Set Based Computing Methods in Optimization and Control

Boris Houska

ShanghaiTech University

Overview

- Set arithmetics
- Robust model predictive control

Affine set parameterization

Notation:

- Basis set: $\mathbb{E} \subseteq \mathbb{R}^m$
- Coefficients: $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$
- Set: $A * \mathbb{E} + b$

	Complexity
Intervals	$\mathbf{O}(n)$
Zono-/ Polytopes	$\mathbf{O}(nm)$
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Intervals:

$A \in \mathbb{R}^{n \times n}$, A diagonal

$\mathbb{E} = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq 1\}$

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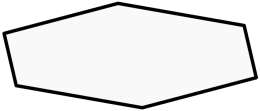

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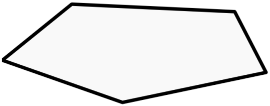
Polytopes:

$A \in \mathbb{R}^{n \times m}$

$\mathbb{E} = \left\{x \in \mathbb{R}^m_+ \mid \sum_i x_i = 1\right\}$



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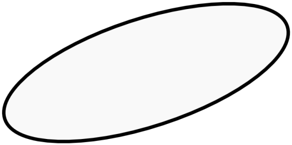

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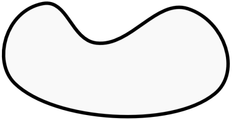
Polynomial Set:

$A \in \mathbb{R}^{n \times \binom{\ell+q}{\ell}}$

$\mathbb{E} = \left\{ (1, \ldots, x_\ell^q)^\mathsf{T} \,\middle|\, x \in [-1,1]^\ell \right\}$



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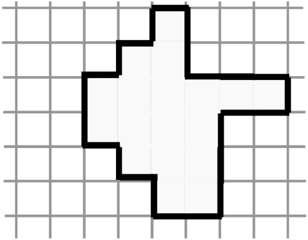

Factorable functions

Notation:

- Library of atom operations: $L = \{+, -, *, \sin, \cos, \log, \ldots\}$
- A function f is called factorable over L, if

$$f = \varphi_N \circ \ldots \circ \varphi_1 \quad \text{with} \quad [\varphi_i]_{\text{last}} \in L \; .$$

Example:

$$\begin{aligned} a_1 &= x_1 * x_2 \\ a_2 &= \sin(a_1) \\ a_3 &= \cos(x_1) \\ f(x) &= a_2 + a_3 \end{aligned}$$

$$\boxed{f(x) = \sin(x_1 * x_2) + \cos(x_1)}$$

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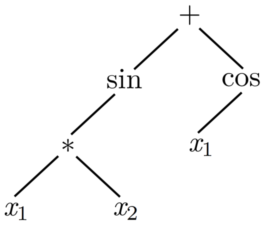

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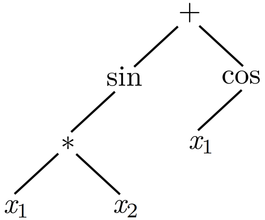

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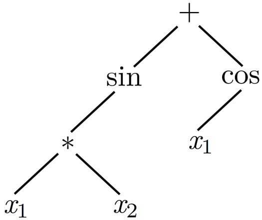

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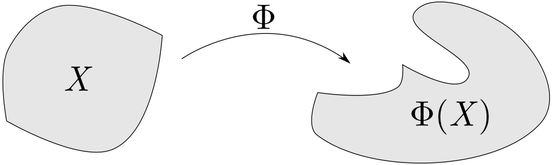
Set arithmetics



- Let f be a given factorable function, $\mathbb{E}$ basis set
- Goal: find enclosure Φ such that

$$\{f(x) \mid x \in X\} \subseteq \Phi(X)$$

with

$$X = A * \mathbb{E} + b \qquad \text{and} \qquad \Phi(X) = C * \mathbb{E} + d$$

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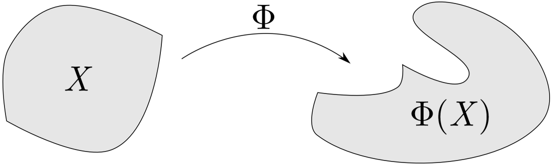

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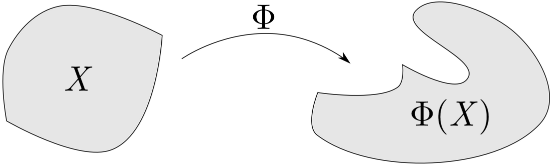

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Construction of set arithmetics

1. Construct enclosures Φ_i of all atom functions $\varphi_i \in L$

$$\{\varphi_i(x) \mid x \in X\} \subseteq \Phi_i(X)$$

2. Enclosure $f = \varphi_N \circ \ldots \circ \varphi_1$ given by $\Phi = \Phi_N \circ \ldots \circ \Phi_1$

Remark: For larger N overestimation might grow (wrapping)

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Set arithmetic software

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- R.E. Moore. Interval Arithmetics, 1966
- G.P. McCormick. Computability of global solutions to factorable nonconvex programs, 1976

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- A.B. Kurzhanski, P. Varaiya. Reachability analysis for uncertain systems—the ellipsoidal technique, 2002
- M.E. Villanueva et.al.. Ellipsoidal arithmetic for multivariate systems, 2015

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- M. Althoff, B.H. Krogh. Zonotope bundles for the efficient computation of reachable sets, 2011
- J.K. Scott. Constrained zonotopes: A new tool for set-based estimation and fault detection, 2016

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- M. Tawarmalani, N.V. Sahinidis. A polyhedral branch-and-cut approach to global optimization, 2005
- R. Misener, C.A. Floudas. ANTIGONE: Algorithms for continuous/integer global optimization of nonlinear equation, 2014

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- M. Berz. From Taylor series to Taylor models, 1997
- A. Bompadre et.al.. Convergence analysis of Taylor and McCormick-Taylor models, 2013

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- A. Townsend, L.N. Trefethen. An extension of Chebfun to two dimensions, 2013
- J. Rajyaguru et.al.., Chebyshev model arithmetic for factorable functions, 2017

Two-Reaction Model of Anaerobic Digestion

Mass-Balance Equations:

$$
\begin{aligned}
\dot{X}_1 &= \left(\mu_1(S_1) - \alpha D\right) X_1 \\
\dot{X}_2 &= \left(\mu_2(S_2) - \alpha D\right) X_2 \\
\dot{S}_1 &= D(S_1^{\mathrm{in}} - S_1) - k_1 \mu_1(S_1) X_1 \\
\dot{S}_2 &= D(S_2^{\mathrm{in}} - S_2) + k_2 \mu_1(S_1) X_1 \\
&\quad - k_3 \mu_2(S_2) X_2 \\
\dot{Z} &= D(Z^{\mathrm{in}} - Z) \\
\dot{C} &= D(C^{\mathrm{in}} - C) + k_4 \mu_1(S_1) X_1 \\
&\quad + k_5 \mu_2(S_2) X_2 - q_{\mathrm{CO_2}}
\end{aligned}
$$

Biomass specific growth rates:

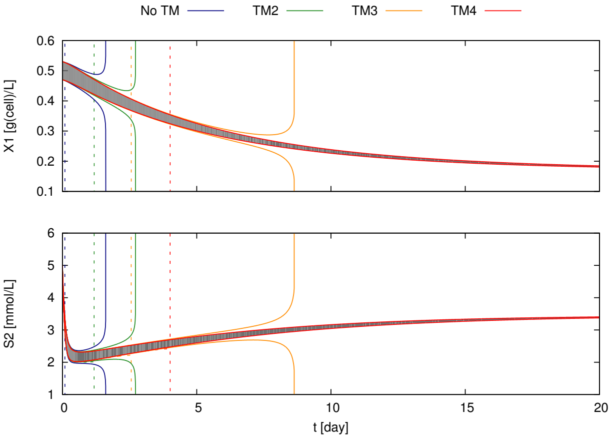
$$
\begin{aligned}
\mu_1(S_1) &:= \bar{\mu}_1 \frac{S_1}{S_1 + K_{S_1}} \\
\mu_2(S_2) &:= \bar{\mu}_2 \frac{S_2}{S_2 + K_{S_2} + S_2^2 / K_{I_2}}
\end{aligned}
$$

Gas-liquid mass transfer:

$$
\begin{aligned}
q_{\mathrm{CO_2}} &:= k_{\mathrm{L}} a (C + S_2 - Z - K_{\mathrm{H}} P_{\mathrm{CO_2}}) \\
P_{\mathrm{CO_2}} &:= \frac{\phi_{\mathrm{CO_2}} - \sqrt{\phi_{\mathrm{CO_2}}^2 - 4 K_{\mathrm{H}} P_{\mathrm{t}} (C + S_2 - Z)}}{2 K_{\mathrm{H}}} \\
\phi_{\mathrm{CO_2}} &:= C + S_2 - Z + K_{\mathrm{H}} P_{\mathrm{t}} \\
&\quad + \frac{k_6}{k_{\mathrm{L}} a} \mu_2(S_2) X_2
\end{aligned}
$$

Goal: Compute reachable sets for uncertain initial conditions

Taylor models with interval / ellipsoidal remainder



Taylor models with $q \geq 4$ + Ellipsoids = stable set integrator

- B. Houska, M.E. Villanueva, B. Chachuat. Stable Set-Valued Integration of Nonlinear Dyn. using Affine Set Parameterizations, 2015

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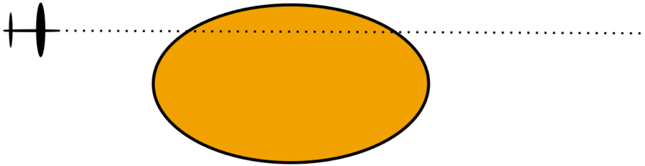
Model Predictive Control (MPC)



Certainty equivalent MPC:

- minimize distance to dotted line
- subject to: system dynamics and constraints

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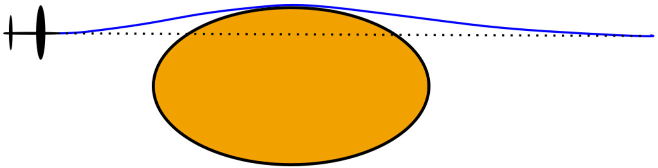

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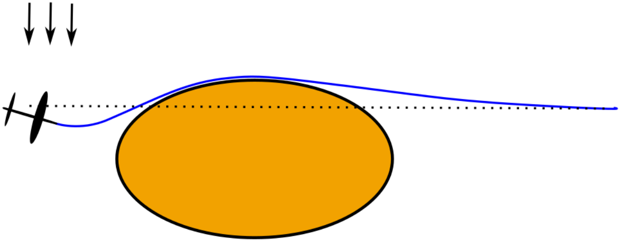

Repeat:

- wait for new measurement
- re-optimize the trajectory

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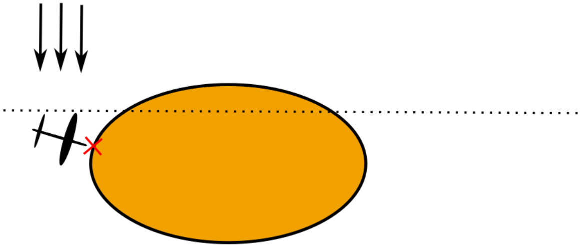
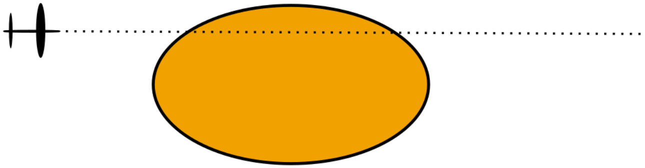


Problem:

- certainty equivalent prediction is optimistic
- infeasible (worst-case) scenarios possible

What is Robust MPC?



Main idea:

- take all possible uncertainty scenarios into account
- important: we can react to uncertainties

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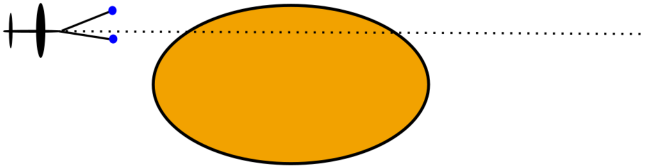

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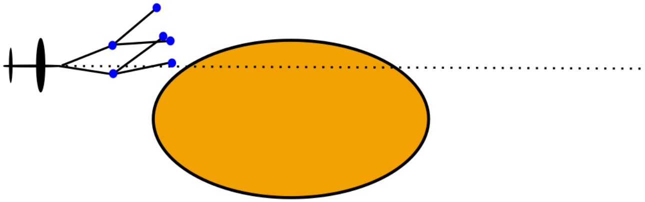

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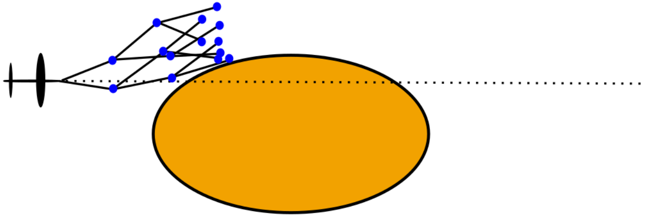

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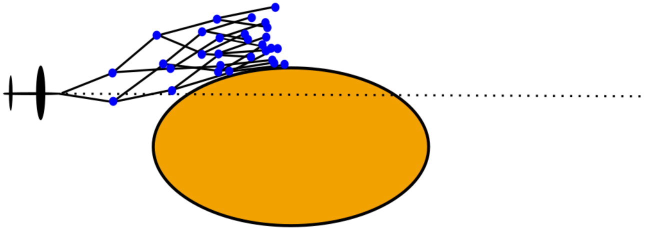
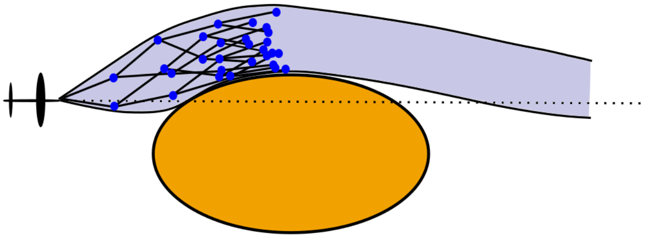


Problem:

- exponentially exploding amount of scenarios possible
- much more expensive than certainty equivalent MPC

Tube-based Robust MPC [Langson'04, Rakovic'05,...]



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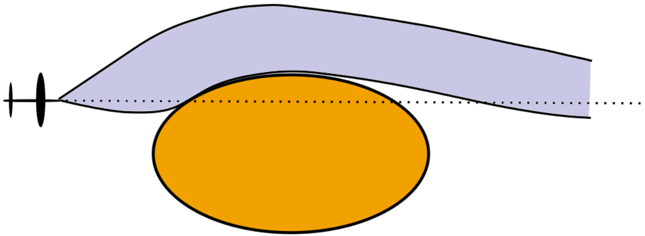

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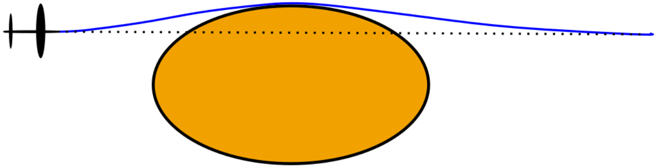
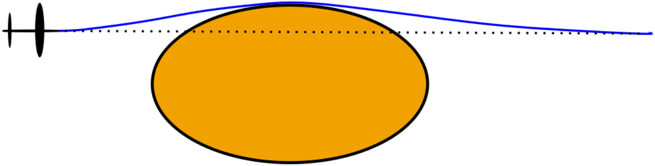
- no exponential scenario tree, but set enclosures needed

Notation: closed-loop system



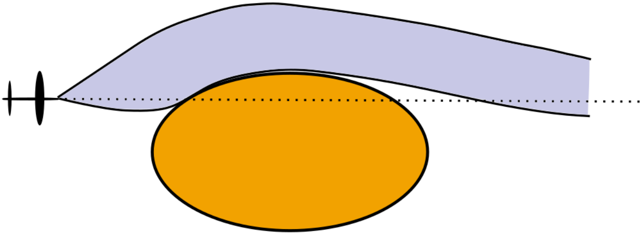
$$\dot{x}(t) = f(x(t), \mu(t, x(t)), w(t))$$

Notation: constraints



$\mu(t, x(t)) \in \mathbb{U}\,, \quad x(t) \in \mathbb{X}\,, \quad w(t) \in \mathbb{W}$ (all compact sets)

Notation: set-valued tubes



$$X(t, x_0, \mu) = \left\{ x_t \in \mathbb{R}^{n_x} \;\middle|\; \begin{array}{l} \exists x \in W_{1,2}^{n_x},\ \exists w \in L_2^{n_w} \;:\; \forall \tau \in [0,t], \\ \dot{x}(\tau) = f(x(\tau), \mu(\tau, x(\tau)), w(\tau)) \\ x(0) = x_0\ ,\ x(t) = x_t \\ w(\tau) \in \mathbb{W} \end{array} \right\}$$

Mathematical Formulation of Robust MPC

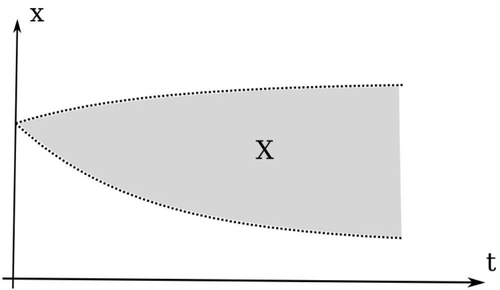
Optimize over future feedback policy μ:

$$\inf_{\mu:\mathbb{R}\times\mathbb{X}\to\mathbb{U}} \int_0^T \ell(X(t,x_0,\mu))\,\mathrm{d}t + \mathcal{M}\,(\,X(T,x_0,\mu)\,)$$

$$\text{s.t.} \qquad X(t,x_0,\mu) \subseteq \mathbb{X} \quad \text{for all} \;\; t\in[0,T]\;.$$

- ℓ denotes scalar performance criterion
- $\mathcal{M}$ denotes terminal cost
- x_0 denotes current measurement
- T denotes finite prediction horizon

Differential Inequalities

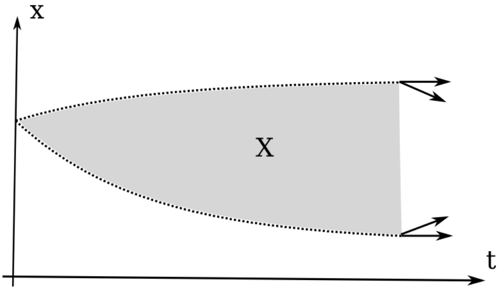


Scalar case:

- uncertain scalar ODE without controls:

$$\dot{x}(t) = f(x(t), w(t)) \quad \text{with} \quad x(0) = x_0$$

Differential Inequalities

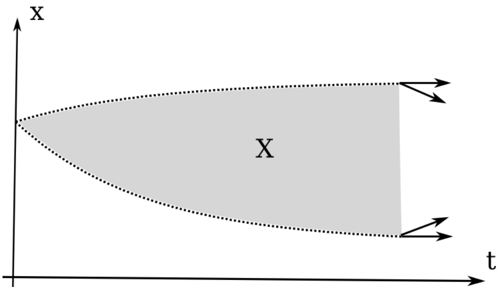


Scalar case:

- Interval $X(t) = \left[x^{\mathsf{L}}(t), x^{\mathsf{U}}(t)\right]$ is robust forward invariant if

$$\begin{aligned} \dot{x}^{\mathsf{L}}(t) &\leq \min_{w \in \mathbb{W}} \ f(x^{\mathsf{L}}(t), w) \\ \dot{x}^{\mathsf{U}}(t) &\geq \max_{w \in \mathbb{W}} \ f(x^{\mathsf{U}}(t), w) \end{aligned} \qquad \text{(Differential Inequalities)}$$

Min-Max Differential Inequalities

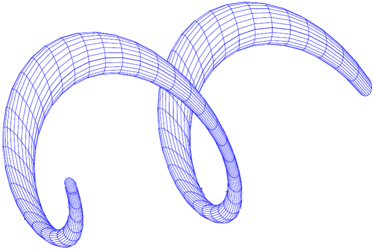


Scalar case with controls:

- Interval $X(t) = \left[x^{\mathsf{L}}(t), x^{\mathsf{U}}(t)\right]$ is robust forward invariant if

$$\begin{aligned}
\dot{x}^{\mathsf{L}}(t) &\leq \max_{u\in\mathbb{U}} \min_{w\in\mathbb{W}} \; f(x^{\mathsf{L}}(t), u, w) \\
\dot{x}^{\mathsf{U}}(t) &\geq \min_{u\in\mathbb{U}} \max_{w\in\mathbb{W}} \; f(x^{\mathsf{U}}(t), u, w) \\
x^{\mathsf{L}}(t) &\leq x^{\mathsf{U}}(t)
\end{aligned}$$

Generalized Differential Inequalities

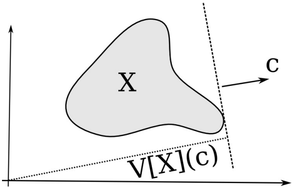


General case:

- The state vector $x(t)$ may have more than one component,

$$\dot{x}(t) = f(x(t), u(t), w(t)) \quad \text{with} \quad x(0) = x_0$$

Generalized Differential Inequalities



Definition:

- The support function of a compact set X is denoted by

$$V[X](c) = \max_{x \in X} c^{\mathsf{T}} x$$

Generalized Differential Inequalities

Theorem [Villanueva et al., 2017]:

- If f Lipschitz, $X(t) \subseteq \mathbb{X}$ convex and compact, and

$$\dot{V}[X(t)](c) \geq \min_{u \in \mathbb{U}} \max_{x,w} \left\{ c^{\mathsf{T}} f(x,u,w) \;\middle|\; \begin{array}{l} x \in X(t) \\ c^{\mathsf{T}} x = V[X(t)](c) \\ w \in \mathbb{W} \end{array} \right\}$$

  for a.e. (t, c), then $X(t)$ is a robust forward invariant tube.

- M.E. Villanueva et.al., Robust MPC via min-max differential inequalities. Automatica, 2017.

Application to Robust MPC

Conservative reformulation:

$$\inf_{X} \int_t^{t+T} \ell(X(\tau))\,\mathrm{d}\tau$$

$$\text{s.t.} \quad \begin{cases} X(t) = \{\hat{x}_t\}\,, \\ X(\tau) \subseteq \mathbb{X} \\ \dot{V}[X(t)](c) \geq \min\limits_{u \in \mathbb{U}} \max\limits_{x,w} \left\{ c^{\mathsf{T}} f(x,u,w) \;\middle|\; \begin{array}{l} x \in X(t) \\ c^{\mathsf{T}} x = V[X(t)](c) \\ w \in \mathbb{W} \end{array} \right\} \\ \text{optional terminal constraints} \end{cases}$$

- Parameterize set $X(t)$; not the feedback law μ!

Example: Ellipsoidal Parameterization

Affine tube parameterization:

$$X(t) = Q_x(t)^{\frac{1}{2}}\mathbb{E} + q_x(t) \qquad \text{with} \qquad \mathbb{E} = \{\ x\ \mid \|x\|_2 \leq 1\}$$

Support function:

$$V[X(t)](c) = \sqrt{c^\mathsf{T} Q_x(t) c} + q_x(t)$$

Assumption: control and uncertainty sets are ellipsoids

$$\mathbb{U} = Q_u(t)^{\frac{1}{2}}\mathbb{E} + q_u(t) \qquad \text{and} \qquad \mathbb{W} = Q_w(t)^{\frac{1}{2}}\mathbb{E} + q_w(t)$$

... and substitute all in the Min-Max Differential Inequality (DI)

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Application of Kurzhanski's ellipsoidal calculus to Min-Max DI

Dynamic system:

$$\dot{x} = f(x, u, w) = Ax + Bu + Cw + \text{nonlinear terms}$$

Center of the ellipsoid $X(t) = Q_x(t)^{\frac{1}{2}}\mathbb{E} + q_x(t)$ (with $v \in \mathbb{R}^{n_u}$):

$$\dot{q}_x = f(q_x, v, q_w)$$

Parameteric ellipsoidal tube (with orthogonal S and $\lambda > 0, \gamma > 0$)

$$\begin{aligned} \dot{Q}_x &= AQ_x + Q_x A^\mathsf{T} + Q^{\frac{1}{2}} S R[v,\gamma] B^\mathsf{T} + B R[v,\gamma] S^\mathsf{T} Q^{\frac{1}{2}} \\ &\quad + \frac{1}{\lambda} Q_x + \lambda C Q_w C^\mathsf{T} + \text{nonlinear terms} \end{aligned}$$

where

$$R[v,\gamma] = (1-\gamma) Q_u + (1-\gamma^{-1})[v - q_u][v - q_u]^\mathsf{T}$$

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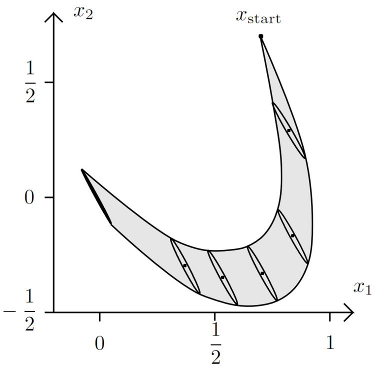
Ellipsoidal Tube MPC

Complete reformulation as implemented for a small $\epsilon > 0$:

$$
\inf_{q_x, v, Q_x, S, \lambda, \gamma} \int_t^{t+T} \hat{\ell}(q_x, v, Q_x)\,\mathrm{d}\tau
$$

$$
\text{s.t.} \quad \begin{cases}
q_x(t) = \{\hat{x}_t\}\,,\ Q_x(t) = \epsilon^2 I \\
\mathcal{E}(q_x, Q_x) \subseteq \mathbb{X} \\
\dot{q}_x = f(q_x, v, q_w) \\
\dot{Q}_x = AQ_x + Q_x A^{\mathsf{T}} + Q^{\frac{1}{2}} S R[v,\gamma] B^{\mathsf{T}} + B R[v,\gamma] S^{\mathsf{T}} Q^{\frac{1}{2}} \\
\qquad\qquad + \frac{1}{\lambda} Q_x + \lambda C Q_w C^{\mathsf{T}} + \text{nonlinear terms} \\
SS^{\mathsf{T}} = I\ ,\ \lambda \geq \epsilon \mathbf{1}\ ,\ \gamma \geq \epsilon \mathbf{1} \\
+ \text{ optional terminal constraints / cost}
\end{cases}
$$

Numerical Example



Spring-mass-damper system:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} x_2(t) + w_1(t) \\ -\frac{k_0 \exp(-x_1) x_1(t)}{M} - \frac{h_d x_2(t)}{M} + \frac{u(t)}{M} + \frac{w_2(t)}{M} \end{pmatrix}$$

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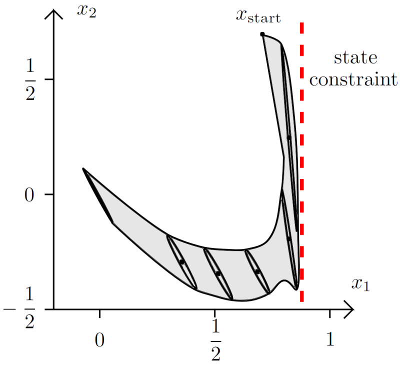

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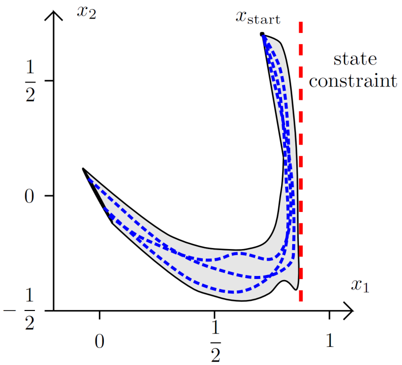

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Conclusions and Open Problems

Set Based Computing—State-of-the-Art

- Many existing software tools for different set arithmetics:

  Intervals, Ellipsoids, Zonotopes, Polynomial image sets, ...
- Maturity of set integrators improved a lot during last decade:
  - Affine set parameterizations $\Rightarrow$ stable set integrator
  - Can deal with nonlinear system up to approx. 10 states
  - Applications in robust and global optimal control

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- Bound all possible scenarios by one single tube
- Tube MPC variants: Rigid-, Homothetic-, Elastic- Tube MPC, ...
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- most formulations either very conservative or non-convex
- no generic software packages—often tailored implementation needed
- stability theory is incomplete (non-convex economic MPC)

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References

- B. Houska, M.E. Villanueva, B. Chachuat.
  Stable Set-Valued Integration of Nonlinear Dynamic Systems using Affine Set Parameterizations.
  SINUM, 2015.
- M.E. Villanueva, B. Houska, B. Chachuat.
  Unified Framework for the Propagation of Continuous-Time Enclosures for Parametric Nonlinear ODEs.
  JOGO, 2015.
- M.E. Villanueva, R. Quirynen, M. Diehl, B. Chachuat, B. Houska.
  Robust MPC via Min-Max Differential Inequalities.
  AUTOMATICA, 2017.

References

- J. Rajyaguru, M.E. Villanueva, B. Houska, B. Chachuat.
  Chebyshev models arithmetic for factorable functions.
  JOGO, 2016.
- B. Houska, M.E. Villanueva.
  Robust Optimization for MPC.
  In S. Raković & W.S. Levine (Eds.), **Handbook of MPC**, 2019.