Numerical Optimal Control Overview

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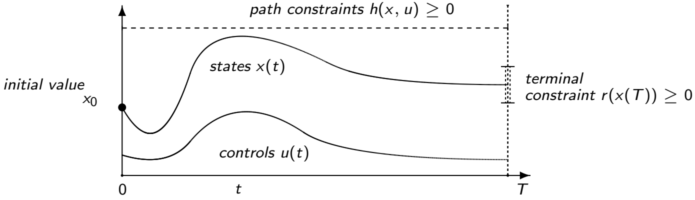
Simplified Optimal Control Problem in ODE



$$\underset{x(\cdot),u(\cdot)}{\text{minimize}} \quad \int_0^T L(x(t), u(t))\, dt \;+\; E\,(x(T))$$

subject to

$$\begin{aligned} x(0) - x_0 &= 0, && && \text{(fixed initial value)} \\ \dot{x}(t) - f(x(t), u(t)) &= 0, && t \in [0, T], && \text{(ODE model)} \\ h(x(t), u(t)) &\geq 0, && t \in [0, T], && \text{(path constraints)} \\ r\,(x(T)) &\geq 0 && && \text{(terminal constraints)} \end{aligned}$$

More general optimal control problems

Many features left out here for simplicity of presentation:

- multiple dynamic stages
- differential algebraic equations (DAE) instead of ODE
- explicit time dependence
- constant design parameters
- multipoint constraints $r(x(t_0), x(t_1), \ldots, x(t_{\mathrm{end}})) = 0$

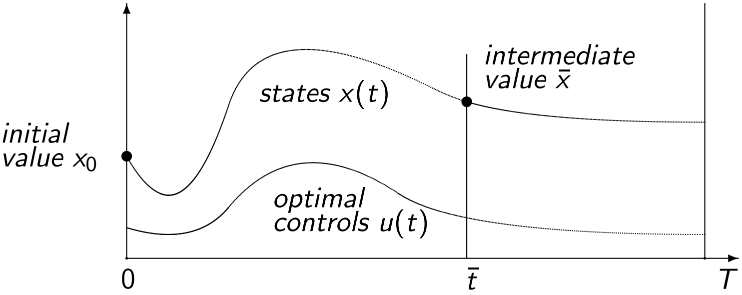
Optimal Control Family Tree

Three basic families:

- Hamilton-Jacobi-Bellmann equation / dynamic programming
- Indirect Methods / calculus of variations / Pontryagin
- Direct Methods (control discretization)

Principle of Optimality

Any subarc of an optimal trajectory is also optimal.



Subarc on $[\bar{t}, T]$ is optimal solution for initial value $\bar{x}$.

Dynamic Programming Cost-to-go

IDEA:

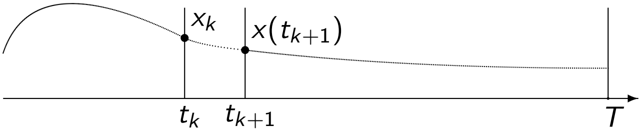
- Introduce **optimal-cost-to-go** function on $[\bar{t}, T]$

$$J(\bar{x}, \bar{t}) := \min_{x,u} \int_{\bar{t}}^{T} L(x,u)dt + E(x(T)) \quad \text{s.t.} \quad x(\bar{t}) = \bar{x}, \ldots$$

- Introduce grid $0 = t_0 < \ldots < t_N = T$.
- Use **principle of optimality** on intervals $[t_k, t_{k+1}]$:

$$J(x_k, t_k) = \min_{x,u} \int_{t_k}^{t_{k+1}} L(x,u)dt + J(x(t_{k+1}), t_{k+1})$$

$$\text{s.t.} \quad x(t_k) = x_k, \ldots$$



Dynamic Programming Recursion

Starting from $J(x, t_N) = E(x)$, compute recursively backwards, for $k = N-1, \ldots, 0$

$$J(x_k, t_k) := \min_{x,u} \int_{t_k}^{t_{k+1}} L(x,u)dt + J(x(t_{k+1}), t_{k+1}) \;\text{ s.t. }\; x(t_k) = x_k, \ldots$$

by solution of short horizon problems for all possible x_k and tabulation in state space.

Dynamic Programming Recursion

Starting from $J(x, t_N) = E(x)$, compute recursively backwards, for $k = N - 1, \ldots, 0$

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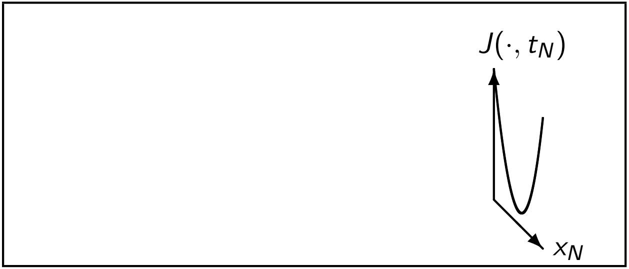

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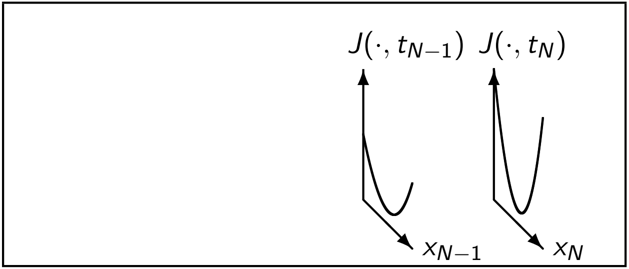
by solution of short horizon problems for all possible x_k and tabulation in state space.



Dynamic Programming Recursion

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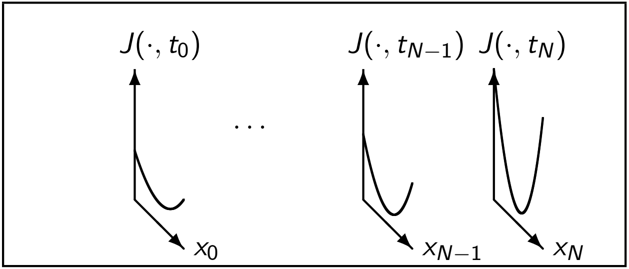
by solution of short horizon problems for all possible x_k and tabulation in state space.



Hamilton-Jacobi-Bellman (HJB) Equation

- Dynamic Programming with infinitely small timesteps leads to **Hamilton-Jacobi-Bellman (HJB) Equation**:

$$-\frac{\partial J}{\partial t}(x,t) = \min_u \left(L(x,u) + \frac{\partial J}{\partial x}(x,t) f(x,u) \right) \quad \text{s.t.} \quad h(x,u) \geq 0.$$

- Solve this partial differential equation (PDE) backwards for $t \in [0, T]$, starting at the end of the horizon with

$$J(x,T) = E(x).$$

- **NOTE:** Optimal controls for state x at time t are obtained from

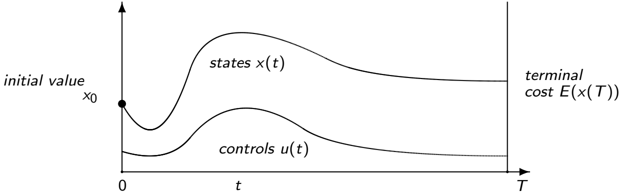
$$u^*(x,t) = \arg\min_u \left(L(x,u) + \frac{\partial J}{\partial x}(x,t) f(x,u) \right) \quad \text{s.t.} \quad h(x,u) \geq 0.$$

Dynamic Programming / HJB

- "Dynamic Programming" applies to discrete time, "HJB" to continuous time systems.
- Pros and Cons
  - \+ Searches whole state space, finds global optimum.
  - \+ Optimal feedback controls precomputed.
  - \+ Analytic solution to some problems possible (linear systems with quadratic cost $\rightarrow$ Riccati Equation)
- "Viscosity solutions" (Lions et al.) exist for quite general nonlinear problems.
  - \- But: in general intractable, because partial differential equation (PDE) in high dimensional state space: "curse of dimensionality".
  - Possible remedy: Approximate J e.g. in framework of neuro-dynamic programming [Bertsekas 1996].
- Used for practical optimal control of small scale systems e.g. by Bonnans, Zidani, Lee, Back, ...

Indirect Methods

For simplicity, regard only problem without inequality constraints:



$$\underset{x(\cdot),u(\cdot)}{\text{minimize}} \quad \int_0^T L(x(t), u(t))\, dt \quad + \quad E\left(x(T)\right)$$

subject to

$$\begin{aligned} x(0) - x_0 &= 0, & & \text{(fixed initial value)} \\ \dot{x}(t) - f(x(t), u(t)) &= 0, & t \in [0, T], \quad & \text{(ODE model)} \end{aligned}$$

Pontryagin's Minimum Principle

OBSERVATION: In HJB, optimal controls

$$u^*(t) = \arg\min_u \left(L(x,u) + \frac{\partial J}{\partial x}(x,t) f(x,u) \right)$$

depend only on derivative $\frac{\partial J}{\partial x}(x,t)$, not on J itself!

IDEA: Introduce **adjoint variables**

$$\lambda(t) \quad \hat{=} \quad \frac{\partial J}{\partial x}(x(t),t)^T \in \mathbb{R}^{n_x}$$

and get controls from **Pontryagin's Minimum Principle**

$$u^*(t,x,\lambda) = \arg\min_u \left(\underbrace{L(x,u) + \lambda^T f(x,u)}_{\textbf{Hamiltonian} =: H(x,u,\lambda)} \right)$$

QUESTION: How to obtain $\lambda(t)$?

Adjoint Differential Equation

- Differentiate HJB Equation

$$-\frac{\partial J}{\partial t}(x,t) = \min_u H(x, u, \frac{\partial J}{\partial x}(x,t)^T)$$

with respect to x and obtain:

$$-\dot{\lambda}^T = \frac{\partial}{\partial x}\left(H(x(t), u^*(t,x,\lambda), \lambda(t))\right).$$

- Likewise, differentiate $J(x,T) = E(x)$ and obtain terminal condition

$$\lambda(T)^T = \frac{\partial E}{\partial x}(x(T)).$$

How to obtain explicit expression for controls?

- In simplest case,

$$u^*(t) = \arg\min_u H(x(t), u, \lambda(t))$$

  is defined by

$$\frac{\partial H}{\partial u}(x(t), u^*(t), \lambda(t)) = 0$$

  (Calculus of Variations, Euler-Lagrange).
- In presence of path constraints, expression for $u^*(t)$ changes whenever active constraints change. This leads to state dependent switches.
- If minimum of Hamiltonian locally not unique, "singular arcs" occur. Treatment needs higher order derivatives of H.

Necessary Optimality Conditions

Summarize optimality conditions as **boundary value problem**:

$$
\begin{aligned}
x(0) &= x_0, && \textit{initial value}\\
\dot{x}(t) &= f(x(t), u^*(t)), \quad t \in [0, T], && \textit{ODE model}\\
-\dot{\lambda}(t) &= \frac{\partial H}{\partial x}(x(t), u^*(t), \lambda(t))^T, \quad t \in [0, T], && \textit{adjoint equations}\\
u^*(t) &= \arg\min_u H(x(t), u, \lambda(t)), \quad t \in [0, T], && \textit{minimum principle}\\
\lambda(T) &= \frac{\partial E}{\partial x}(x(T))^T. && \textit{adjoint final value.}
\end{aligned}
$$

Solve with so called

- gradient methods,
- shooting methods, or
- collocation.

Indirect Methods

- "First optimize, then discretize"
- Pros and Cons
  + Boundary value problem with only $2 \times n_x$ ODE.
  + Can treat large scale systems.
  - Only necessary conditions for local optimality.
  - Need explicit expression for $u^*(t)$, singular arcs difficult to treat.
  - ODE strongly nonlinear and unstable.
  - Inequalities lead to ODE with state dependent switches.

    Possible remedy: Use interior point method in function space inequalities, e.g. Weiser and Deuflhard, Bonnans and Laurent-Varin
- Used for optimal control e.g. in satellite orbit planning at CNES...

Direct Methods

- "First discretize, then optimize"
- Transcribe infinite problem into finite dimensional, **Nonlinear Programming Problem (NLP)**, and solve NLP.
- Pros and Cons:
  - \+ Can use state-of-the-art methods for NLP solution.
  - \+ Can treat inequality constraints and multipoint constraints much easier.
  - \- Obtains only suboptimal/approximate solution.
- Nowadays most commonly used methods due to their easy applicability and robustness.

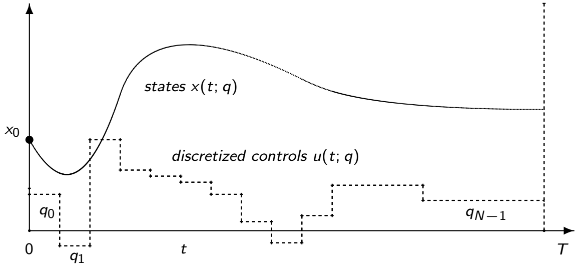
Direct Methods Overview

We treat three direct methods:

- Direct Single Shooting (sequential simulation and optimization)
- Direct Collocation (simultaneous simulation and optimization)
- Direct Multiple Shooting (simultaneous resp. hybrid)

Direct Single Shooting [Hicks1971,Sargent1978]

Discretize controls $u(t)$ on fixed grid $0 = t_0 < t_1 < \ldots < t_N = T$, regard states $x(t)$ on $[0, T]$ as dependent variables.



Use numerical integration to obtain state as function $x(t; q)$ of finitely many control parameters $q = (q_0, q_1, \ldots , q_{N-1})$

NLP in Direct Single Shooting

After control discretization and numerical ODE solution, obtain NLP:

$$\underset{q}{\text{minimize}} \quad \int_0^T L(x(t;q), u(t;q))\, dt + E\left(x(T;q)\right)$$

subject to

$$\begin{aligned} & h(x(t_i;q), u(t_i;q)) \geq 0, \\ & \quad i = 0, \ldots, N, && \textit{(discretized path constraints)} \\ & r\left(x(T;q)\right) \geq 0. && \textit{(terminal constraints)} \end{aligned}$$

Solve with finite dimensional optimization solver, e.g. Sequential Quadratic Programming (SQP).

Solution by Standard SQP

Summarize problem as

$$\min_{q} F(q) \text{ s.t. } H(q) \geq 0.$$

Solve e.g. by Sequential Quadratic Programming (SQP), starting with guess q^0 for controls. $k := 0$

1. Evaluate $F(q^k), H(q^k)$ by ODE solution, and derivatives!
2. Compute correction Δq^k by solution of QP:

$$\min_{\Delta q} \nabla F(q_k)^T \Delta q + \frac{1}{2} \Delta q^T A^k \Delta q \text{ s.t. } H(q^k) + \nabla H(q^k)^T \Delta q \geq 0.$$

3. Perform step $q^{k+1} = q^k + \alpha_k \Delta q^k$ with step length α_k determined by line search.

ODE Sensitivities

How to compute the sensitivity $\dfrac{\partial x(t;q)}{\partial q}$ of a numerical ODE solution $x(t;q)$ with respect to the controls q?

Four ways:

1. External Numerical Differentiation (END)
2. Variational Differential Equations
3. Automatic Differentiation
4. Internal Numerical Differentiation (IND)

Numerical Test Problem

$$\underset{x(\cdot),u(\cdot)}{\text{minimize}} \quad \int_0^3 x(t)^2 + u(t)^2 \, dt$$

subject to

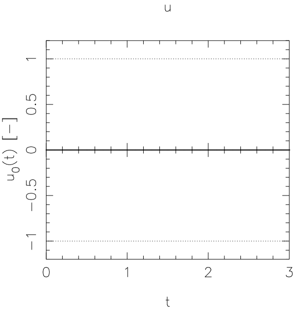
$$\begin{aligned}
x(0) &= x_0, & & \text{(initial value)}\\
\dot{x} &= (1+x)x + u, & t \in [0,3], \quad & \text{(ODE model)}\\
\begin{bmatrix} 1 - x(t) \\ 1 + x(t) \\ 1 - u(t) \\ 1 + u(t) \end{bmatrix} &\geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, & t \in [0,3], \quad & \text{(bounds)}\\
x(3) &= 0. & & \text{(zero terminal constraint).}
\end{aligned}$$

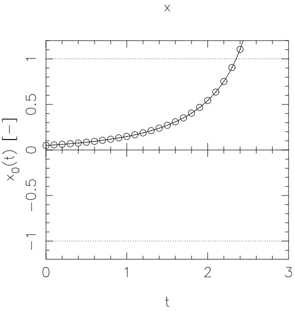
Remark: Uncontrollable growth for $(1 + x_0)x_0 - 1 \geq 0 \Leftrightarrow x_0 \geq 0.618$.

Single Shooting Optimization for $x_0 = 0.05$

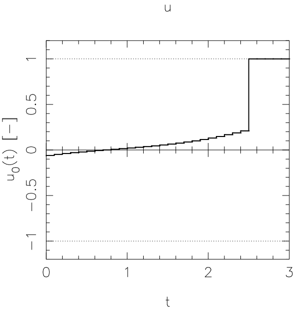
- Choose $N = 30$ equal control intervals.
- Initialize with steady state controls $u(t) \equiv 0$.
- Initial value $x_0 = 0.05$ is the maximum possible, because initial trajectory explodes otherwise.

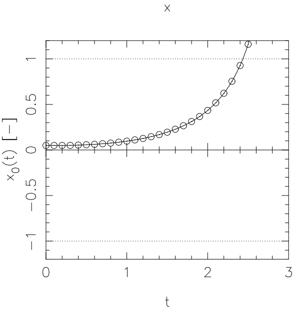
Single Shooting: Initialization



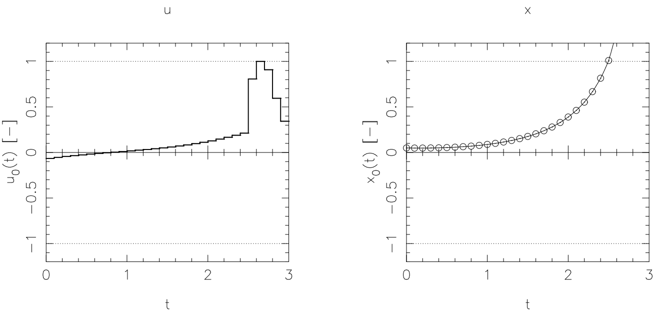


Single Shooting: First Iteration

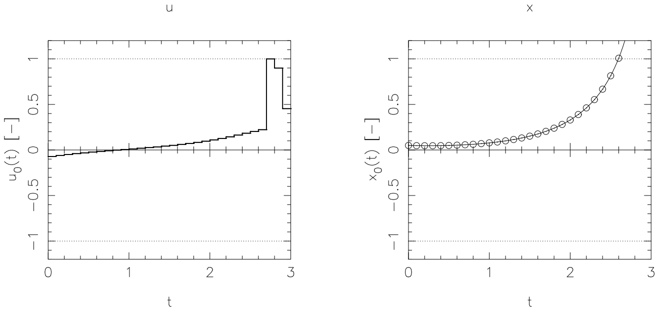




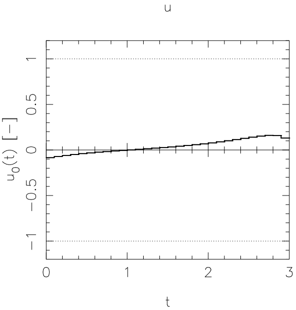
Single Shooting: 2nd Iteration

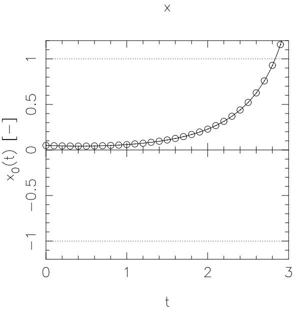


Single Shooting: 3rd Iteration

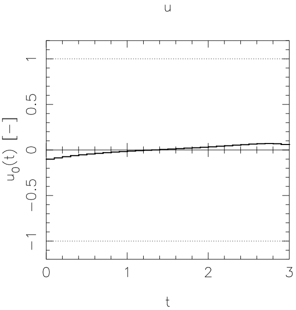


Single Shooting: 4th Iteration

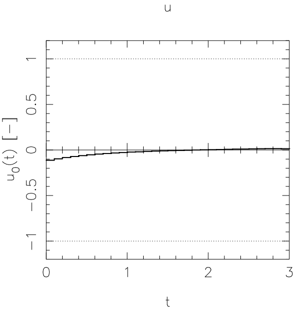




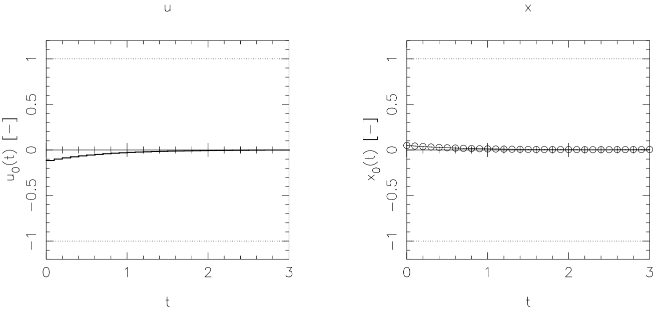
Single Shooting: 5th Iteration



Single Shooting: 6th Iteration



Single Shooting: 7th Iteration and Solution



Direct Single Shooting: Pros and Cons

- **Sequential** simulation and optimization.

+ Can use state-of-the-art ODE/DAE solvers.
+ Few degrees of freedom even for large ODE/DAE systems.
+ Active set changes easily treated.
+ Need only initial guess for controls q.

- Cannot use knowledge of x in initialization (e.g. in tracking problems).
- ODE solution $x(t; q)$ can depend very nonlinearly on q.
- Unstable systems difficult to treat.

- Often used in engineering applications e.g. in packages gOPT (PSE), DYOS (Marquardt), ...

Direct Collocation (Sketch) [Tsang1975]

- Discretize controls and states on **fine** grid with node values $s_i \approx x(t_i)$.
- Replace infinite ODE

$$0 = \dot{x}(t) - f(x(t), u(t)), \quad t \in [0, T]$$

by finitely many equality constraints

$$c_i(q_i, s_i, s_{i+1}) = 0, \quad i = 0, \ldots, N-1,$$

$$\text{e.g.} \quad c_i(q_i, s_i, s_{i+1}) := \frac{s_{i+1} - s_i}{t_{i+1} - t_i} - f\left(\frac{s_i + s_{i+1}}{2}, q_i\right)$$

- Approximate also integrals, e.g.

$$\int_{t_i}^{t_{i+1}} L(x(t), u(t))dt \approx l_i(q_i, s_i, s_{i+1}) := L\left(\frac{s_i + s_{i+1}}{2}, q_i\right)(t_{i+1} - t_i)$$

NLP in Direct Collocation

After discretization obtain large scale, but sparse NLP:

$$\underset{s,q}{\text{minimize}} \quad \sum_{i=0}^{N-1} l_i(q_i, s_i, s_{i+1}) \;+\; E\left(s_N\right)$$

subject to

$$\begin{aligned}
s_0 - x_0 &= 0, & & \text{(fixed initial value)} \\
c_i(q_i, s_i, s_{i+1}) &= 0, & i = 0, \ldots, N-1, & \quad \text{(discretized ODE model)} \\
h(s_i, q_i) &\geq 0, & i = 0, \ldots, N, & \quad \text{(discretized path constraint} \\
r\left(s_N\right) &\geq 0. & & \text{(terminal constraints)}
\end{aligned}$$

Solve e.g. with SQP method for sparse problems.

What is a sparse NLP?

General NLP:

$$\min_{w} F(w) \;\text{ s.t.}$$
$$G(w) = 0,$$
$$H(w) \geq 0.$$

is called sparse if the Jacobians (derivative matrices)

$$\nabla_w G^T = \frac{\partial G}{\partial w} = \left(\frac{\partial G}{\partial w_j} \right)_{ij} \quad \text{and} \quad \nabla_w H^T$$

contain many zero elements.
In SQP methods, this makes QP much cheaper to build and to solve.

Direct Collocation: Pros and Cons

- **Simultaneous** simulation and optimization.

+ Large scale, but very sparse NLP.
+ Can use knowledge of x in initialization.
+ Can treat unstable systems well.
+ Robust handling of path and terminal constraints.

- Adaptivity needs new grid, changes NLP dimensions.

- Successfully used for practical optimal control e.g. by Biegler and Wächter (IPOPT), Betts,

Direct Multiple Shooting [Bock 1984]

- Discretize controls piecewise on a coarse grid

$$u(t) = q_i \quad \text{for} \quad t \in [t_i, t_{i+1}]$$

- Solve ODE on each interval $[t_i, t_{i+1}]$ numerically, starting with artificial initial value s_i:

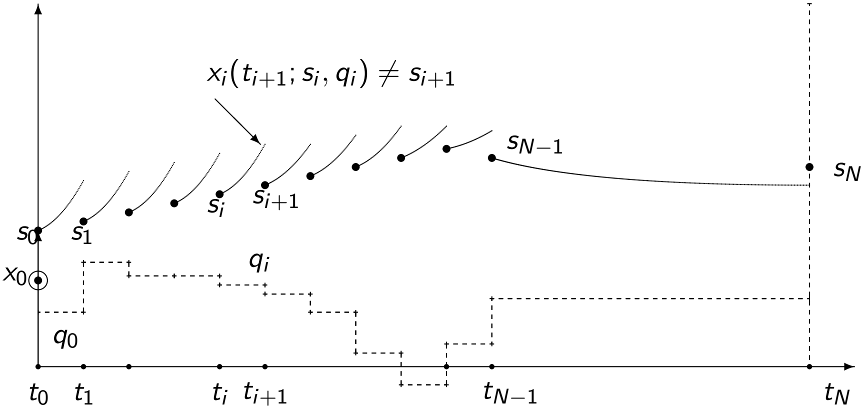
$$\begin{aligned} \dot{x}_i(t; s_i, q_i) &= f(x_i(t; s_i, q_i), q_i), \quad t \in [t_i, t_{i+1}], \\ x_i(t_i; s_i, q_i) &= s_i. \end{aligned}$$

  Obtain trajectory pieces $x_i(t; s_i, q_i)$.

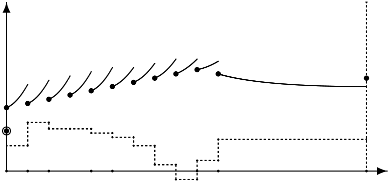
- Also numerically compute integrals

$$l_i(s_i, q_i) := \int_{t_i}^{t_{i+1}} L(x_i(t_i; s_i, q_i), q_i) dt$$

Sketch of Direct Multiple Shooting



NLP in Direct Multiple Shooting



$$\underset{s,q}{\text{minimize}} \quad \sum_{i=0}^{N-1} l_i(s_i, q_i) \;+\; E\left(s_N\right)$$

subject to

$$\begin{aligned}
s_0 - x_0 &= 0, && \text{(initial value)}\\
s_{i+1} - x_i(t_{i+1}; s_i, q_i) &= 0,\ i = 0, \ldots, N-1, && \text{(continuity)}\\
h(s_i, q_i) &\geq 0,\ i = 0, \ldots, N, && \text{(discretized path constraints}\\
r\left(s_N\right) &\geq 0. && \text{(terminal constraints)}
\end{aligned}$$

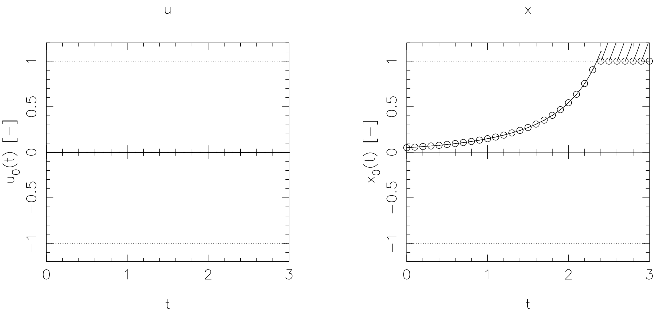
Structured NLP

- Summarize all variables as $w := (s_0, q_0, s_1, q_1, \dots, s_N)$.
- Obtain structured NLP

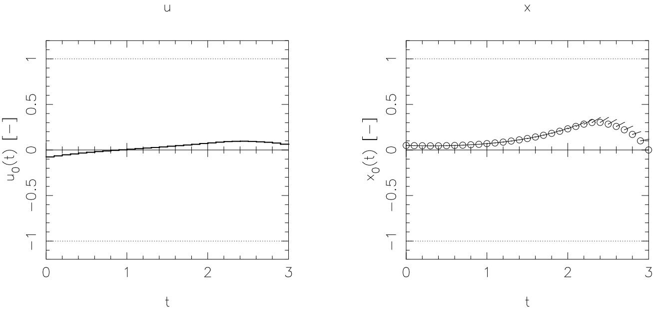
$$\boxed{\min_{w} \; F(w) \quad \text{s.t.} \quad \left\{ \begin{array}{rcl} G(w) & = & 0 \\ H(w) & \geq & 0. \end{array} \right.}$$

- Jacobian $\nabla G(w^k)^T$ contains dynamic model equations.
- Jacobians and Hessian of NLP are block sparse, can be exploited in numerical solution procedure.

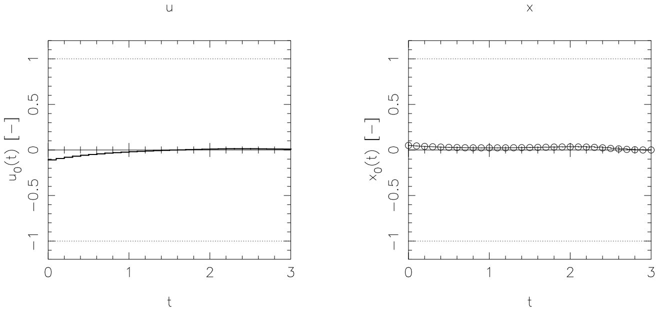
Test Example: Initialization with $u(t) \equiv 0$



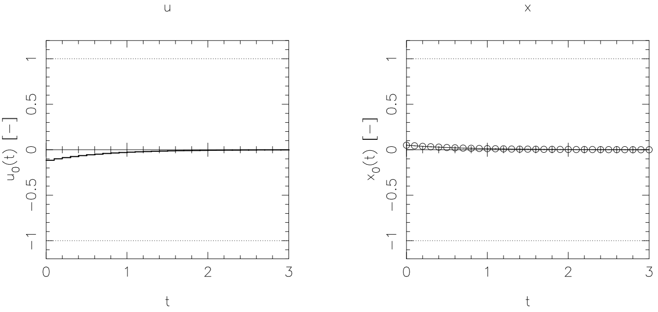
Multiple Shooting: First Iteration



Multiple Shooting: 2nd Iteration



Multiple Shooting: 3rd Iteration and Solution



Direct Multiple Shooting: Pros and Cons

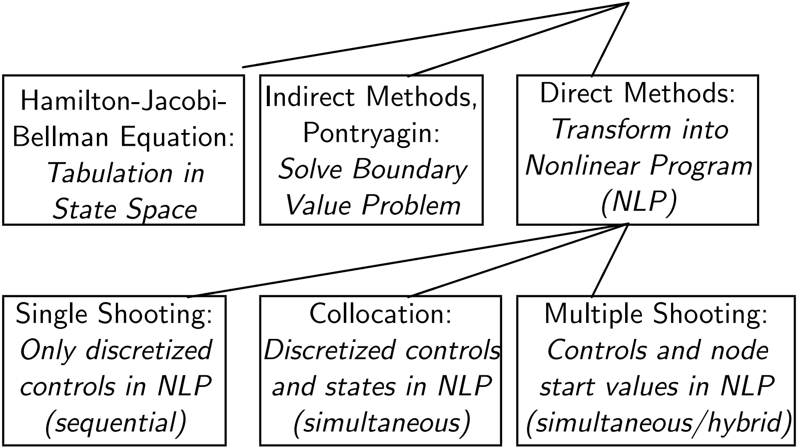
- **Simultaneous** simulation and optimization.

+ uses **adaptive** ODE/DAE solvers
+ but NLP has **fixed dimensions**
+ can use knowledge of x in initialization (here bounds; more important in online context).
+ can treat unstable systems well.
+ robust handling of path and terminal constraints.
+ easy to parallelize.

- not as sparse as collocation.

- Used for practical optimal control e.g by Franke (ABB) ("HQP"), Terwen (Daimler); Bock et al. ("MUSCOD-II"); in ACADO Toolkit; ...

Conclusions: Optimal Control Family Tree



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