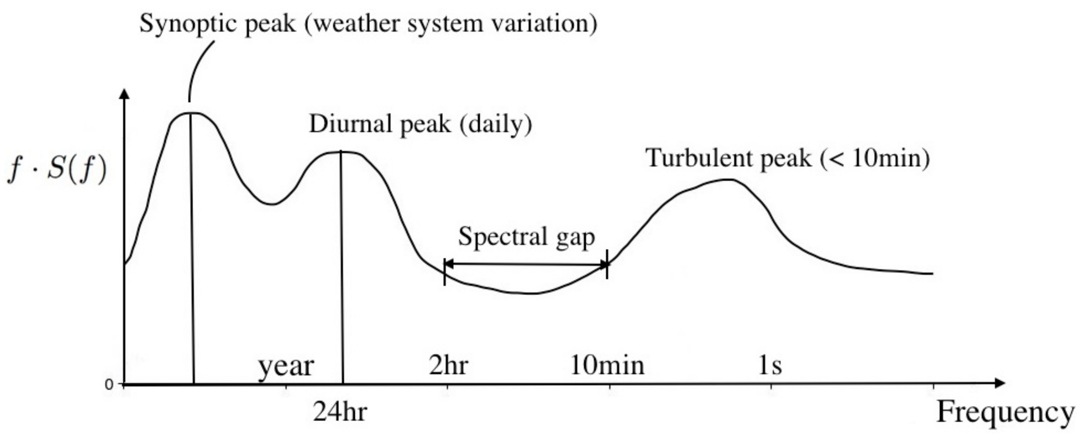


Figure 2.21 Density $S(f)$ (Fourier Transform)

Turbulence is most relevant at time scales below 10 minutes sampling time. The **turbulence intensity** in a time window of length 10 minutes is defined as $\boxed{\frac{\sigma_u}{\overline{U}}}$, where $\overline{U}$ is the mean over 10 minutes and σ_u is the standard deviation of e.g. 1 second samples.

$$\overline{U} = \frac{1}{N} \sum_{i=1}^{N} U_i \tag{2.23}$$

$$\sigma_u^2 = \frac{1}{(N-1)} \sum_{i=1}^{N} (U_i - \overline{U})^2 \tag{2.24}$$

Another interesting quantity is **autocorrelation**:

It helps to characterize repeating patterns, such as periodic wind patterns. The **autocorrelation function,** $r(t)$, can be computed for the discrete values $t = k\Delta t$ from a time series as follows:

$$r(k\Delta t) := \frac{1}{\sigma_u^2(N-k)} \sum_{i=1}^{N-k} (U_i - \overline{U})(U_{i+k} - \overline{U}), \tag{2.25}$$

where Δt is the sampling time, k is the lag number and $k\Delta t$ is the lag time. Between the discrete time values, it can be interpolated, so that the function $r(t)$ is defined for all $t > 0$.

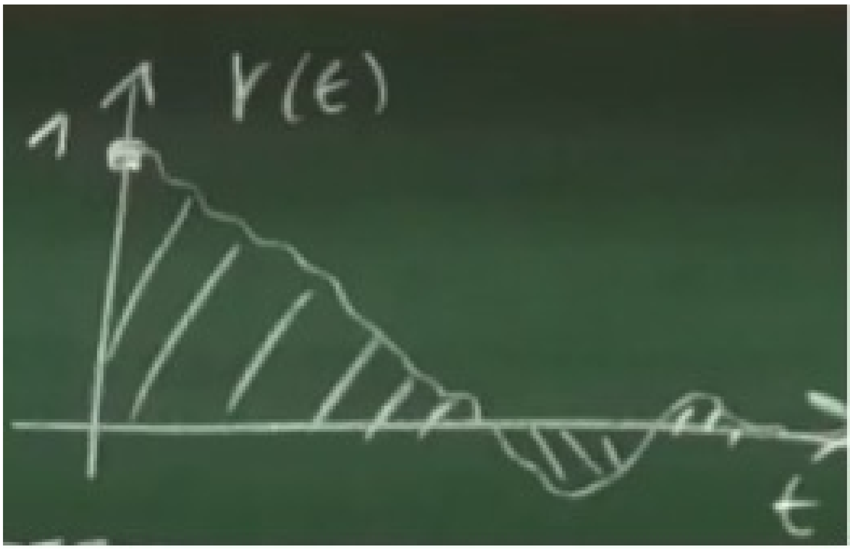


Figure 2.22

Figure 2.22 shows a typical autocorrelation function where the wind is strongly autocorrelated at very short lag times and not so strongly at longer lag times. This is as expected because we expect wind one second ago to have a big influence on the current wind, while not so much for the wind from a day ago. An interesting number is the **integral time scale** T that is defined as the following integral:

$$\boxed{T := \int_0^\infty r(t)\, \mathrm{d}t}$$

Related to it is the **integral length scale** L that is $\boxed{L = \overline{U} \cdot T}$ $\approx$ size of turbulent interruption

Note: The Fourier transform of the autocorrelation function equals (up to factors) the power spectral density (PSD).

Chapter 3

Aerodynamics of Wind Turbines

3.1 Wakes

Like a boat passing through water, and disturbing the water, leaving a wake, a wind turbine disturbs the flow of wind blowing across it.

Figure 3.1 Photo of the wakes behind turbines in a wind park. Foto: Vattenfall

3.2 Actuator Disc Model and Betz' Limit (Momentum Theory)

The wind is slower approaching, at and after the wind turbine. Figure 3.2 is a side view of a wind turbine.A stream tube is defined as a tube whose boundaries are parallel to the local fluid velocity:

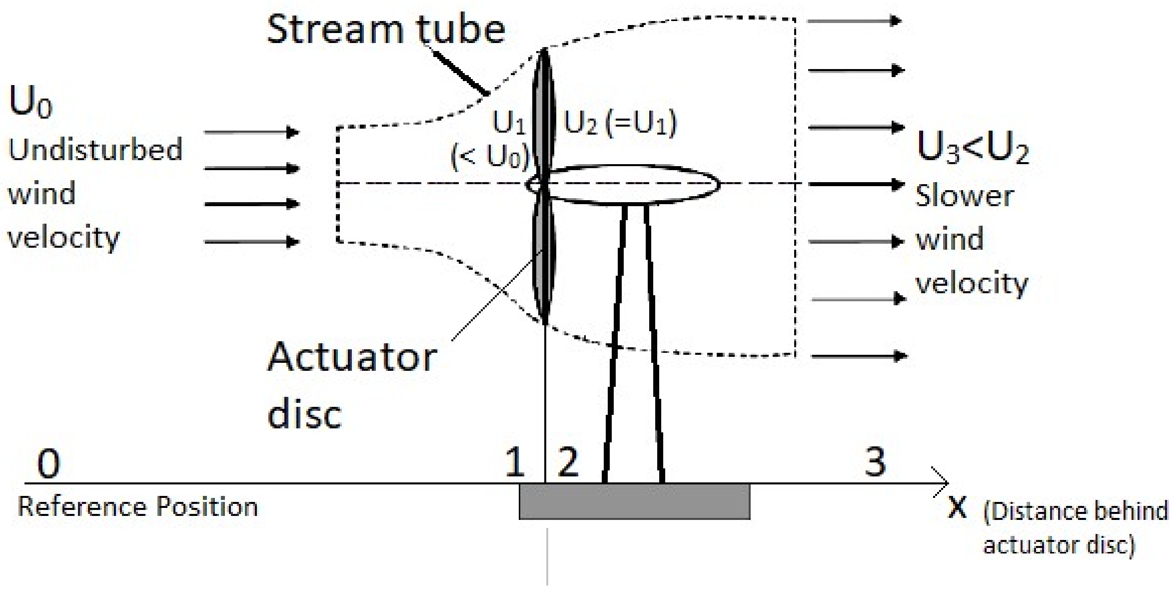


Figure 3.2 Position 0 is assumed to be infinitely far upstream of position 1; position 3 is assumed to be infinitely far downstream of position 2

First guess (not achievable): $P_{\text{air}} = \frac{1}{2}\rho A u_0^3$, P_{air} is the power in the air that would flow through the actuator disc if the actuator disc weren't actually there.

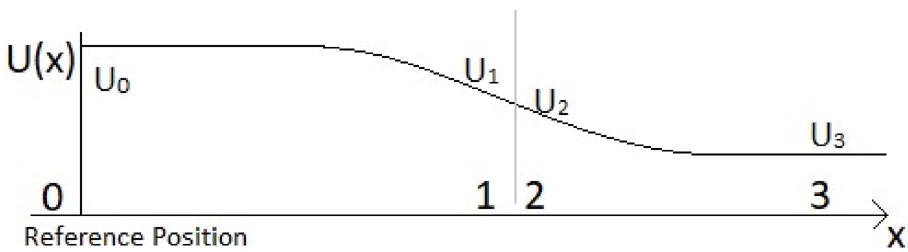


Figure 3.3 Axial wind velocity slows down as it approaches the turbine and is slowed down further as it passed through.

We define four positions in the direction x of the wind flow and the corresponding wind velocities $u(x)$: x_0 far upwind of the turbine, x_1 just before the turbine, x_2 just after the turbine, and x_3 very far after the turbine. The velocities are given by $u(x_0) = u_0$, $u(x_1) = u_1 = u_2 = u(x_2)$, $u(x_3) = u_3$. We know that the velocity must be continuous as inflow into the turbine's "actuator disk" must equal the outflow, and the cross sectional area for both equals the area of the disk.

Note: We assume there is no interaction of the stream tube with the outside.

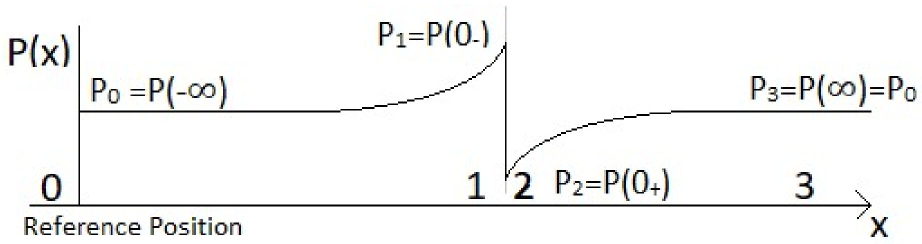


Figure 3.4 Pressure is built up as the wind approaches the wind turbine, and drops after passing the turbine.

- **Mass flow through turbine**:

$$\dot{m} = \rho \cdot A \cdot u_1 \left[\frac{\mathrm{kg}}{\mathrm{s}} = \frac{\mathrm{kg}}{\mathrm{m}^3} \cdot \mathrm{m}^2 \cdot \frac{\mathrm{m}}{\mathrm{s}} \right] \tag{3.1}$$

(Assume incompressible air $\rightarrow \rho$ is constant)

- **Thrust of turbine (force against wind)**:
Change of the pressure times area gives the trust force:

$$T = A(P_1 - P_2) \tag{3.2}$$

This force equals the change of momentum:

$$T = \dot{m}(u_0 - u_3) \tag{3.3}$$

- **Power extraction**:

$$P = T \cdot u_1 \tag{3.4}$$

Or equivalently, by the change of kinetic energy:

$$P = \dot{m}\left(\frac{1}{2}{u_0}^2 - \frac{1}{2}{u_3}^2\right) \tag{3.5}$$

Given that $u_1 = u_2, P_3 = P_0$, u_0 and P_0, the remaining unknowns are u_1, u_2, P_1 and P_2? They are derived by the following equations:

First we have the thrust equation:

$$\boxed{T = A(P_1 - P_2) = \dot{m}(u_0 - u_3)} \tag{3.6}$$

And then from Bernoulli equation (which is only valid when no energy is extracted, among other assumptions) we get: $P + \frac{1}{2}\rho u^2 = \text{constant}$. Therefore, in the wind flowing before passing the disc we have that:

$$\boxed{P_0 + \tfrac{1}{2}\rho {u_0}^2 = P_1 + \tfrac{1}{2}\rho {u_1}^2} \tag{3.7}$$

After passing the disc (note, energy is lost at the disc):

$$\boxed{P_2 + \tfrac{1}{2}\rho {u_1}^2 = P_0 + \tfrac{1}{2}\rho {u_3}^2} \tag{3.8}$$

Eliminating P_1 & P_2 from Eq.(3.6) via Eq.(3.7) & Eq.(3.8):

$$P_1 = P_0 + \frac{1}{2}\rho({u_0}^2 - {u_1}^2) \tag{3.9}$$

$$P_2 = P_0 + \frac{1}{2}\rho({u_3}^2 - {u_1}^2) \tag{3.10}$$

$$P_1 - P_2 = \frac{1}{2}\rho({u_0}^2 - {u_3}^2) \tag{3.11}$$

With Eq.(3.6):

$$T = A(P_1 - P_2) \tag{3.12a}$$

$$\cancel{A}\,\frac{1}{2}\,\cancel{\rho}\,\cancel{(u_0 - u_3)}\,(u_0 + u_3) = \cancel{\rho A}\,u_1 \cdot \cancel{(u_0 - u_3)} \tag{3.12b}$$

$$\Rightarrow \boxed{\tfrac{1}{2}(u_0 + u_3) = u_1} \tag{3.12c}$$

Induction Factor: $a \in [0, \frac{1}{2}]$

$u_1 = (1 - a)u_0$

With Eq.(3.12c):

$u_3 = (1 - 2a)u_0$

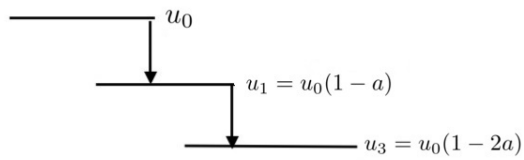


Figure 3.5

Now we can compute power & thrust as a function of a:

$$P = \rho \cdot u_1 \cdot A \cdot (\frac{1}{2}u_0^2 - \frac{1}{2}u_3^2) \tag{3.13a}$$

$$= \frac{1}{2} \cdot \rho \cdot u_0^3 \cdot A \cdot (1-a)(1-(1-2a)^2) \tag{3.13b}$$

$$= \frac{1}{2}\ \rho\ A\ u_0^3 \cdot \underbrace{4a\ (1-a)^2}_{C_\mathrm{P}(a) \text{ Power Coefficient}} \tag{3.13c}$$

$$T = \dot{m}(u_0 - u_3) = \rho \cdot u_1 \cdot A\ (u_0 - u_3) \tag{3.14a}$$

$$= \frac{1}{2} \cdot \rho \cdot A \cdot u_0^2 \cdot 2 \cdot (1-a)(1-(1-2a)) \tag{3.14b}$$

$$= \frac{1}{2}\ \rho\ A\ u_0^2 \cdot \underbrace{4a\ (1-a)}_{C_\mathrm{T}(a) \text{ Thrust Coefficient}} \tag{3.14c}$$

Note that we have $C_\mathrm{P}(a) = (1-a)C_\mathrm{T}(a)$ in agreement with Eq. (3.4).

Maximize power extraction:

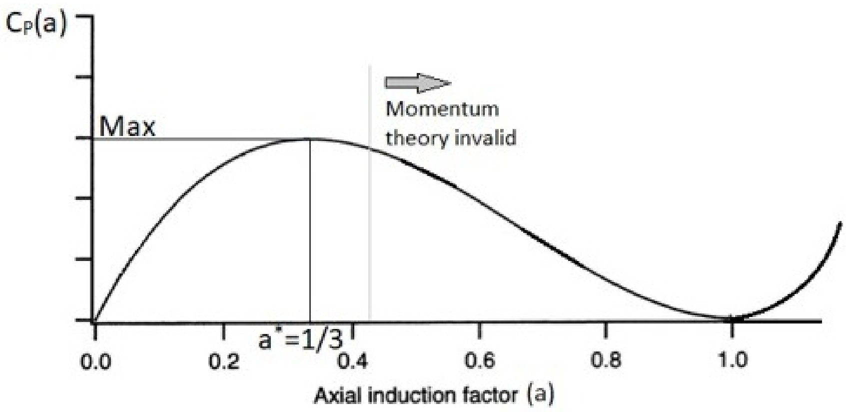


Figure 3.6

Since $\frac{\mathrm{d}C_\mathrm{P}}{\mathrm{d}a} = 2(1-a)\cdot 4a + (1-a)^2 \cdot 4 = 0 \Leftrightarrow 2a = 1-a \Leftrightarrow a^* = \frac{1}{3}$, where a^* is the optimal induction factor, we get:

$C_\mathrm{P}(a^*) = (\frac{2}{3})^2 \cdot 4 \cdot \frac{1}{3} = \frac{16}{27} \approx 0.59$ **(Betz' limit)**

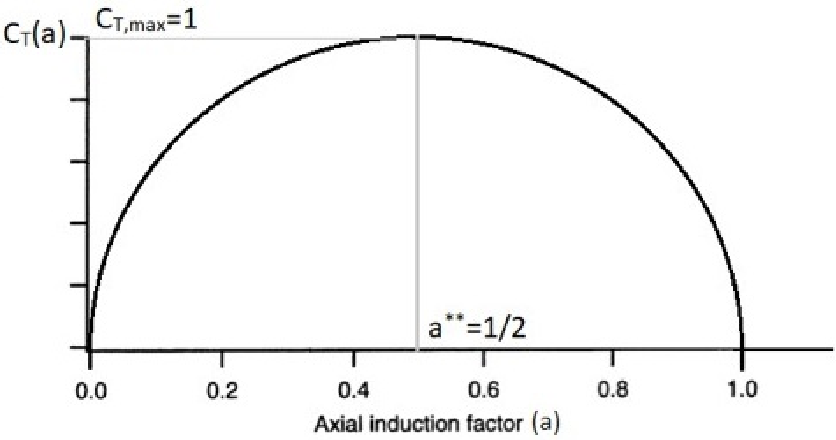
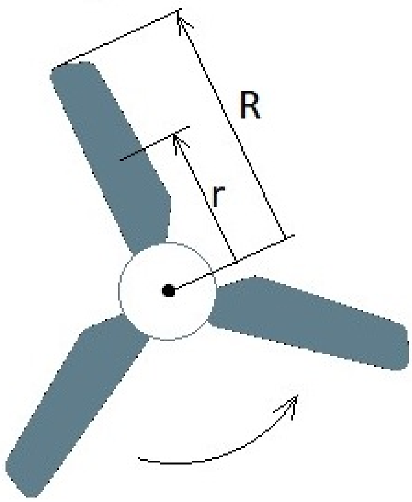


Figure 3.7

Because $C_{\mathrm{T}}(a) = 4a(1-a)$, $C_{\mathrm{T}}(a^{**}) = 4 \cdot \frac{1}{2} \cdot (1 - \frac{1}{2}) = 1$.

3.3 Wake Rotation & Rotor Disc Theory



Tip speed: $R \cdot \Omega$

Speed at $r < R$: $r \cdot \Omega$

$\Omega = \frac{2\pi}{T}$ (T: period of rotation)

Figure 3.8

Air is deflected in the tangential direction by the blade. **Tangential induction** depends on r. V_1,the tangential induced velocity is found using: $\boxed{V_1 = r \cdot \Omega \cdot a'}$, where a' is the tangential induction factor. Figure 3.9 shows that with an initial tangential velocity of zero, the tangential velocity downwind will be in the opposite direction of the blade's rotation. Then, the tangential velocity downwind is $\boxed{V_3 = r \cdot \Omega \cdot (2a')}$, where the change of the tangential momentum equals $\dot{m} \cdot (V_3 - 0)$.

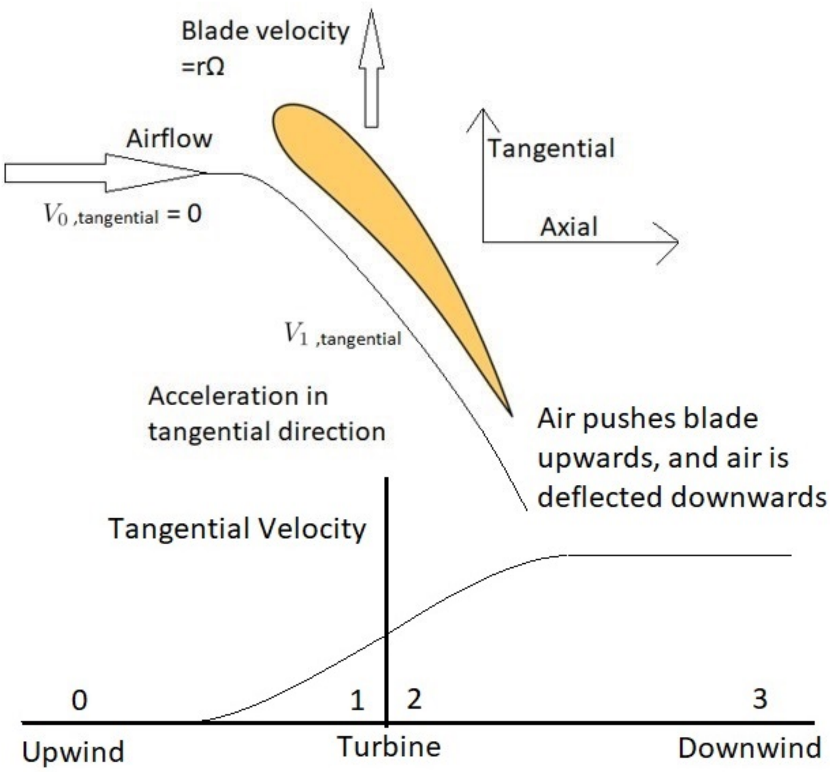


Figure 3.9

To compute $V_3(r)$ with given a, Ω, R and U_∞, regard an infinitesimal annulus of area $\mathrm{d}A$:

$$\boxed{\mathrm{d}A = 2\pi \cdot r \cdot \mathrm{d}r} \tag{3.15}$$

$$\int_0^R 2\pi \cdot r \cdot \mathrm{d}r = \pi R^2 \tag{3.16}$$

The infinitesimal power extracted:

$$\mathrm{d}P = \frac{1}{2}\rho \cdot U_\infty^3 \cdot \mathrm{d}A \cdot C_\mathrm{P}(a) \tag{3.17}$$

To harvest this power via rotary motion with the angular velocity Ω, we need a tangential force $\mathrm{d}F$. Thus:

$$\mathrm{d}P = \mathrm{d}F \cdot r \cdot \Omega \tag{3.18}$$

Because $F = \dot{m}\Delta V$ due to the momentum change ($\dot{m} = \rho \cdot A \cdot U_\infty(1-a)$),

$$\mathrm{d}F = \rho \cdot \mathrm{d}A \cdot U_\infty \cdot (1-a)(V_3 - 0) \tag{3.19}$$

From Eq. 3.17, 3.18 and 3.19 we get:

$$\frac{1}{2}\cancel{\rho}\, U_\infty^3 \cdot \cancel{\mathrm{d}A} \cdot C_\mathrm{P}(a) = r \cdot \Omega \cdot \cancel{\rho} \cdot \cancel{\mathrm{d}A} \cdot U_\infty(1-a) \cdot V_3 \tag{3.20a}$$

$$\frac{1}{2}U_\infty^2 C_\mathrm{P}(a) = r \cdot \Omega \cdot (1-a) \cdot V_3 \tag{3.20b}$$

$$\Rightarrow V_3 = \frac{2U_\infty^2 \cdot a \cdot (1-a)}{r \cdot \Omega} \tag{3.21}$$

Since $a'(r) = \frac{V_3(r)}{2 \cdot r \cdot \Omega}$, we know the tangential induction factor:

$$a'(r) = \frac{U_\infty^2 \cdot a(1-a)}{r^2 \cdot \Omega^2} \tag{3.22}$$

which means $V_3 \propto \frac{1}{r}$ (for a = constant). And with the local speed ratio $\lambda_r = \mu\lambda = \mu\frac{R\Omega}{U_\infty} = \frac{r\Omega}{U_\infty}$, where $\mu = \frac{r}{R}$, we also have:

$$a'(r) = \frac{a(1-a)}{\lambda_r^2} \tag{3.23}$$

We conclude that the wake rotates more if the turbine moves relatively slower (low λ) and higher λ leads to less wake rotation.

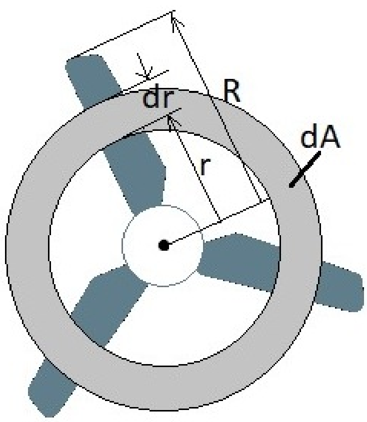


Figure 3.10

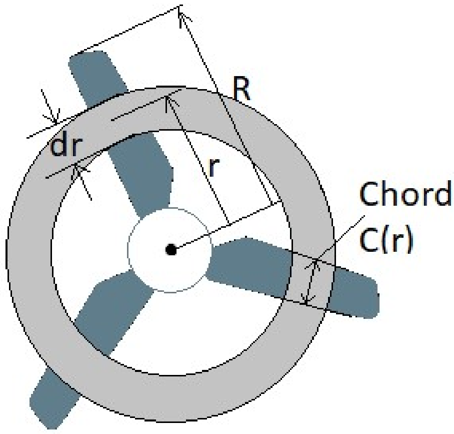


Figure 3.11

3.4 Blade element momentum theory (BEM)

Regard the annuli independent from each other like rotor discs (see figure 3.11), and assume that aerodynamic lift & drag accounting to 2-D airfoil theory. The “solidity” at radius r is defined as:

$$\sigma_r := \frac{B \cdot c(r)}{2\pi r}$$

where B is the number of blades, therefore the overall solidity is the total blade area divided by the disc area:

$$\sigma := \frac{B \cdot \int_0^R c(r)\mathrm{d}r}{\pi R^2} \tag{3.24}$$

Geometry & speeds:
Note: a & a' can depend on r, thus $a = a(r), a' = a'(r)$

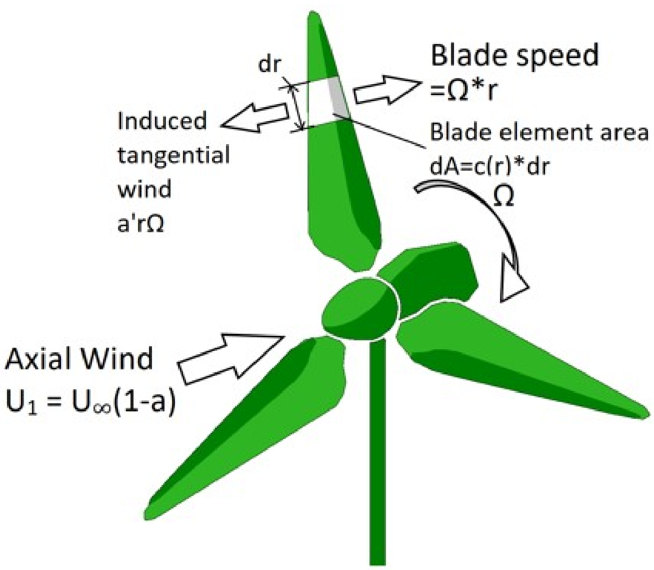


Figure 3.12

Blade element top view at r:

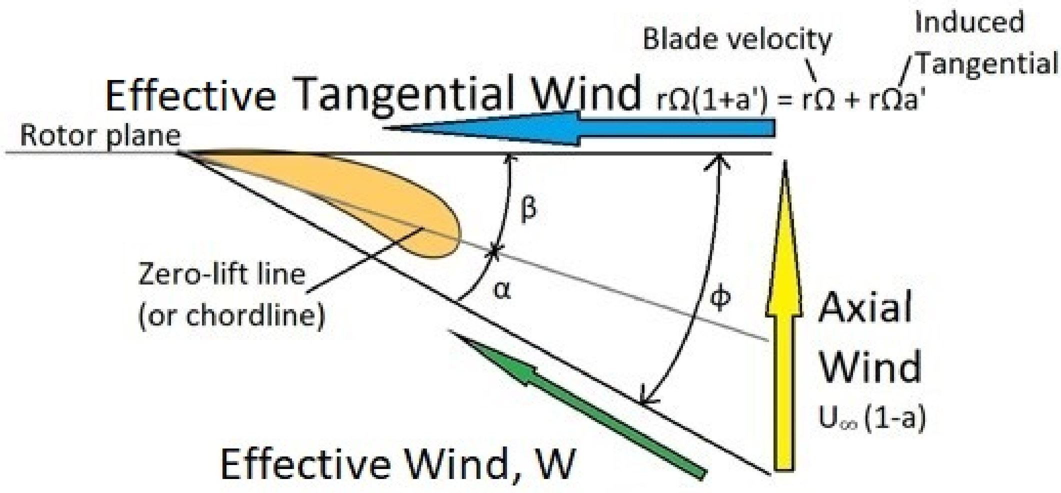
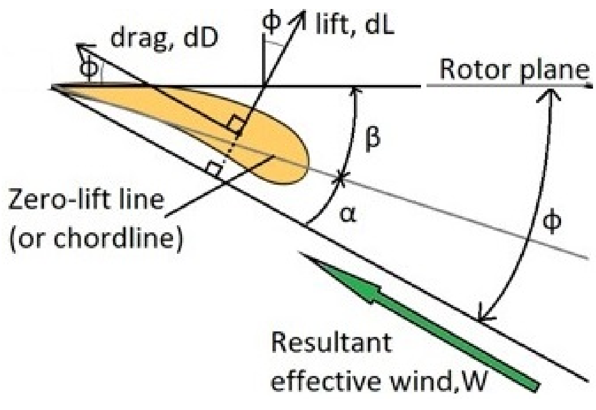


Figure 3.13 : β is the set pitch angle at radius r, α is the angle of attack, $\phi = \alpha + \beta$ is the flow angle

Effective wind magnitude:

$$W = \sqrt{U_\infty^2(1-a)^2 + r^2\Omega^2(1+a')^2} \tag{3.25}$$

With 2D-lift coefficient $c_\mathrm{l}(\alpha)$, 2D-drag coefficient $c_\mathrm{d}(\alpha)$ and the area of blade element $\mathrm{d}A_B = c \cdot \mathrm{d}r$ we get the lift and drag of blade element:



$$\begin{cases} dL = \frac{1}{2}\,\rho\, W^2\, \mathrm{d}A_B \cdot c_\mathrm{l} \\ dD = \frac{1}{2}\,\rho\, W^2\, \mathrm{d}A_B \cdot c_\mathrm{d} \end{cases}$$

Figure 3.14

Since $\sin\phi = \frac{U_\infty(1-a)}{W}$ and $\cos\phi = \frac{r\cdot\Omega\cdot(1+a')}{W}$, we can also have the following equations:

- Axial force on all blade elements:

$$\mathrm{d}F_A = B \cdot (\mathrm{d}L \cdot \cos\phi + \mathrm{d}D \cdot \sin\phi) \tag{3.26}$$

- Tangential force on all blade elements:

$$\mathrm{d}F_T = B \cdot (\mathrm{d}L \cdot \sin\phi - \mathrm{d}D \cdot \cos\phi) \tag{3.27}$$

(positive if the blade element produce power)

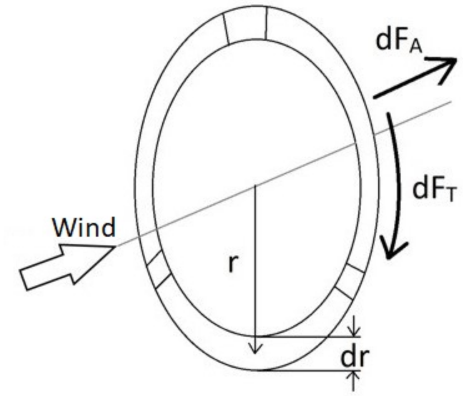


Figure 3.15

Axial and tangential force cause induction a & a' due to momentum balance (as before).

- Axial force:

$$\begin{aligned} \mathrm{d}F_A &= \mathrm{d}\dot{m} \cdot (2au_\infty) && (3.28a) \\ &= \rho \cdot 2\pi r \cdot \mathrm{d}r \cdot u_\infty(1-a) \cdot 2au_\infty && (3.28b) \end{aligned}$$

$$\Rightarrow \mathrm{d}F_A = \frac{1}{2}\rho U_\infty^2 \cdot 2\pi r \cdot \mathrm{d}r \cdot 4a(1-a) \tag{3.29}$$

- Tangential momentum change:

$$\mathrm{d}F_T = \mathrm{d}\dot{m}(2a' \cdot r \cdot \Omega) = \frac{1}{2}\rho U_\infty \cdot r \cdot \Omega \cdot 2\pi r \ \mathrm{d}r \cdot 4a'(1-a) \tag{3.30}$$

From Eq.3.26 = Eq.3.29, Eq.3.27 = Eq.3.30, we get two equations for two unkowns a & a' which need to be solved numerically.

Let us first simplify our equations:

Eq. 3.26 = Eq. 3.29:

$$\frac{1}{2}\rho W^2 \cdot B \cdot c(c_\mathrm{l}\cos\phi + c_\mathrm{d}\sin\phi)\mathrm{d}r = \frac{1}{2}\rho 2\pi r \mathrm{d}r U_\infty^2 4a(1-a) \tag{3.31a}$$

$$\Rightarrow W^2 \cdot B \cdot c(c_\mathrm{l}\cos\phi + c_\mathrm{d}\sin\phi) = 2\pi r U_\infty^2 4a(1-a) \tag{3.31b}$$

Eq. 3.27 = Eq. 3.30:

$$\frac{1}{2}\rho W^2 \cdot B \cdot c(c_\mathrm{l}\sin\phi - c_\mathrm{d}\cos\phi)\mathrm{d}r = \frac{1}{2}\rho 2\pi r \mathrm{d}r U_\infty r\Omega 4a'(1-a) \tag{3.32a}$$

$$\Rightarrow W^2 \cdot B \cdot c(c_\mathrm{l}L\sin\phi - c_\mathrm{d}\cos\phi) = 2\pi r^2 U_\infty \Omega 4a'(1-a) \tag{3.32b}$$

Use solidity $\sigma_r = \frac{B \cdot c}{2\pi r}$, local speed ratio $\lambda_r = \frac{r\Omega}{U_\infty}$ and W:

$$W = \sqrt{U_\infty^2 \lambda_r^2 (1+a')^2 + U_\infty^2 (1-a)^2} \tag{3.33a}$$

$$= U_\infty \sqrt{\lambda_r^2 (1+a')^2 + (1-a)^2} \tag{3.33b}$$

with the expressions:

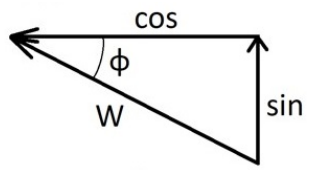


Figure 3.16

$$\sin\phi = \frac{(1-a)}{\sqrt{\lambda_r^2(1+a')^2+(1-a)^2}}$$
$$\cos\phi = \frac{\lambda_r(1+a')}{\sqrt{\lambda_r^2(1+a')^2+(1-a)^2}}$$

Therefore we get the equivalent formula:

From Eq. 3.31b:

$$\begin{aligned}(\lambda_r(1+a')^2 + (1-a)) \cdot \sigma_r \cdot (c_{\mathrm{l}} \frac{\lambda_r(1+a')}{\lambda_r^2(1+a')^2+(1-a)^2} \\ + c_{\mathrm{d}} \frac{(1-a)}{\sqrt{\lambda_r^2(1+a')^2+(1-a)^2}}) = 4a(1-a)\end{aligned} \tag{3.34a}$$

$$\begin{aligned}\Rightarrow \sqrt{\lambda_r^2(1+a')^2+(1-a)^2} \cdot \sigma_r \cdot (c_{\mathrm{l}} \lambda_r (1+a' + c_{\mathrm{d}}(1-a))) \\ = 4a(1-a)\end{aligned} \tag{3.34b}$$

From Eq. 3.32b:

$$\begin{aligned}\frac{\sqrt{\lambda_r^2(1+a')^2+(1-a)^2} \cdot \sigma_r}{\lambda_r} (c_{\mathrm{l}}(1-a) - c_{\mathrm{d}} \lambda_r (1+a')) \\ = 4a'(1-a)\end{aligned} \tag{3.35}$$

Dividing both eq. 3.34b by eq. 3.35 for each side gives:

$$\lambda_r \cdot \frac{c_{\mathrm{l}}\lambda_r(1+a') + c_{\mathrm{d}}(1-a)}{c_{\mathrm{l}}(1-a) - c_{\mathrm{d}}\lambda_r(1+a')} = \frac{a}{a'} \tag{3.36}$$

Recall from rotor disc theory $a' = \frac{a(1-a)}{\lambda_r^2}$:

$$a' = \frac{a}{\lambda_r} \frac{(1-a) - \frac{c_{\mathrm{d}}}{c_{\mathrm{l}}}\lambda_r(1+a')}{\lambda_r(1+a') + \frac{C_D}{C_L}(1-a)} \tag{3.37a}$$

$$= \frac{a(1-a)}{\lambda_r^2} \left(\frac{1 - \frac{c_{\mathrm{d}}}{c_{\mathrm{l}}} \lambda_r \frac{1+a'}{1-a}}{1 + \frac{c_{\mathrm{d}}}{c_{\mathrm{l}}}(1-a) \cdot \frac{1}{\lambda_r}} \right) \tag{3.37b}$$

And if we get the quadratic equation in a':

$$a'^2 + \left(1 + \frac{c_\mathrm{d}}{c_\mathrm{l}} \cdot \frac{1}{\lambda_r}\right) a' - \left(\frac{a(1-a)}{\lambda_r^2} - \frac{c_\mathrm{d}}{c_\mathrm{l}}\frac{a}{\lambda_r}\right) \tag{3.38}$$

There will be only positive solution meaningful:

$$a' = -\frac{1 + \frac{c_\mathrm{d}}{c_\mathrm{l}} \cdot \frac{1}{\lambda_r}}{2} + \sqrt{\frac{(1 + \frac{c_\mathrm{d}}{c_\mathrm{l}} \cdot \frac{1}{\lambda_r})^2}{4} + \frac{a(1-a)}{\lambda_r^2} - \frac{C_D}{c_\mathrm{l}}\frac{a}{\lambda_r}} \tag{3.39}$$

For $C_D = 0$, we set:

$$a' = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{a(1-a)}{\lambda_r^2}} \tag{3.40a}$$

$$= -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4a(1-a)}{\lambda_r^2}} \tag{3.40b}$$

$$= -\frac{1}{2} + \frac{1}{2} + \frac{1}{4}\frac{4a(1-a)}{\lambda_r^2} \tag{3.40c}$$

$$= \frac{a(1-a)}{\lambda_r^2} + O(\lambda_r^{-4}) \tag{3.40d}$$

Note that by Taylor series $\sqrt{1+x} = 1 + \frac{1}{2}x + O(x^2)$

3.4.1 BEM example

Let us now assume a few typical values: $\lambda_r \in [1, 7]$ and $B = 3$. For $\lambda_R = 7$, $a = \frac{1}{3}$ for all r (extracting maximum power due to Betz' limit):

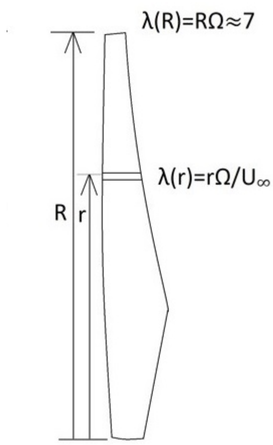


Figure 3.17

Let's assume that the 2D lift and drag coefficients of each radial airfoil can be chosen to be $c_\mathrm{l} = 1$ and $c_\mathrm{d} = 0.01$.

$a' \approx \frac{a(1-a)}{\lambda_r^2} = \frac{2}{9} \cdot \frac{1}{\lambda_r^2}$

Get local solidity from Eq. 3.34b:

$$\begin{aligned}
\sigma_r &= \frac{4a(1-a)}{C_L\lambda_r(1+a') + C_D(1-a)} \cdot \frac{1}{\sqrt{\lambda_r^2(1+a')^2 + (1-a)^2}} \\
&= \frac{8}{9}\frac{1}{\lambda_r(1+a')\underbrace{+0.00\bar{6}}_{\approx 0}} \cdot \frac{1}{\lambda_r(1+a')\sqrt{1+\underbrace{\frac{4}{9}\frac{1}{(1+a')\lambda_r)}}_{\approx 0}}} \\
&\approx \frac{8}{9}\frac{1}{\lambda_r^2}
\end{aligned}$$

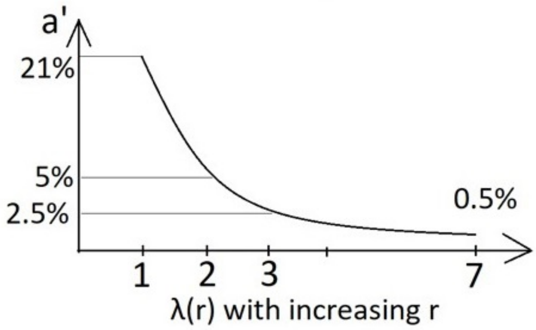


Figure 3.18

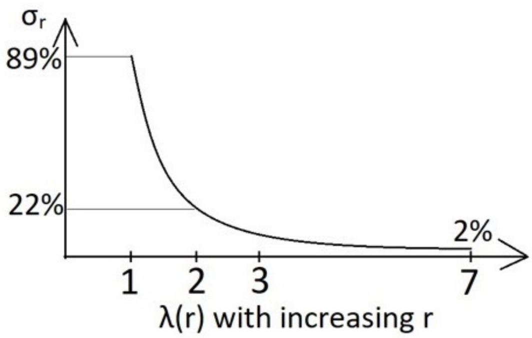


Figure 3.19

What does this mean for chord length c?

Since $\sigma_r = \frac{B \cdot c}{2\pi r} = \frac{8}{9}\frac{1}{\lambda_r^2}$, we know:

$$\begin{aligned} c &= \frac{2\pi r}{B} \cdot \frac{8}{9} \cdot \frac{R^2}{r^2} \cdot \frac{1}{\lambda_R^2} \\ &= \frac{2\pi R}{B\lambda_R^2} \cdot \frac{8}{9} \cdot \frac{1}{\mu} \\ &\approx \frac{2\pi R}{B} \cdot 2\% \cdot \frac{1}{\mu} \approx 4\% \frac{R}{\mu} \end{aligned}$$

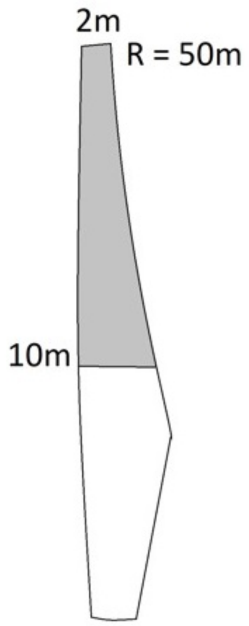


Figure 3.20

For $R = 50\,\mathrm{m}$,
we get $c(R) = 2\,\mathrm{m}$ and
$c(10\,\mathrm{m}) = 10\,\mathrm{m}$.

$\boxed{\text{Most important equation to remember:}}$

$$\left.\begin{array}{l} \lambda_r = \frac{r}{R} \cdot \lambda \gg 1 \\[1em] \frac{C_L}{C_D} \gg 1 \end{array}\right\} \text{Assumptions}$$

Axial momentum balance:
Force on blade area:

$$F_B = \frac{1}{2}\,\rho \cdot A_B\,\underbrace{(\lambda_r U_\infty)^2}_{W^2} \cdot C_L$$

Equals thrust on annulus:

$$\begin{aligned} F_A &= \rho A_A U_\infty (1-a)(2a \cdot U_\infty) \\ &= \frac{1}{2}\rho A_A \cdot U_\infty^2 \cdot \underbrace{4a(1-a)}_{C_T(a)} \end{aligned}$$

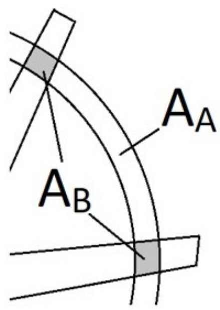


Figure 3.21

Local solidity:

$$\sigma_r = \frac{A_B}{A_A} = \frac{B \cdot c(r)}{2\pi r}$$

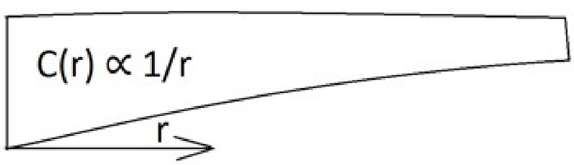
Optimal chord:

$$\cancel{\frac{1}{2}\rho U_\infty^2} \cdot A_B \cdot C_L \cdot \lambda_r^2 = \cancel{\frac{1}{2}\rho U_\infty^2} \cdot A_A \cdot 4a(1-a)$$

$$\Rightarrow \boxed{\sigma_r \cdot C_L = \frac{4a(1-a)}{\lambda_r^2} = \frac{8}{9} \cdot \frac{1}{\lambda_r^2}} \; a = \frac{1}{3} \text{ (optimal)}$$

$$\Rightarrow \frac{B \cdot c(r)}{2\pi r} C_L(r) = \frac{8}{9}\frac{1}{\lambda^2}\frac{R^2}{r^2}$$

$$\Rightarrow c(r) = \frac{1}{B \cdot C_L(r)} \frac{2\pi \cdot 8 \cdot R^2}{9 \cdot \lambda^2 \cdot r} \propto \frac{1}{r}$$



For a fixed c_l (and the above assumptions), the optimal chord is inversely proportional to radius

Figure 3.22

3.4.2 Practical blade design

In practice "linear taper" is often used (see figure 3.23), meaning that the chord length is an affine function of the radius, avoiding the excessive growth of the chord length close to the hub. To compensate for the lower solidity, one can increase the angle of attack in order to increase $C_L(r)$ accordingly. Here the drag loss in the inner part of the blade is less important to us.

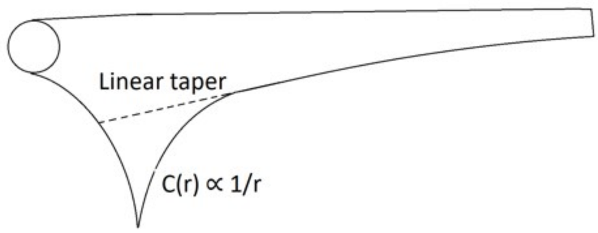


Figure 3.23

3.4.3 What is the flow angle?

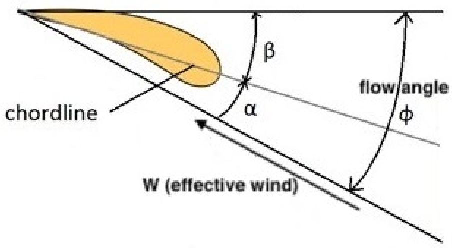
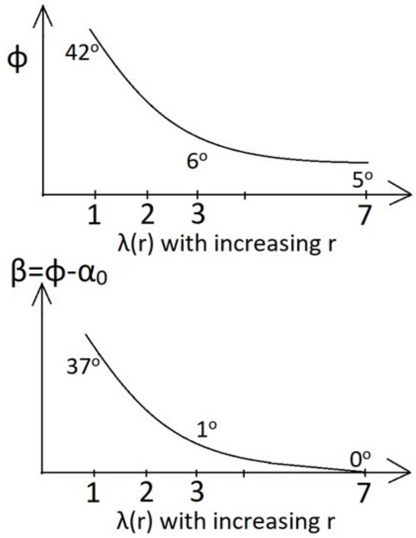


Figure 3.24

The flow angle $\phi = \beta + \alpha$, where β is the twist, therefore $\alpha = \phi - \beta$ (fixed α for the best $\frac{C_L}{C_D}$).

$$\sin\phi = \frac{(1-a)}{\lambda_r(1+a')\sqrt{1+\underbrace{\frac{(1-a)^2}{\lambda^2(1+a')^2}}_{\approx 0}}} \approx \frac{1-a}{\lambda_r(1+a')} \approx \frac{2}{3}\frac{1}{\lambda_r}\ (a = \tfrac{2}{3}, a' = 0)$$



$\phi = \sin^{-1}(\frac{2}{3} \cdot \frac{R}{r \cdot \lambda}) \approx \frac{2}{3}\frac{R}{\lambda} \cdot \frac{1}{r}$

(inversely proportional to r)

Figure 3.25 Refer to Page 36 for angle definitions.
Here α_0 is an assumed constant angle of attack.

3.5 The power harvesting factor zeta

Recall the definition of the power harvesting factor ζ:

$$\zeta = \frac{\text{power actually harvested}}{\text{wind power through blade area}} = \frac{P}{\frac{1}{2}\rho A_{\mathrm{B}} U_0^3}$$

Here we have a radial dependence of ζ, i.e., we write $\zeta(r)$

$$\zeta(r) = \frac{C_{\mathrm{p}} 2\pi r \mathrm{d}r \frac{1}{2}\rho U_0^3}{\sigma_r 2\pi r \mathrm{d}r \frac{1}{2}\rho U_0^3} = \frac{C_{\mathrm{p}}}{\sigma_r} \tag{3.41}$$

Most basic model (neglecting blade drag and other losses):

$$C_{\mathrm{p}}(a) = 4a(1-a)^2$$

$$\sigma_r C_{\mathrm{L}} \lambda_r^2 = 4a(1-a) \rightarrow \sigma_r = \frac{4a(1-a)}{\lambda_r^2 C_{\mathrm{L}}}$$

$$\zeta = \frac{C_{\mathrm{p}}}{\sigma_r} = \frac{4a(1-a)^2 \lambda_r^2 C_{\mathrm{L}}}{4a(1-a)} = (1-a) C_{\mathrm{L}} \lambda_r^2 \tag{3.42}$$

At optimal $a = \frac{1}{3} \rightarrow \boxed{\zeta = \frac{2}{3} C_{\mathrm{L}} \lambda_r^2}$ for optimal choice.

Case A - at outer blade element (tip): $\lambda_r^2 \approx 7$, $C_{\mathrm{L}} \approx 1$, $\zeta \approx 32$
Case B - half way: $\zeta \approx 8$

Chapter 4

Mechanics & Dynamics of Wind Turbines

Loads and Forces:

Sources:

- Aerodynamics (lift & drag)
- Gravity
- Inertia (gyroscopic & centrifugal)
- Electro mechanical (generator torque)
- Operational (brakes, yaw and pitch actuator)

Type of Loads:

- Steady (static & rotational)
- Cyclic: multiples (harmonics) of rotation frequency
  - "1P" once per revolution
  - "3P" 3 times per revolution
  - "B.P" B times per revolution (If B = number of blades, B.P = "Blade passing frequency")
- Resonant (vibration of tower & blades)
- Transient (start, stop, yew)
- Stochastic (wind)

4.1 Steady loads in normal operation

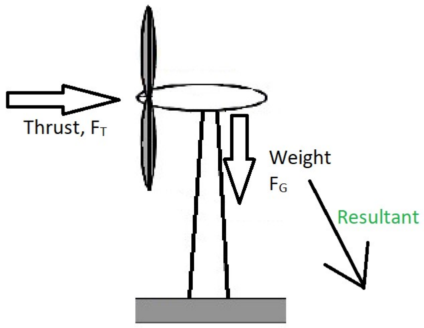


Figure 4.1

$$F_T \approx \frac{P}{\frac{2}{3}U_\infty} \tag{4.1}$$

$$F_G = m_N \cdot g \tag{4.2}$$

Example: $P = 6\,\mathrm{MW}$, $U_\infty = 9\,\mathrm{m/s}$, $m_N = 360\,\mathrm{t}$

$$F_T \approx \frac{P}{\frac{2}{3}U_\infty} = \frac{6\,\mathrm{MW}}{\frac{2}{3}\cdot 9\,\mathrm{m/s}} = 1\,\mathrm{MN}$$

$$\begin{aligned} F_G &= m_N \cdot g \\ &= 360 \cdot 10^3\mathrm{kg} \cdot 9.81\,\mathrm{m/s^2} \\ &= 3.6\,\mathrm{MN} \end{aligned}$$

4.2 Stress and strain

Regard material under tension:

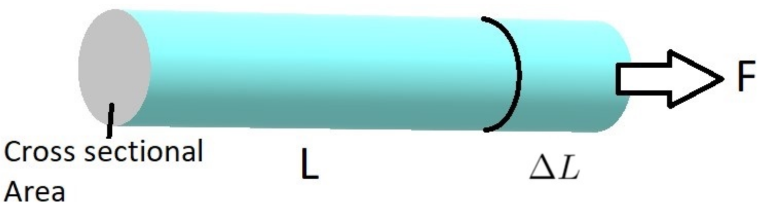


Figure 4.2

Stress:

$$\sigma = \frac{F}{A}\ [\mathrm{Pa}] \tag{4.3}$$

Strain:

$$\varepsilon = \frac{\Delta L}{L}\ [-] \tag{4.4}$$

Stress-strain curve:

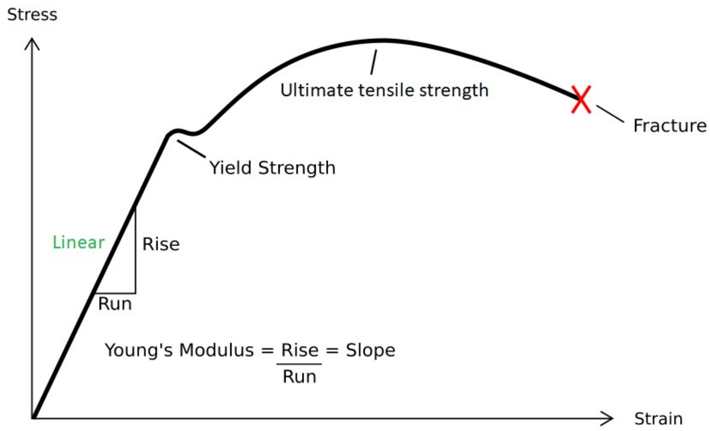


Figure 4.3

Example steel: Young's modulus $E = 200\,\text{GPa}$, Yield strength $Y = 250\,\text{MPa}$, [Ultimate tensile stregth $U = 500\,\text{MPa}$]

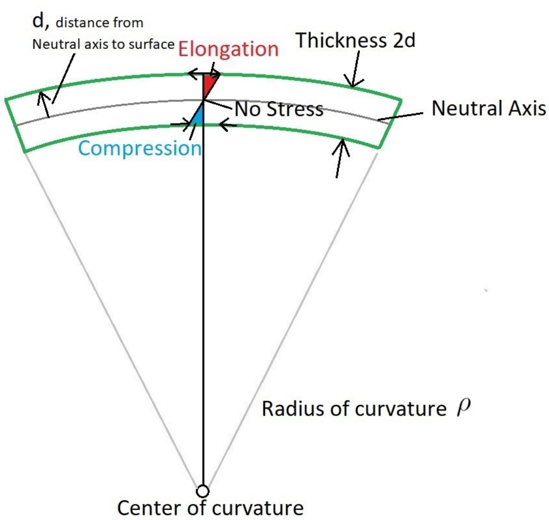
At which strain does a steel start to deform plastically/ permanently?

$$\sigma_Y = E \cdot \varepsilon_Y \tag{4.5}$$

$$\sigma_Y = Y \tag{4.6}$$

$$\varepsilon_Y = \frac{Y}{E} \tag{4.7}$$

When does a beam start to deform?



$$\varepsilon_Y = \frac{d}{\rho} \tag{4.8}$$

Figure 4.4

4.3 (Static) beam bending (Euler-Bernoulli theory)

Hooke's law:

$$\boxed{\sigma = E \cdot \varepsilon} \tag{4.9}$$

σ = stress [Pa]
E = Young's modulus [Pa]
ε = strain (deformation) [%]

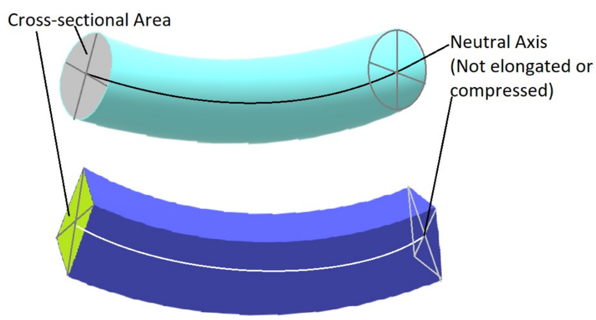


Figure 4.5

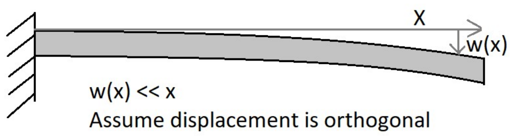


Figure 4.6

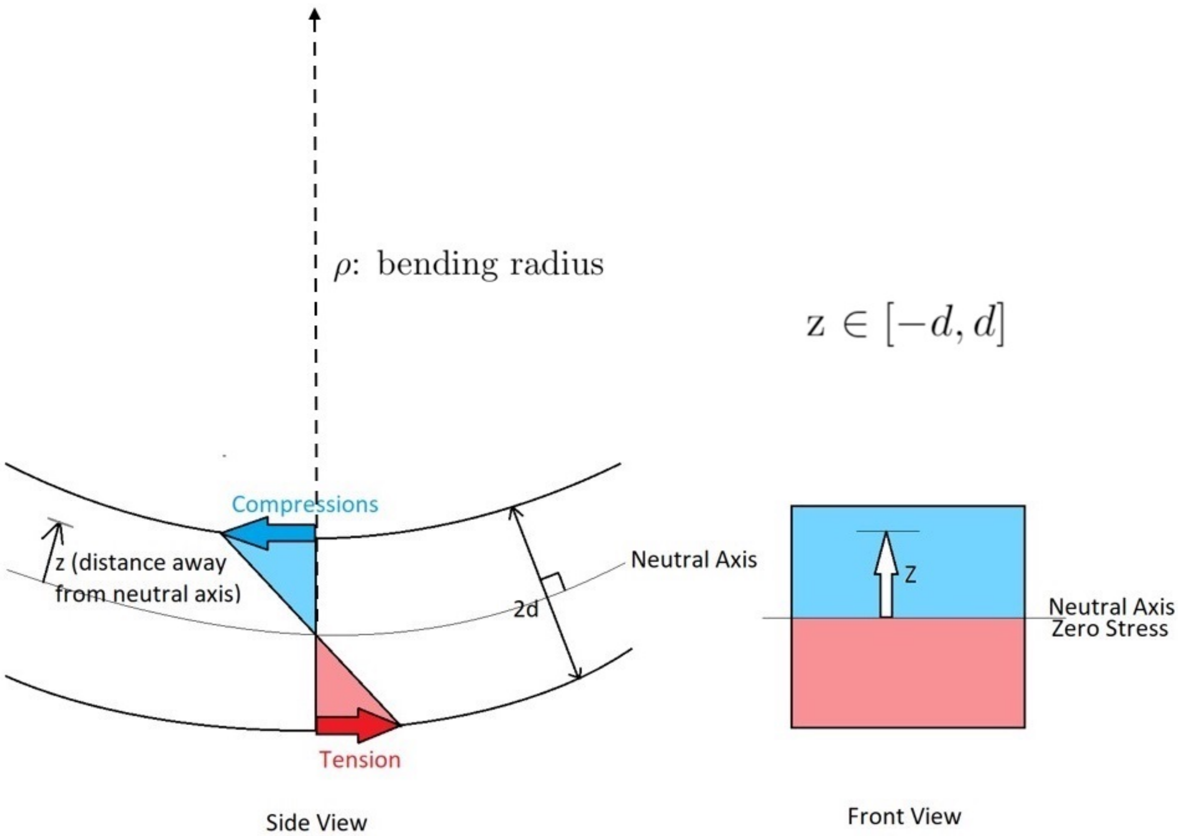


Figure 4.7

Strain:

$$\varepsilon = \frac{z}{\rho} \tag{4.10}$$

Because $\frac{1}{\rho} = \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}$ we get:

$$\varepsilon = z \cdot \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} \tag{4.11}$$

Bending moment:

$$M(x) = \int_{-d}^{d} z \cdot \sigma(z) \cdot \mathrm{d}A \tag{4.12a}$$

$$= \int_{-d}^{d} z \cdot E \cdot z \cdot (\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}) \cdot \mathrm{d}A \tag{4.12b}$$

$$= E(\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}) \underbrace{\int_{-d}^{d} z^2 \mathrm{d}A}_{:= I \text{ (second moment of area)}} \tag{4.12c}$$

$$= E \cdot I \cdot \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} \tag{4.12d}$$

$$\Rightarrow M = E \cdot I \cdot \frac{1}{\rho} \tag{4.13}$$

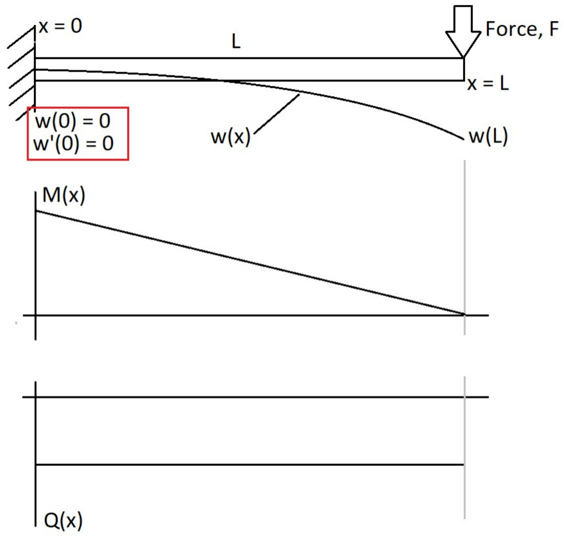
Static beam equation/ Euler Bernoulli:

$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(E(x)I(x)\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}\right) = q(x)} \tag{4.14}$$

"Shear force" $= \frac{\mathrm{d}M(x)}{\mathrm{d}x} = Q(x)$

"Distributed load" $= \frac{\mathrm{d}^2 M(x)}{\mathrm{d}x^2} = \frac{\mathrm{d}Q(x)}{\mathrm{d}x} = q(x)$

Example 1 - Cantilever beam with end load:



$$M(x) = E \cdot I \cdot \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = F(L - x)$$

$$Q(x) = \frac{\mathrm{d}M}{\mathrm{d}x} = -F$$

$$\frac{\mathrm{d}Q(x)}{\mathrm{d}x} = 0$$

E, I is constant and there is no distributed load, $q(x) = 0$
(No gravity of the beam)

Figure 4.8

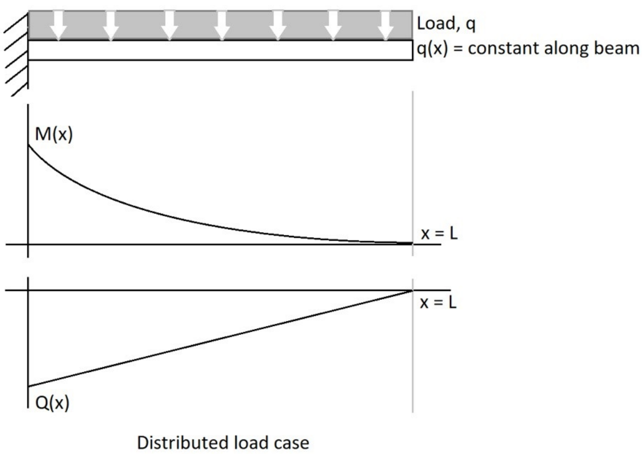
$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \frac{F}{E \cdot I} \cdot (L - x)$

$w(x) = \frac{F}{E \cdot I}(L\frac{x^2}{2} - \frac{x^3}{6} + c_0 + c_1 x)$, with initial value $c_1 = 0$ and $c_0 = 0$

$\Leftrightarrow w(x) = \frac{Fx^2}{6EI}(3 \cdot L - x)$

$$\underbrace{W(L)}_{\text{displacement}} = \frac{F \cdot L^2}{6EI}(2 \cdot L) = \overbrace{F}^{\text{force}} \cdot \underbrace{\frac{L^3}{3 \cdot E \cdot I}}_{\text{spring constant}}$$

Example 2 - Cantilever beam with constant loading:



$q(x) \equiv q \equiv$ constant
E, I is constant.

Figure 4.9

$$\frac{q}{E \cdot I} = \frac{\mathrm{d}^4 w(x)}{\mathrm{d}x^4} \tag{4.15}$$

$$\Leftrightarrow w(x) = \frac{q}{E \cdot I} \cdot (\frac{x^4}{24} + c_3 x^3 + c_2 x^2 + c_1 x + c_0) \tag{4.16}$$

Boundary conditions: $w(0) = 0 \Rightarrow c_0 = 0$, $\frac{\mathrm{d}w(0)}{\mathrm{d}x} = 0 \Rightarrow c_1 = 0$

$$M(x) = \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \cdot E \cdot I \tag{4.17}$$

$M(L) = 0 \Rightarrow \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}(L) = 0$

$$Q(x) = \frac{\mathrm{d}M(x)}{\mathrm{d}x} \tag{4.18}$$

$Q(L) = 0 \Rightarrow \frac{\mathrm{d}^3 w}{\mathrm{d}x^3}(L) = 0$

From Eq 4.17:

$$\frac{x^2}{2} + 6c_3 x + 2c_2 \bigg|_{x=L} = 0 \tag{4.19}$$

From Eq 4.18:

$$x + 6c_3|_{x=L} = 0 \Rightarrow c_3 = -\frac{1}{6} \tag{4.20}$$

From Eq 4.19:

$$\frac{L^2}{2} - L^2 + 2c_2 = 0 \Leftrightarrow c_2 = \frac{1}{4}L^2 \tag{4.21}$$

$$w(x) = \frac{q}{E \cdot I}(\frac{x^4}{24} + \frac{L}{6}x^3 + \frac{1}{4}L^2 x^2) \tag{4.22a}$$

$$= \frac{qx^2}{EI \cdot 24}(x^2 - 4Lx + 6L^2) \tag{4.22b}$$

$$M(x) = E \cdot I \cdot \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 9 \cdot (\frac{1}{2}x^2 - Lx + \frac{L^2}{2}) \tag{4.23}$$

$$Q(x) = q(x - L) \tag{4.24}$$

4.3.1 Moment at the blade root

Euler-Bernoulli:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(E(x)I(x)\frac{\mathrm{d}^2w(x)}{\mathrm{d}x^2}\right) = q(x) =: M(x) - \text{“Bending Moment”}$$

$$\frac{\mathrm{d}M(x)}{\mathrm{d}x} = Q(x) \text{ - “Shear force”}$$

Recall: maximum strain $\epsilon_{\max} = \frac{Y}{E}$

$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2} \approx \frac{1}{\rho}$$

and

$$\frac{d}{\rho} = E$$

Maximum moment we can support:

$$\epsilon_{\max} = \frac{\mathrm{d}^2w}{\mathrm{d}x^2}(0) = \frac{d(0)M(0)}{I(0)E(0)} \tag{4.25}$$

$$M_{\max} = \epsilon_{\max}\frac{IE}{d} = \frac{IY}{d} \tag{4.26}$$

4.3.2 Loads at blade root (in flapwise direction)

For a blade in an ideal design, the distributed load $q(r)$ is given by $\frac{1}{B}$ the thrust of the corresponding annulus:

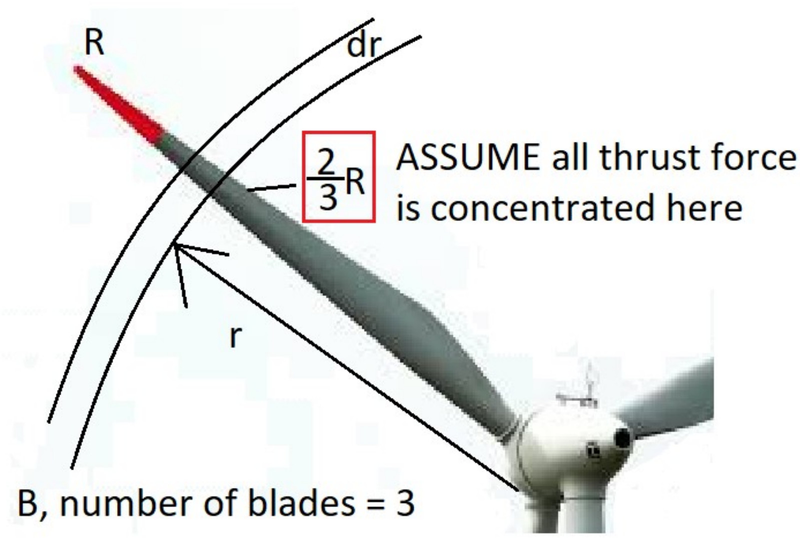


Figure 4.10

$$\mathrm{d}F = \underbrace{4a(1-a)}_{C_\mathrm{T}(a)} \cdot \frac{1}{2}\rho U_\infty^2 \cdot 2\pi r \cdot \mathrm{d}r \tag{4.27a}$$

$$= \underbrace{C_\mathrm{T}(a) \cdot \frac{1}{2}\rho U_\infty^2 \cdot 2\pi r}_{=B\cdot q(r)} \cdot \mathrm{d}r \tag{4.27b}$$

$$q(r) = C_\mathrm{T}(a) \cdot \frac{1}{2}\rho U_\infty^2 \cdot 2\pi r \cdot \frac{1}{B} \tag{4.28}$$

The bending moment at the bladeroot ($r = 0$) can be computed by integration:

$$M(0) = \int_0^R q(r) \cdot r\mathrm{d}r \tag{4.29a}$$

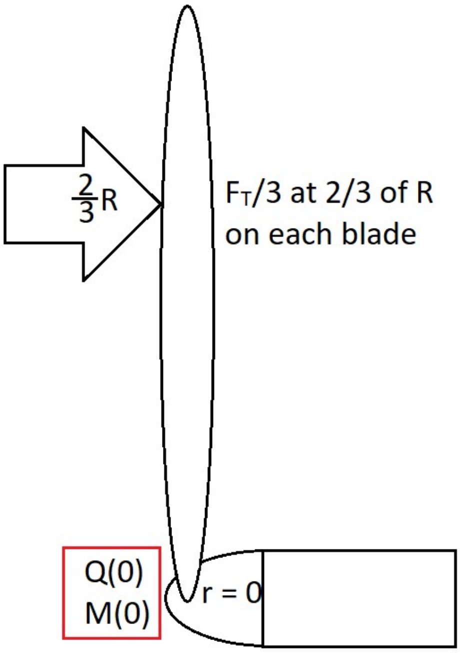
$$= \frac{1}{B} \cdot C_\mathrm{T}(a) \cdot \frac{1}{2}\ \rho\ U_\infty^2 \cdot 2\pi \cdot \underbrace{\int_0^R r^2 \mathrm{d}r}_{:= \frac{R^3}{3}} \tag{4.29b}$$

$$= \frac{1}{B} \cdot C_\mathrm{T}(a) \cdot \frac{1}{2}\ \rho\ U_\infty^2 \cdot \frac{2}{3}\pi R^3 \tag{4.29c}$$

$$= \frac{1}{B}\frac{2}{3} \cdot R \cdot \underbrace{C_\mathrm{T}(a) \cdot \frac{1}{2}\ \rho\ U_\infty^2 \cdot (\pi R^2)}_{:= F_T \text{ (Total force on actuator disc)}} \tag{4.29d}$$

Shear force at blade root is trivially given by $Q(0) = \frac{F_T}{B}$.

Easy to remember: $\frac{F_T}{B}$ (total force on blade) $\times\ \frac{2}{3}R$ ($\frac{2}{3}$ of radius) equals moment $M(0)$. If we assume all forces acting on $\frac{2}{3}R$, we get the right bending moment.



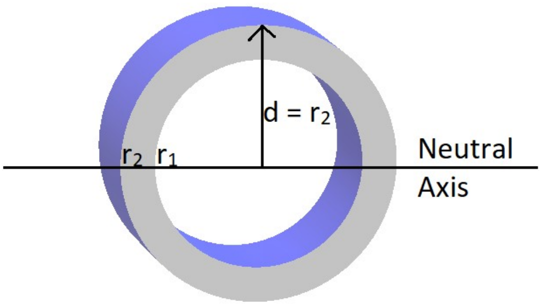
$$F_T \approx \frac{P}{(1-a)U_\infty} \tag{4.30}$$

Example: 1 MN for 6 MW at $U_\infty = 9\,\mathrm{m/s}$ and $R = 75\,\mathrm{m}$:

$M(0) = \frac{2}{3}R \cdot \frac{F_T}{3} = 50\,\mathrm{m} \cdot \frac{1}{3}\mathrm{MN}$
$\approx 16\,\mathrm{MN \cdot m}$

Figure 4.11

What is the maximum bending stress at blade root? Regard the annulus cross-section:



$r_2 - r_1 = b \ll r_2$

$I = \frac{\pi}{4}{r_2}^4 - \frac{\pi}{4}{r_1}^4 \approx \pi {r_2}^3 \cdot b$

$\frac{I}{r_2} = \pi {r_2}^2 \cdot b$

$\sigma_{\max} = \frac{r_2}{I} M(0) = \frac{M(0)}{\frac{I}{r_2}} = \frac{M(0)}{\pi {r_2}^2 \cdot b}$

Figure 4.12

If $r_2 = 1\,\mathrm{m}$, $\sigma_{\max} = 250\,\mathrm{MPa}$, how thick should the blade root shell be?

$$b = \frac{M(0)}{\pi {r_2}^2} \cdot \frac{1}{\sigma_{\max}} = 5\,\frac{\mathrm{MN \cdot m}}{\mathrm{m}^2} \cdot \frac{1}{250\,\mathrm{MPa}} = \frac{1}{50}\mathrm{m} = 2\,\mathrm{cm}$$

4.4 Oscillations & eigenmodes

4.4.1 Intro: spring-mass-damper-system

$$\boxed{m\ddot{x} + \beta\dot{x} + kx = F(t)} \tag{4.31}$$

$$\left[\begin{array}{l} x : \text{displacement, } m : \text{mass} \\ F(t) : \text{external force} \\ kx : \text{spring force} \\ \beta : \text{(viscous/ linear) damping} \end{array}\right.$$

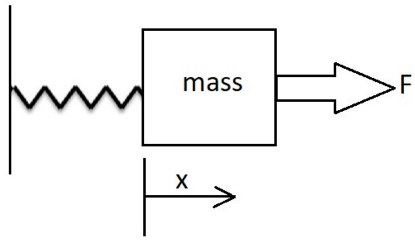


Figure 4.13

For $F(t) = F_0 \cdot e^{j\omega t}$, where $F_0 > 0$ and we take the real part of the solution in design, then the solution is given by:

$$x(t) = x_0 \cdot e^{j\omega t},\ x_0 \in \mathbb{C} \tag{4.32}$$

$$\dot{x} = (j\omega) \cdot x_0 e^{j\omega t} \tag{4.33}$$

$$\ddot{x} = -\omega^2 x_0 e^{j\omega t} \tag{4.34}$$

$$-m\omega^2 x_0 e^{j\omega t} + \beta j\omega x_0 e^{j\omega t} + kx_0 e^{j\omega t} = F_0 e^{j\omega t} \tag{4.35}$$

$$x_0 \cdot (\underbrace{k - m\omega^2}_{\text{real}} + \underbrace{j\beta\omega}_{\text{imaginary}}) = F_0 \tag{4.36}$$

x_0 is a complex number with magnitude:

$$|x_0| = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \beta^2\omega^2}} \tag{4.37}$$

Maximum $|x_0|$ is approximately taken at natural resonant “Eigen frequency” ω_{NR} with:

$$k - m\,{\omega_{\text{NR}}}^2 = 0 \Leftrightarrow \omega_{\text{NR}} = \sqrt{\frac{k}{m}} \tag{4.38}$$

How much can F_0 be amplified?

Spring force $F_0^{\text{spring}} = k \cdot x$

$$|F_0^{\text{spring}}| = k|x_0| = \frac{F_0}{\sqrt{(1-(\frac{\omega}{\omega_{\text{NR}}})^2)^2 + \frac{\beta^2}{k^2}\omega^2}} \tag{4.39}$$

At $\omega = \omega_{\text{NR}}$ we get:

$$\frac{|F_0^{\text{spring}}|}{F_0} = \frac{k}{\beta\ \omega_{\text{NR}}} \tag{4.40}$$

That is, the smaller the damping, the higher the amplification.

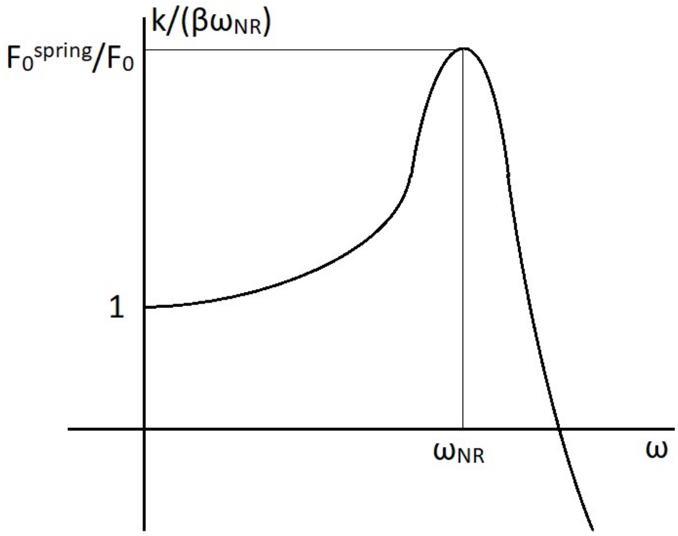


Figure 4.14 Bode diagram of $\frac{|F_0^{\text{spring}}(\omega)|}{F_0}$

Amplification factors can be 5 – 10, so resonance shall typically be avoid. At very low frequencies, spring force equals applied force, i.e., static analysis is sufficient (see section 4.2).

4.4.2 Eigenmodes

For spring-mass-damper systems with more than one degree of freedom. The displacement can be described by a vector $\mathbf{w(t)} \in \mathbb{R}^n$ and the equation of motion becomes:

$$\boxed{\mathbf{M\ddot{w}} + \mathbf{D\dot{w}} + \mathbf{Kw} = \mathbf{F(t)}} \tag{4.41}$$

$$\left[\begin{array}{l} \mathbf{M} : \text{mass matrix}, \ \in \mathbb{R}^{n\times n} \\ \mathbf{D} : \text{damping matrix} \\ \mathbf{K} : \text{stiffness matrix}, \ \in \mathbb{R}^{n\times n} \end{array}\right.$$

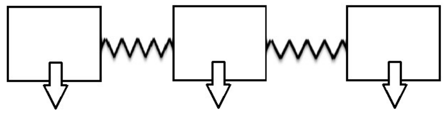


Figure 4.15

If damping is neglected ($\mathbf{D} = 0$), natural resonances must satisfy $\bar{\mathbf{w}} \in \mathbb{R}^n$:

$$\mathbf{w}(t) = \bar{\mathbf{w}} \cdot e^{j\omega t} \tag{4.42}$$

$$\mathbf{M\ddot{w}} + \mathbf{Kw} = 0 \tag{4.43}$$

That is,

$$-\omega^2 \mathbf{M}\bar{\mathbf{w}} + \mathbf{K}\bar{\mathbf{w}} = 0 \Leftrightarrow (\mathbf{M^{-1}K} - \omega^2 \mathbf{I})\bar{\mathbf{w}} = 0 \tag{4.44}$$

This is an eigenvalue equation for matrix $\mathbf{M^{-1}K} \in \mathbb{R}^{n\times n}$, and we know there are n eigenvalues with n eigenvectors $\bar{\mathbf{w}}$ ("eigenmodes"). As both $\mathbf{M}$ and $\mathbf{K}$ are positive definite, eigenvalues of $\mathbf{M^{-1}K}$ are real & positive. We are often only interested in the eigenmodes with lowest eigenfrequency.

4.4.3 Rayleighs method

Assume we have a good guess of an eigenmode vector, $\bar{\mathbf{w}} \in \mathbb{R}^n$. To find the corresponding $\omega^2 \in \mathbb{R}$, we can use the equation:

$$\mathbf{K}\bar{\mathbf{w}} = \omega^2 \mathbf{M}\bar{\mathbf{w}} \tag{4.45}$$

Eq. 4.45 is overdertermined if $\bar{\mathbf{w}}$ is fixed, but to multiply eq. 4.45 by $\frac{1}{2}\bar{\mathbf{w}}^T$ gives:

$$\underbrace{\frac{1}{2}\bar{\mathbf{w}}^\top \mathbf{K}\bar{\mathbf{w}}}_{\text{elastic/potential energy at max. displacement}} = \overbrace{\omega^2 \cdot \frac{1}{2}\bar{\mathbf{w}}^\top \mathbf{M}\bar{\mathbf{w}}}^{\text{kinetic energy at max. speed}} \tag{4.46}$$

$$\omega = \sqrt{\frac{\frac{1}{2}\bar{\mathbf{w}}^\top \mathbf{K}\bar{\mathbf{w}}}{\frac{1}{2}\bar{\mathbf{w}}^\top \mathbf{M}\bar{\mathbf{w}}}} := f(\bar{\mathbf{w}}) \tag{4.47}$$

If the guess of $\bar{\mathbf{w}}$ is good, this method can give surprisingly accurate estimation of ω. (To check, one can insert ω & $\bar{\mathbf{w}}$ in eq. 4.45).

What is the error of Rayleighs method?

Assume $\omega_0 \in \mathbb{R}$ and $\omega_0 \in \mathbb{R}^n$ are the true eigen-pair, i.e., they satisfy:

$$\mathbf{K}\mathbf{w_0} = \omega_0^2 \mathbf{M}\mathbf{w_0} \tag{4.48}$$

$\bar{\mathbf{w}} = \mathbf{w_0} + \Delta\mathbf{w}$ with $\Delta\mathbf{w}$ is the error of our guess. We then get:

$$\omega^2 = \underbrace{\frac{\frac{1}{2}\bar{\mathbf{w}}^\top \mathbf{K}\bar{\mathbf{w}}}{\frac{1}{2}\bar{\mathbf{w}}^\top \mathbf{M}\bar{\mathbf{w}}}}_{:= f(\bar{\mathbf{w}})} = \overbrace{\frac{\frac{1}{2}\mathbf{w_0}^\top \mathbf{K}\mathbf{w_0}}{\frac{1}{2}\mathbf{w_0}^\top \mathbf{M}\mathbf{w_0}}}^{:= f(\mathbf{w_0}) = {\omega_0}^2} + \nabla f(\mathbf{w_0})^\top \Delta\mathbf{w} + O(\|\Delta\mathbf{w}\|^2) \tag{4.49}$$

But here:

$$\nabla f(\mathbf{w_0}) = \frac{(\frac{1}{2}\mathbf{w_0}^\top \mathbf{M}\mathbf{w_0})\mathbf{K}\mathbf{w_0} - (\frac{1}{2}\mathbf{w_0}^\top \mathbf{K}\mathbf{w_0})\mathbf{M}\mathbf{w_0}}{(\frac{1}{2}\mathbf{w_0}^\top \mathbf{M}\mathbf{w_0})^2} \tag{4.50a}$$

$$= \frac{\mathbf{K}\mathbf{w_0} - {\omega_0}^2 \mathbf{M}\mathbf{w_0}}{(\frac{1}{2}\mathbf{w_0}^\top \mathbf{M}\mathbf{w_0})} = 0 \tag{4.50b}$$

Thus, the error is of second order:

$$\omega^2 = \omega_0^2 + O(\|\Delta w\|^2) \tag{4.51}$$

Example 1: for $\mathbf{M}$ & $\mathbf{K}$

$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0$

$m_1\ddot{x}_1 + k_1 \cdot x_1 - k_2(x_2 - x_1) = 0$

$\mathbf{w} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$

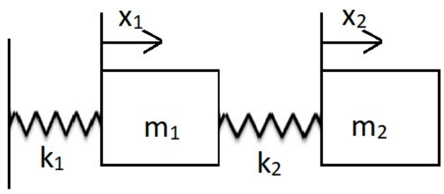


Figure 4.16

$$\underbrace{\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}}_{:= \mathbf{M} \in \mathbb{R}^{2\times 2}} \ddot{\mathbf{w}} + \overbrace{\begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}}^{:= \mathbf{K} \in \mathbb{R}^{2\times 2}} \mathbf{w} = 0 \tag{4.52}$$

Example 2:

$\mathbf{w(t)} = \bar{\mathbf{w}} \cdot e^{j\omega t}$

Assume $m_2 \gg m_1$, $k_1 \approx k_2$

$\bar{\mathbf{w}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (eigenvector)

$\mathbf{w(t)} = \begin{bmatrix} e^{j\omega t} \\ 2 \cdot e^{j\omega t} \end{bmatrix}$

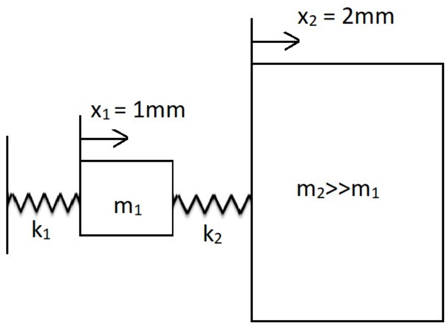


Figure 4.17

$$E_{\text{kin}} = \frac{1}{2}\bar{\mathbf{w}}^{\top}\mathbf{M}\bar{\mathbf{w}} = \frac{1}{2}(m_1 + 4m_2)\omega^2 {A_0}^2 \tag{4.53}$$

$$E_{\text{potential}} = \frac{1}{2}\bar{\mathbf{w}}^{\top}\mathbf{K}\bar{\mathbf{w}} \tag{4.54a}$$

$$= \frac{1}{2}{A_0}^2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}^{\top} \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{4.54b}$$

$$= \frac{1}{2}A_0 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}^{\top} \begin{bmatrix} (k_1 + k_2) - 2k_2 \\ -k_2 + 2k_2 \end{bmatrix} \tag{4.54c}$$

$$= \frac{1}{2}A_0(k_1 - k_2) + 2k_2 \tag{4.54d}$$

$$= \frac{1}{2}A_0(k_1 + k_2) \tag{4.54e}$$

$$\omega^2 = \frac{k_1 + k_2}{m_1 + 4m_2} \approx \frac{k_1 + k_2}{4} \cdot \frac{1}{m_2} \approx \frac{k_1}{2} \cdot \frac{1}{m_2} \tag{4.55}$$

4.4.4 Dynamic beam equation

Euler-Bernoulli & Lagrange produced equation 4.56, the Dynamic Beam Equation. Note, this equation also depends on time, and hence the "dynamic" beam equation.

$$\boxed{\frac{\partial^2}{\partial x^2}\left(E(x) \cdot I(x)\frac{\partial^2 w}{\partial x^2}\right) = q(x,t) - \mu(x) \cdot \frac{\partial^2 w}{\partial t^2}} \tag{4.56}$$

$$\left[\begin{array}{l} \mu(x) : \text{mass density per length} \\ q(x,t) : \text{distributed load} \\ w(x,t) : \text{time varying solution (no damping)} \end{array}\right.$$

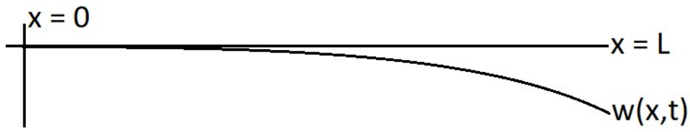


Figure 4.18

Note that this is a linear PDE, which after spacial discretization we get:

$$\boxed{Kw = -M \cdot \ddot{w}} \tag{4.57}$$

$$E_{\text{kin}} = \frac{1}{2}\int_0^L M(x) \cdot \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x \tag{4.58}$$

$$E_{\text{ela}} = \frac{1}{2}\int_0^L E(x)I(x) \cdot \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x \tag{4.59}$$

4.4.5 Tower eigenmodes

Both nacelles and towers have mass. For example, MHI-VESTAS V164 9.5 MW:

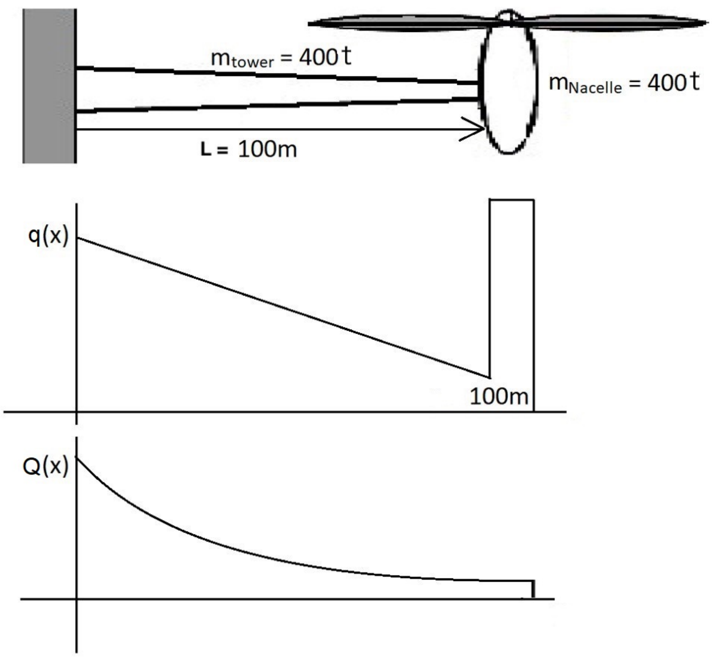


Figure 4.19

So the eigenmodes need to be computed for a very unequal mass distribution. The lowest two eigenmodes look approximately as follows:

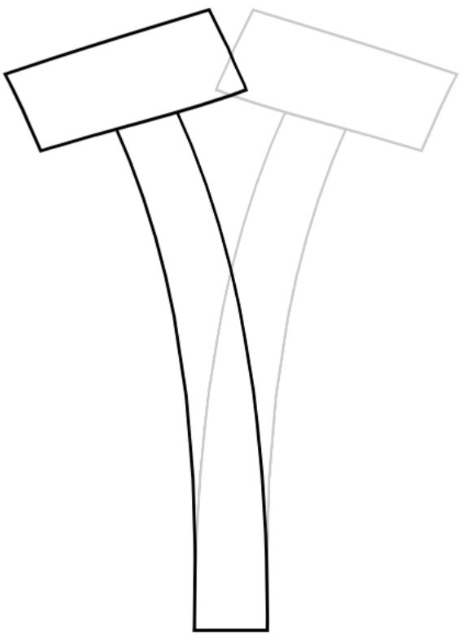


Figure 4.20 Lowest eigenmode

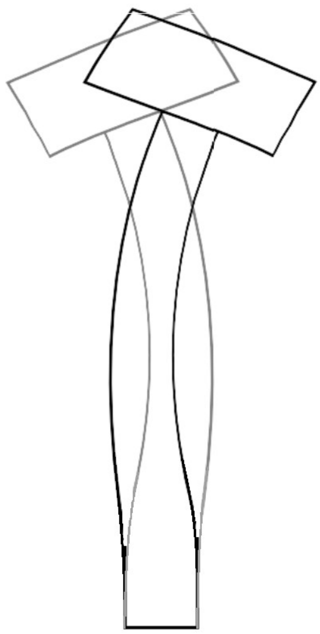


Figure 4.21 2nd-lowest eigenmode

Kinetic energy:

$$E_{\text{kin}} = \frac{1}{2}\int\limits_0^L \mu(x)\left(\frac{\partial w(x,t)}{\partial t}\right)^2 \mathrm{d}x \tag{4.60}$$

Elastic poetential energy:

$$E_{\text{ela}} = \frac{1}{2}\int\limits_0^L E(x)I(x)\left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 \mathrm{d}x \tag{4.61}$$

Assuming for the example above $w(x,t) = \bar{w}(x)\cdot e^{j\omega t}$ with $\bar{w}(x) = A_0 \cdot \frac{x^2}{L^2}$ for a rough approximation of the lowest eigenmode, and assuming constant mass $\mu(x)$, $E(x)$ and $I(x)$ throughout the tower, we would get the following estimation by using Rayleigh's method:

$$E_{\text{kin}} = \omega^2 \cdot \left(\frac{1}{2}\int\limits_0^L \frac{m_{\text{tower}}}{L}\left(\frac{A_0}{L^2}\right)^2 \left(x^2\right)^2 \mathrm{d}x + \frac{1}{2}m_{\text{nacelle}}{A_0}^2\right) \tag{4.62a}$$

$$= \frac{\omega^2}{2}{A_0}^2\left(\frac{m_{\text{tower}}}{L^5}\int\limits_0^L x^4 \mathrm{d}x + m_{\text{nacelle}}\right) \tag{4.62b}$$

$$= \omega^2\frac{{A_0}^2}{2}\left(\frac{1}{5}m_{\text{tower}} + m_{\text{nacelle}}\right) \tag{4.62c}$$

$$E_{\text{ela}} = \frac{1}{2}\int\limits_0^L E\cdot I\left(\frac{{A_0}^2}{L^2}\right)\mathrm{d}x \tag{4.63a}$$

$$= \frac{{A_0}^2}{2}E\cdot I\left(\frac{4}{L^4}\right)L \tag{4.63b}$$

$$= \frac{{A_0}^2}{2}E\cdot I\frac{4}{L^3} \tag{4.63c}$$

Equating $E_{\text{kin}} = E_{\text{ela}}$ gives:

$$\omega^2\left(\frac{m_{\text{tower}}}{5} + m_{\text{nacelle}}\right) = \frac{4EI}{L^3} \tag{4.64}$$

Site and weight of wind turbines:

Example 1: VESTAS V90, 1.8 MW

Tower height: 120 m

Blade length: $R = 45\,\text{m}$

$$\left.\begin{array}{l}\text{Nacelle weight: } 75\,\text{t}\\[1em] \text{3 blades weight: } 40\,\text{t}\end{array}\right\} = 115\,\text{t}$$

Tower weight: 152 t

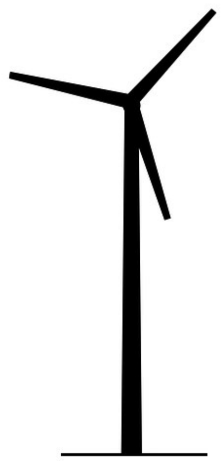


Figure 4.22

Example 2: MHI-VESTAS V164, 9.5 MW

Tower height: 105 m

Blade length: $R = 82\,\text{m}$

$$\left.\begin{array}{l}\text{Nacelle weight: } 390\,\text{t}\\[1em] \text{3 blades weight: } 105\,\text{t}\end{array}\right\} \approx 150\,\text{t}$$

Tower weight: 400 t

Base diameter: 6.5 m

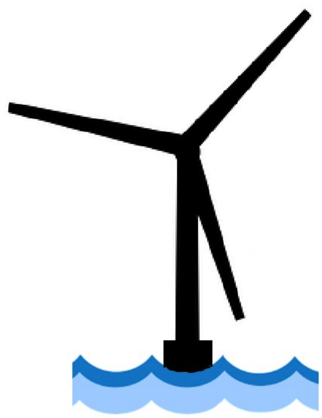


Figure 4.23

4.4.6 Stiff & soft towers

Lowest excitation freuquencies:

$1P$: "Rotor rotation frequency" (blade excitation, blade asymmetries)

$B.P$: "Blade passing frequency" (with B = number of blades)

With tip speed ratio $\lambda = \frac{R \cdot \Omega}{U_\infty}$, radius R and wind speed U_∞ we have:

$$\omega_{1P} = \Omega = \frac{\lambda \cdot U_\infty}{R} \tag{4.65}$$

$$\omega_{B.P} = B \cdot \Omega = B \cdot \frac{\lambda U_\infty}{R} \tag{4.66}$$

Note that we always have $\boxed{\omega_{B.P} = B \cdot \omega_{1P}}$. ω_{1P} typically varies with wind speed. There would be problems if ω_{1P} or $\omega_{B.P}$ become equal to tower eigenfrequencies, so we have to avoid resonance with **(a) tower design** and **(b) controller design**.

Given the range of operational speeds, the tower can be operated in three frequency domains (see figure 4.27):

Ⓐ: "soft-soft" if its lowest ω_{tower} is in region Ⓐ.

Ⓑ: "soft-stiff" if ω_{tower} is in region Ⓑ.

Ⓒ: "stiff-stiff" if ω_{tower} is in region Ⓒ, i.e., higher than $B.P$. In this case, all eigenfrequencies are above $\omega_{B.P}^{\max}$

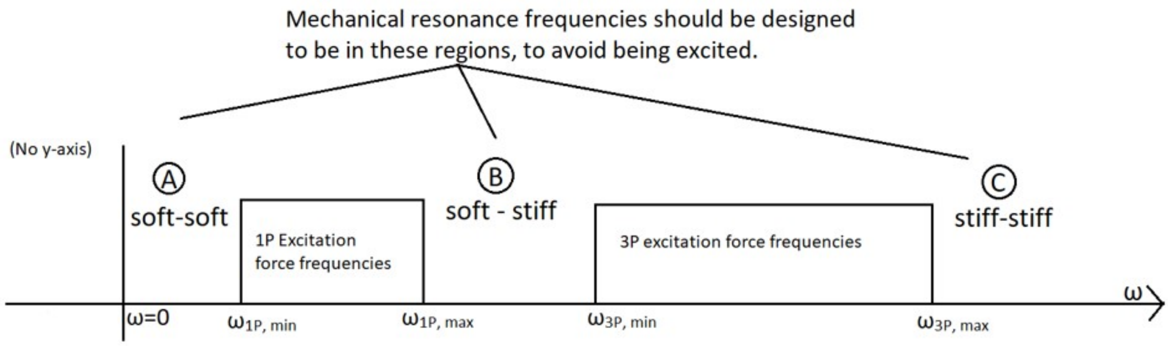


Figure 4.24

Note: The lowest eigenfrequency of tower matters!

Example: MHI-VASTAS V164

$$\begin{aligned} \lambda &= 8 \\ R &= 80\,\mathrm{m} \\ U_\infty &= 10\,\mathrm{m/s} \\ \Omega &= 1\,\mathrm{rad/s} \end{aligned}$$

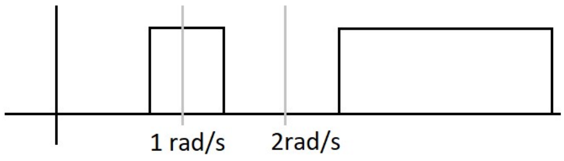


Figure 4.25

4.5 Blade oscillation & centrifugal stiffening

Blade oscillations mostly occur “flapwise”, i.e., forward-backward.

Figure 4.26

Interestingly, due to rotation, the blades “stiffen” and has higher eigenfrequencies than it would have without rotating. Let’s see why.

4.5.1 Rotating, hinged beam (no elasticity)

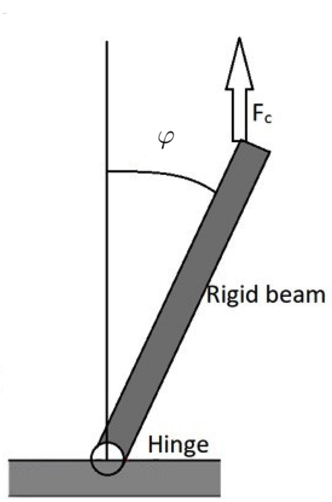


Figure 4.27

Moment of inertia:

$$I = \int_0^R \mu(r) \cdot r^2 \mathrm{d}r \tag{4.67}$$

Flapwise oscillation angle: φ

Rotating frequency: Ω

Restoring moment: $M(\varphi)$

$$\boxed{I\ddot{\varphi} = M(\varphi)} \tag{4.68}$$

Moment $M(\varphi)$ comes from centrifugal force:

$$M(\varphi) = -\int_0^R \mu(r)\Omega^2 \cdot r \cdot \underbrace{\cos(\varphi)}_{\approx 1} \cdot \overbrace{\sin(\varphi)}^{\approx \varphi} \cdot r\mathrm{d}r \tag{4.69a}$$

$$\approx -\varphi \cdot \Omega^2 \int_0^R \mu(r) r^2 \mathrm{d}r \tag{4.69b}$$

$$= -\varphi \cdot \Omega^2 \cdot I \tag{4.69c}$$

With eq. 4.68 this gives:

$$I\ddot{\varphi} = -\Omega^2 I\varphi \Leftrightarrow \varphi(t) = A\sin(\Omega t) \tag{4.70}$$

Eigenfrequency equals rotor frequency!

4.5.2 Rotating beam with torsional spring

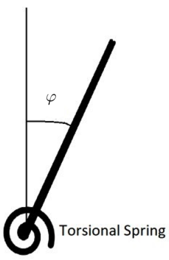


Figure 4.28

$\boxed{\text{Spring constant } K}$

$$\text{Natural resonance: } \omega_{NR} = \sqrt{\frac{K}{I}} \tag{4.71}$$

$$M(\varphi) = -\Omega^2 I\varphi - K\varphi \tag{4.72}$$

$$I\ddot{\varphi} = -(\Omega^2 I + K)\varphi \tag{4.73}$$

$$\ddot{\varphi} = -(\Omega^2 + \frac{K}{I})\varphi = -(\Omega^2 + {\omega_{NR}}^2)\varphi \tag{4.74}$$

$${\omega_R}^2 = \underbrace{{\omega_{NR}}^2 + \Omega^2}_{\text{“centrifugal stiffening”}} \tag{4.75}$$

Chapter 5

Control of Wind Turbines

There are two different ways of controlling the wind turbines:

(a) Passive control by mechanical design. For example:

Figure 5.1 Tail-rotor

Figure 5.2 Vane

(b) Active control by sensor-actuator systems, usually using digital controllers:

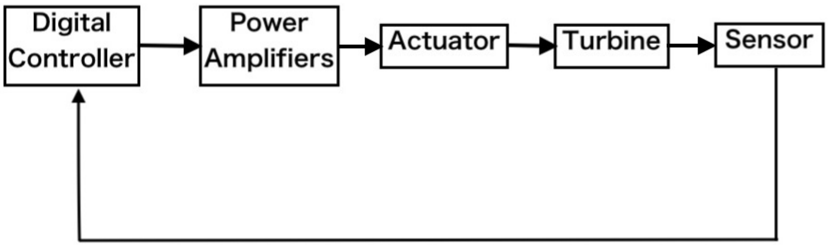


Figure 5.3

5.1 Sensors and Actuators in wind turbines

Sensors:

- A generator speed, rotor speed, wind speed, yaw rate
- Temperature of gearbox oil, generator winding, ambient air, etc
- Blade pitch, blade azimuth, yaw angle, wind direction
- Grid power, current, voltage, grid frequency
- Tower top acceleration, gearbox vibration, shaft torque, blade root bending moment, etc
- Environment (icing, humidity, lightning)

Actuators:

- Generator
- Motors: pitch, yaw
- Linear motors, magnets, switches
- Hydraulic powers and pistons (high power & speed)
- Resistance heaters & fans for temperature control
- Brakes (rotor, yaw)

5.2 Control system architecture

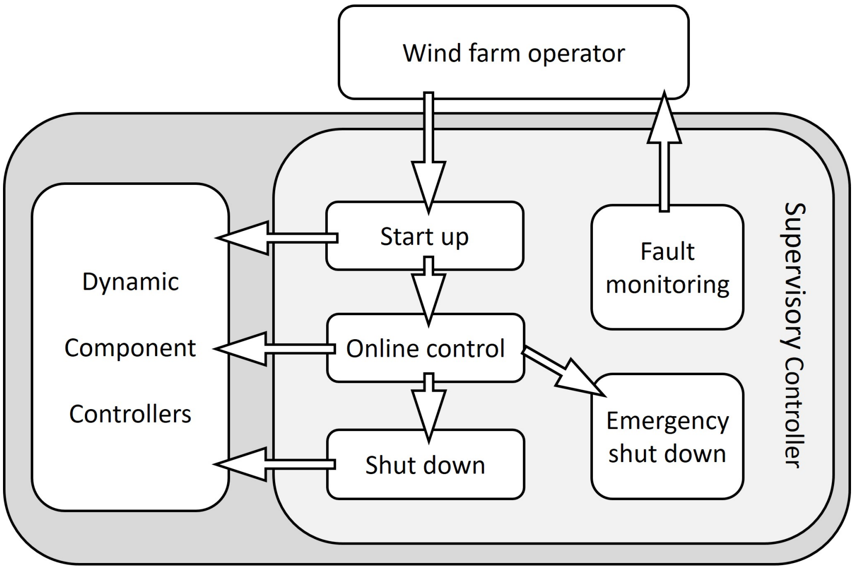


Figure 5.4

Usually, the "supervisory control" is on high level for turbine operating status. And "Dynamic control" is on low level (e.g. torque, pitch, power...etc).

5.3 Control of variable speed turbines

For speed control, main actuators are:

- blade pitch
- generator torque
  (controlled slowly to avoid drive-train oscillations)

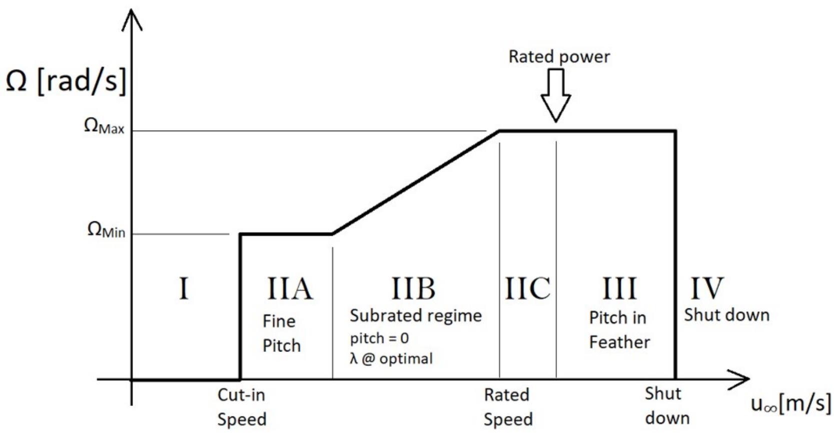


Figure 5.5 Rotation speed as function of wind speed

With the problem that wind speed on rotor discs can not be perfectly known, what is the maximum power production and power coefficient $C_p(\lambda, \beta)$?

$$P = \frac{1}{2}\rho A \cdot {u_\infty}^3 \cdot C_P(\lambda, \beta) \tag{5.1}$$

The equation 5.1 is the power function, where $\lambda = \frac{\Omega R}{u_\infty}$ is the tip speed ratio, β is the collective blade pitch. And the power coefficient C_P is maximized at $\lambda = \lambda^*$ (e.g. $= 7$) and $\beta = \beta^*$ [1] ($C_P^* = C_P(\lambda^*, \beta^*)$).

(Note: * means the optimal value.)

Figure 5.6 shows pitch, torque and λ as function of wind speed. Q_{Gen} is the generator torque. In equilibrium, $Q_{\text{Gen}} = Q_{\text{Aero}}$.

- Region IIA: λ is fixed to $\lambda_{\text{fix}}^{IIA} = \frac{\Omega_{\min} \cdot R}{u_\infty}$ and β is maximized.

  $C_P = C_P(\lambda_{\text{fix}}^{IIA}, \beta)$

- Region IIB (subrated): $\lambda = \lambda^*$ and $\beta = \beta^*$

  $C_P = C_P^* = C_P(\lambda^*, \beta^*)$

  This means that $P = (constant) * u_\infty^3$

- Region IIC & III: λ is again fixed to $\lambda_{\text{fix}}^{IIC} = \frac{\Omega_{\max} \cdot R}{u_\infty}$ and β regulates power.

  (Region III is at maximum power.)

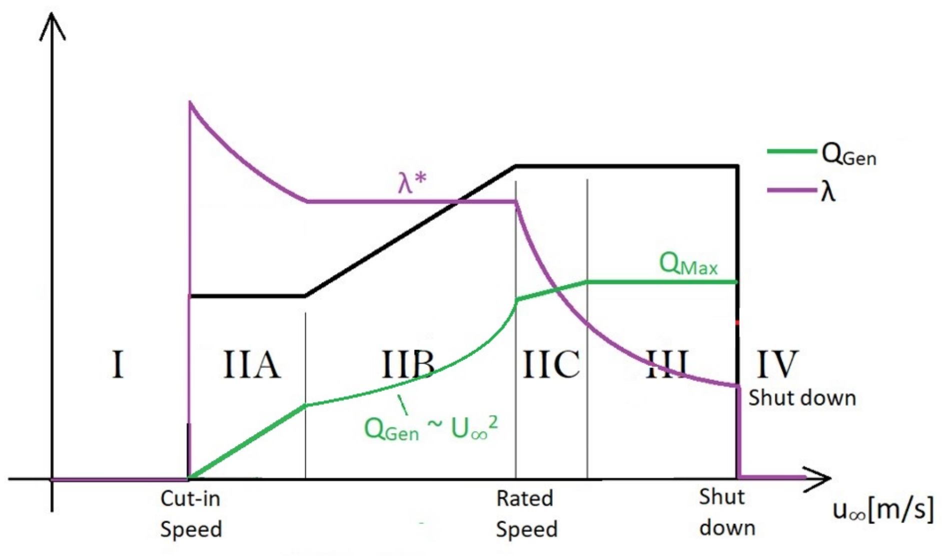


Figure 5.6 Torque, λ V.S. wind speed

[1] The β^* is centered at 0 as the result of a practical choice to design the blades so that the power coefficient is maximized.

5.4 Torque control at partial load (in region IIB)

Torque $Q_{\text{Generator}}$ can be controlled directly and should counteract aerodynamic torque Q_{Aero}. Given rotor inertia I we have the ODE for Ω:

$$I\dot{\Omega} = Q_{\text{Aero}} - Q_{\text{Generator}} \tag{5.2}$$

Q_{Aero} depends on u_∞ & Ω & β and is given by $P_{\text{Aero}} = \Omega \cdot Q_{\text{Aero}}$, where $\Omega = \frac{\lambda}{R} u_\infty$ so that $\lambda = \frac{R\Omega}{u_\infty}$:

$$Q_{\text{Aero}} = \frac{P_{\text{Aero}}}{\Omega}$$

$$= \frac{1}{2}\rho(\pi R^2){u_\infty}^3 \cdot \frac{C_P(\lambda,\beta)\cdot R}{\lambda u_\infty} \tag{5.3a}$$

$$= \frac{1}{2}\rho\pi R^3 {u_\infty}^2 \underbrace{\left[\frac{C_P(\lambda,\beta)}{\lambda}\right]}_{:= C_Q(\lambda,\beta)} \tag{5.3b}$$

$$= \frac{1}{2}\rho\pi R^5 \Omega^2 \left[\frac{C_P(\lambda,\beta)}{\lambda^3}\right] \tag{5.3c}$$

How to choose $Q_{\text{Generator}}$ when only Ω is measured?

Idea: Find the function $Q_{\text{Generator}}(\Omega)$ that brings turbine to an optimal tip speed ratio λ^* (in region IIB)). Intuitively, setting high Q_{Gen} if Ω is too large and small Q_{Gen} if Ω is too small in order to stabilize the rotor speed. At optimal $\Omega^* = \frac{\lambda^* \cdot u_\infty}{R}$ we would have:

$$Q_{\text{Aero}}(\Omega^*, u_\infty, \beta^*) = Q_{\text{Gen}}(\Omega^*) \tag{5.4}$$

So let us generally try the law:

$$Q_{\text{Gen}}(\Omega) := Q_{\text{Aero}}(\Omega, \frac{R\Omega}{\lambda^*}, \beta^*)$$

$$= \frac{1}{2}\rho\pi R^3 \left(\frac{R\Omega}{\lambda^*}\right)^2 \frac{C_P(\lambda^*,\beta^*)}{\lambda^*} \tag{5.5a}$$

$$= \underbrace{\frac{1}{2}\rho\pi R^5 \frac{C_P(\lambda^*,\beta^*)}{(\lambda^*)^3}}_{\text{constant } K_{\text{Gen}}} \cdot \Omega^2 \tag{5.5b}$$

Is this control-loop stable at Ω^*?

From equation 5.2 we know:

$$\dot{\Omega} := f(\Omega) = \frac{1}{I}(Q_{\text{Aero}}(\Omega, u_\infty, \beta^*) - Q_{\text{Gen}}(\Omega)) \tag{5.6}$$

Question 1: Is $f(\Omega^*) = 0$? And is it in steady state?

If $\Omega^* = \frac{\lambda^* u_\infty}{R}$, then by construction $Q_{\text{Aero}}(\Omega^*, u_\infty, \beta^*) = K_{\text{Gen}} \cdot (\Omega^*)^2$ such that indeed $f(\Omega^*) = 0$.

Question 2: If $\frac{\mathrm{d}f}{\mathrm{d}\Omega}(\Omega^*) < 0$, is it stable?

At $\Omega = \Omega^* = \frac{\lambda^* \cdot u_\infty}{r}$ and $u_\infty = \frac{\Omega^* R}{\lambda^*}$, we get:

$$\frac{\mathrm{d}f}{\mathrm{d}\Omega} = \frac{1}{I}\left(\frac{1}{2}\rho\pi R^5\right)\left(-\frac{C_P^*\Omega^*}{(\lambda^*)^3} - \frac{2C_P^*}{(\lambda^*)^3}\Omega^*\right) \tag{5.7a}$$

$$= \underbrace{-\frac{\frac{1}{2}\rho\pi R^5}{I \cdot (\lambda^*)^3} \cdot 3C_P^*}_{\text{constant [-]}} \cdot \Omega^* \tag{5.7b}$$

That is, the settling time is proportional to $\frac{1}{\Omega^*}$ or $\frac{R}{u_\infty}$

5.5 Thrust jump at nominal wind speed

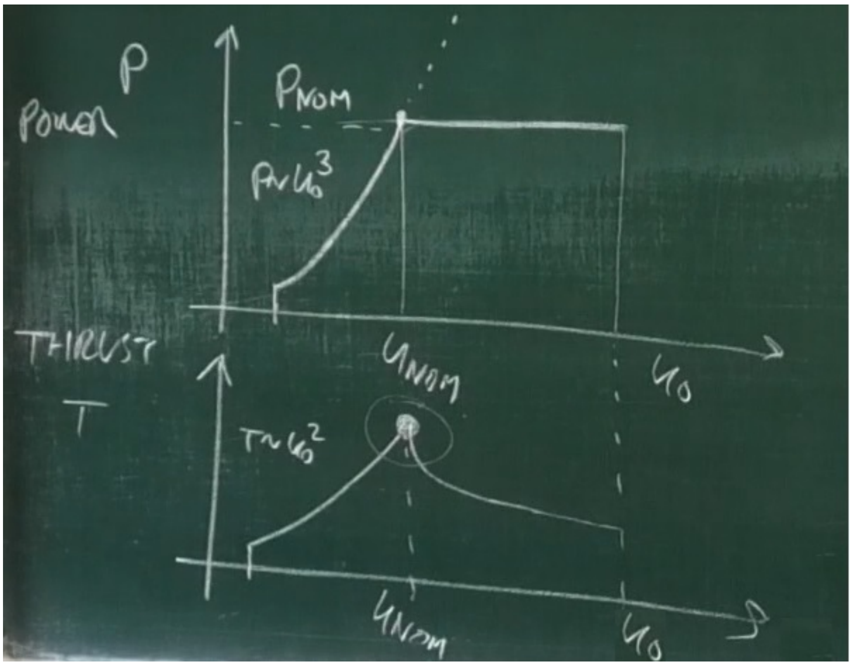


Figure 5.7 Power, Thrust as a function of wind speed

The reason for reduction in thrust for $U_0 > U_{\text{nom}}$ is the reduction of induction a.

Recall some facts from Betz theory:

$$C_{\text{p}} = \frac{P}{\frac{1}{2}\rho A u_0^3} = 4a(1-a)^2$$

$$C_{\text{T}} = \frac{T}{\frac{1}{2}\rho A u_0^2} = 4a(1-a)$$

$$T = \frac{P}{(1-a)u_0}$$

The optimal power harvesting was achieved for induction factor $a^* = \frac{1}{3}$.
How does a depend on U_0?

At the Betz limit we have $C_{\text{p}} := C_{\text{p}}(a^*) = \frac{16}{27}$ and $\boxed{\frac{\mathrm{d}C_{\text{p}}}{\mathrm{d}a}(a^*) = 0}$

For $U_0 \geq U_{\text{nom}}$ we have to ensure $P = P_{\text{nom}}$, which is only possible if we reduce C_{p}. In a pitch-controlled system, the C_{p} is reduced via a reduction of the induction factor a (alternatively, one could also increase the induction factor to reduce the power).

For $C_{\text{p}} < C_{\text{p}}^*$ we get $a < a^*$. Applying Taylor expansion:

$$C_{\text{p}} = 4a(1-a)^2 = C_{\text{p}}^* + \frac{1}{2}\frac{\mathrm{d}^2 C_{\text{p}}}{\mathrm{d}a^2}(a - a^*)^2 + \text{(higher order)} \tag{5.8}$$

$$\frac{\mathrm{d}C_\mathrm{p}}{\mathrm{d}a} = 4(1-a)^2 - 8a(1-a) \rightarrow$$

$$\frac{\mathrm{d}^2C_\mathrm{p}}{\mathrm{d}a^2} = -16(1-a) + 8a \rightarrow \frac{\mathrm{d}^2C_\mathrm{p}}{\mathrm{d}a^2} = -8$$

Plugging it into Eq.(5.8) we get:

$$C_\mathrm{p} = C_\mathrm{p}^* - 4(a-a^*)^2 \rightarrow \boxed{a - a^* = -\sqrt{\frac{C_\mathrm{p}^* - C_\mathrm{p}}{4}}} \tag{5.9}$$

How does T depend on $u_0 - u_\mathrm{nom} > 0$?

$$C_\mathrm{p} = C_\mathrm{p}^* + \frac{\mathrm{d}C_\mathrm{p}}{\mathrm{d}U_0}(U_\mathrm{nom})(u - u_\mathrm{nom}) + (\text{higher order}) \tag{5.10}$$

For $P = P_\mathrm{max}$ and $u_0 = u_\mathrm{nom}$:

$$C_\mathrm{p} = \frac{P_\mathrm{max}}{\frac{1}{2}\rho A u_0^3}$$

$$\rightarrow \frac{\mathrm{d}C_\mathrm{p}}{\mathrm{d}u_0} = -3\frac{P_\mathrm{max}}{\frac{1}{2}\rho A u_0^4} \rightarrow \frac{\mathrm{d}C_\mathrm{p}}{\mathrm{d}u_0} = -\frac{3}{u_\mathrm{nom}}C_\mathrm{p}^*$$

Plugging it into Eq.(5.10) we get:

$$C_\mathrm{p} = C_\mathrm{p}^* - -\frac{3C_\mathrm{p}^*}{u_\mathrm{nom}}(u_0 - u_\mathrm{nom}) \rightarrow$$

$$\boxed{C_\mathrm{p}^* - C_\mathrm{p} = \frac{3C_\mathrm{p}^*}{u_\mathrm{nom}}(u_0 - u_\mathrm{nom})} \tag{5.11}$$

Plugging it into Eq.(5.11) into Eq.(5.9) we get:

$$a - a^* = -\sqrt{\frac{3C_\mathrm{p}^*}{4u_\mathrm{nom}(u_0 - u_\mathrm{nom})}} \tag{5.12}$$

$$\begin{aligned} T = \frac{P}{(1-a(u_0))u_0} &= T_\mathrm{nom} + \frac{\mathrm{d}T}{\mathrm{d}u_0}(u_0 - u_\mathrm{nom}) + \frac{\mathrm{d}T}{\mathrm{d}a}(a(u_0) - a^*) \\ &= T_\mathrm{nom} - \frac{T_\mathrm{nom}}{u_\mathrm{nom}}(u_0 - u_\mathrm{nom}) + \frac{P_\mathrm{max}}{(1-a)^2u_\mathrm{nom}}(a - a^*) \\ &= T_\mathrm{nom}\left(1 - \frac{u_0 - u_\mathrm{nom}}{u_\mathrm{nom}}\right) - \\ \frac{P_\mathrm{max}}{(1-a)^2u_\mathrm{nom}}&\sqrt{\frac{3C_\mathrm{p}^*}{4u_\mathrm{nom}}(u_0 - u_\mathrm{nom})} \end{aligned} \tag{5.13}$$

Chapter 6

Alternative Concepts

6.1 Vertical axis wind turbines

Darrieus rotor:

Figure 6.1

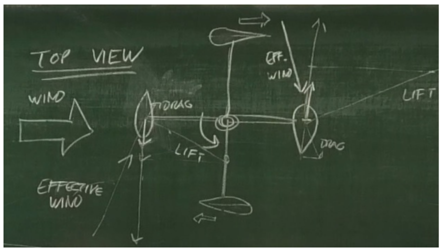


Figure 6.2 Top view

Savonius wind turbine:

Figure 6.3

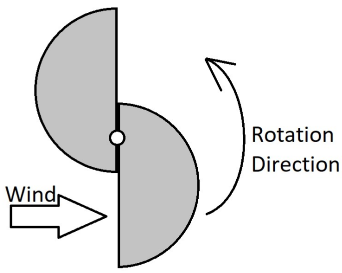


Figure 6.4 Top view

6.2 Airborne wind energy (AWE)

See slides: (click here for slides:
https://www.syscop.de/files/2018ss/WES/lectures/20180711WES-AWE.key.pdf)

Variant 2: Generator on ground (pumping cycle)

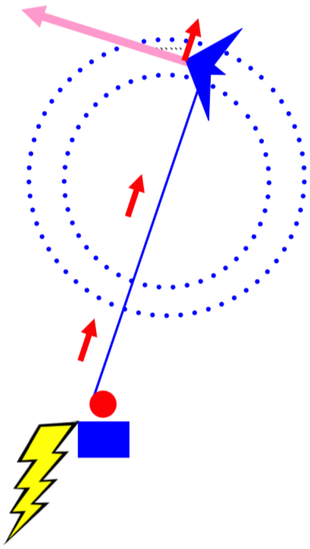


Figure 6.5 Ground based and pumping cycle

We assume:

- the effect of gravity is neglected.
- cable is parallel to wind W.
- kite flies crosswind with high speed.

where:

$$\left[\begin{array}{l} V = \lambda \cdot W \\ \\ W : \text{real wind} \\ \\ V : \text{speed of kite} \\ \\ \alpha : \text{roll out speed as fraction of real wind} \end{array}\right.$$

6.3 Loyd's formula

Regard a kite/airfoil under idealized conditions, which means:

- The tether is parallel to the wind.
- No gravity, steady wind $W \equiv u_\infty$
- Steady crosswind flight with downward components

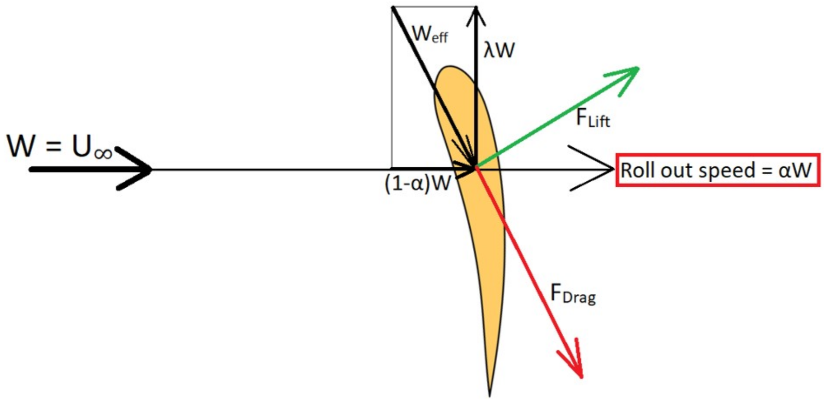


Figure 6.6

Given C_L & C_D, roll out speed αW, wing area A and tip speed ratio λ, the wind & motion vector in x-y-frame are:

$$\vec{W} = \begin{bmatrix} W \\ 0 \end{bmatrix} \tag{6.1}$$

$$\vec{V} = \begin{bmatrix} \alpha W \\ \lambda W \end{bmatrix} \tag{6.2}$$

Effective wind:

$$\vec{V_e} = \vec{W} - \vec{V} = \begin{bmatrix} (1-\alpha)W \\ -\lambda W \end{bmatrix} \tag{6.3}$$

With $V_e := \|V_e\| = W \cdot \sqrt{(1-\alpha)^2 + \lambda^2}$ we get:

$$\vec{F_D} = \frac{1}{2}\rho A \|V_e\|^2 \cdot C_D \frac{\vec{V_e}}{\|V_e\|} \tag{6.4a}$$

$$= \frac{1}{2}\rho A {V_e}^2 \cdot C_D \begin{bmatrix} (1-\alpha) \\ -\lambda \end{bmatrix} \frac{1}{\sqrt{(1-\alpha)^2 + \lambda^2}} \tag{6.4b}$$

$$\overrightarrow{F_L} = \frac{1}{2}\rho A \|V_e\|^2 \cdot C_L \frac{\overrightarrow{V_e}}{\|V_e\|} \tag{6.5a}$$

$$= \frac{1}{2}\rho A {V_e}^2 \cdot C_L \begin{bmatrix} \lambda \\ (1-\alpha) \end{bmatrix} \frac{1}{\sqrt{(1-\alpha)^2 + \lambda^2}} \tag{6.5b}$$

$$\overrightarrow{F_L} + \overrightarrow{F_D} = \frac{1}{2}\rho A {V_e}^2 \frac{1}{\sqrt{(1-\alpha)^2 + \lambda^2}} \begin{bmatrix} C_D(1-\alpha) + C_L\lambda \\ -C_D\lambda + C_L(1-\alpha) \end{bmatrix} \tag{6.6}$$

Steady state means there is no acceleration, that is, no force in the y-direction. Thus we get:

$$\lambda C_D = (1-\alpha) C_L \tag{6.7}$$

$$\boxed{\lambda = \tfrac{C_L}{C_D}(1-\alpha)} \tag{6.8}$$

The generated power is equal to roll out speed, αW times the x-component of tension ofF_T:

$$P = \alpha \cdot W \cdot F_T \tag{6.9a}$$

$$= \alpha \cdot W \frac{1}{2}\rho A W^2 \sqrt{(1-\alpha)^2 + \lambda^2}(C_D(1-\alpha) + C_L\lambda) \tag{6.9b}$$

$$= \frac{1}{2}\rho A W^3 \cdot \alpha(1-\alpha)^2 \left[\frac{C_L^3}{C_D^2} + C_L\right] \tag{6.9c}$$

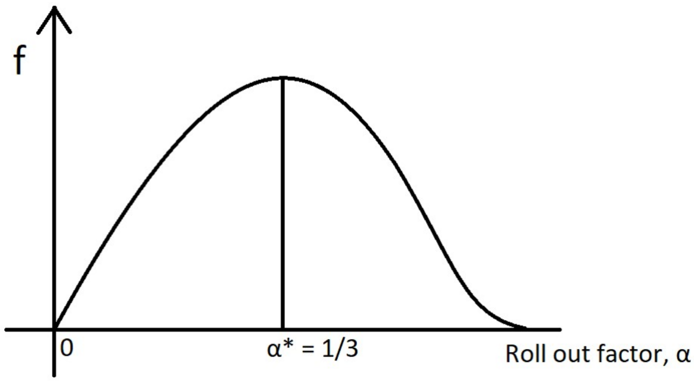
$$= \frac{1}{2}\rho A W^3 \frac{C_L^3}{C_D^2} \left(1 + \frac{C_D^2}{C_L^2}\right) \alpha(1-\alpha)^2 \tag{6.9d}$$

The maximum power is reached if $\alpha(1-\alpha)^2$ is maximized:

$$f(\alpha) = \alpha(1-\alpha)^2 \tag{6.10}$$

$$f(\alpha') = \underbrace{(1-\alpha)^2 - 2\alpha(1-\alpha)}_{:=0} \tag{6.11}$$

According to equation 6.11, we get $(1-\alpha) = 2\alpha \Rightarrow \alpha^* = \frac{1}{3}$.



$$f(\alpha^*) = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27}$$

Figure 6.7

Loyd's formula:

$$\boxed{P = \tfrac{1}{2}\rho A W^3 \cdot \tfrac{4}{27} \cdot \tfrac{C_L^3}{C_D^2} \underbrace{\left(1 + \frac{C_D^2}{C_L^2}\right)}_{\approx 1}} \tag{6.12}$$

Example: Regard $C_L = 1$, $C_D = 0.05$, $W = 10\,\mathrm{m/s}$ and $\rho = 1.2\,\mathrm{kg/m^3}$ we get:

$$\frac{P}{A} = \overbrace{\frac{1}{2}\rho W^3}^{=600\,\mathrm{W/m^2}} \cdot \underbrace{\frac{4}{27} \cdot C_L \frac{C_L^2}{C_D^2}\left(1 + \frac{C_D^2}{C_L^2}\right)}_{:= \zeta \text{ “Harvesting factor zeta”}}$$

$$\rho = \frac{4}{27} \cdot 400(1 + \frac{1}{400}) \approx 59$$

$$\frac{P}{A} = 36\,\mathrm{kW/m^2}$$