An Efficient Algorithm for Tube-based Robust Nonlinear Optimal Control with Optimal Linear Feedback

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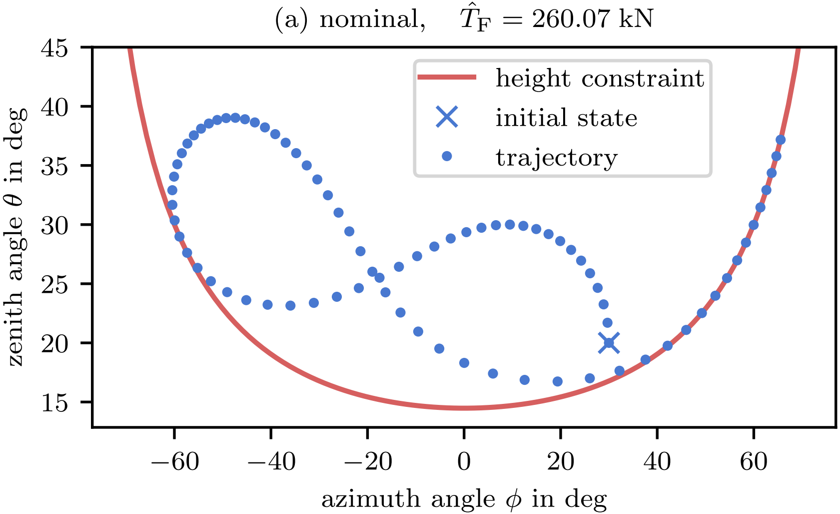
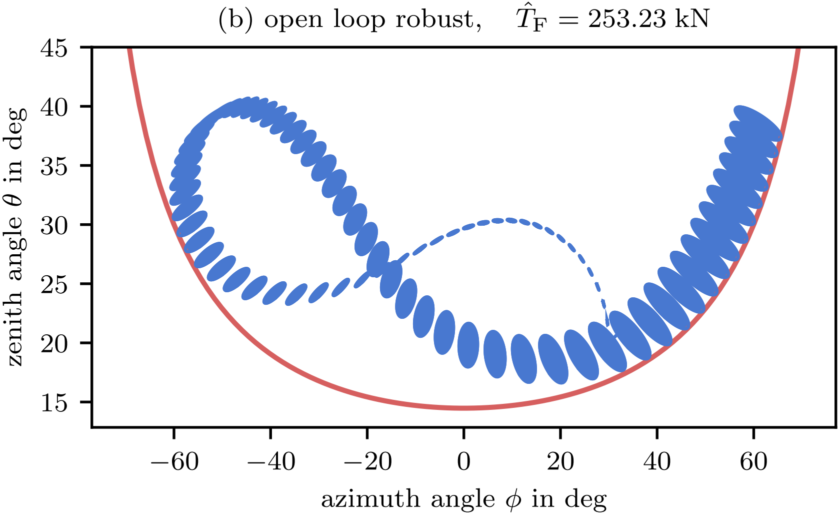
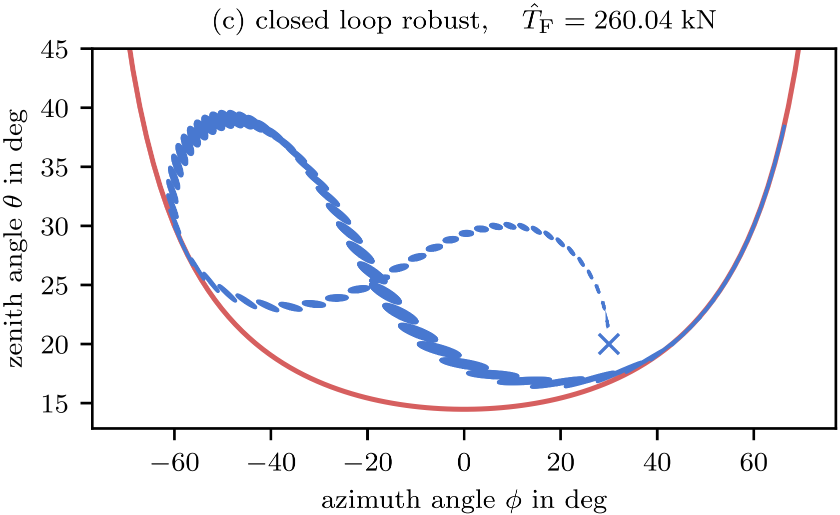

(a) nominal, $\hat{T}_{\mathrm{F}} = 260.07$ kN
height constraint
initial state
trajectory
45
40
35
30
25
20
15
zenith angle θ in deg
−60
−40
−20
0
20
40
60
azimuth angle ϕ in deg

Introduction




(c) closed loop robust, $\hat{T}_{\mathrm{F}} = 260.04$ kN
45
40
35
30
25
20
15
zenith angle θ in deg
−60
−40
−20
0
20
40
60
azimuth angle ϕ in deg

Setting

- Consider a stochastic nonlinear dynamical system

$$x_0 = \bar{\bar{x}}_0, \qquad x_{k+1} = f_k(x_k, u_k, w_k), \qquad k = 0, \ldots, N-1. \tag{1}$$

- $w = (w_0, \ldots, w_{N-1})$ is noise drawn from ellipsoid $w \in \mathcal{E}(\sigma^2 I, 0)$
  - Ellipsoid with shape matrix $V \in \mathbb{S}^n_+$, centered at $c \in \mathbb{R}^n$

$$\mathcal{E}(V, c) := \{V^{\frac{1}{2}} v + c \mid v \in \mathbb{R}^n, v^\top v \leq 1\} \tag{2}$$

  - uncertainty scaling parameter $\sigma \geq 0$
- We are interested in robust constraint satisfaction for all possible trajectories

$$h(x_k, u_k) \leq 0, \quad \forall x_k \in X_k(u), \quad k = 0, \ldots, N-1, \tag{3}$$

$$h(x_N) \leq 0, \quad \forall x_N \in X_N(u), \tag{4}$$

with $X_k(u)$, $k = 0, \ldots, N$, the set of all reachable states at k given control trajectory u.

Nominal OCP

$$
\begin{aligned}
\min_{\bar{x},\,\bar{u}} \quad & \sum_{k=0}^{N-1} l_k(\bar{x}_k, \bar{u}_k) + E(\bar{x}_N) && \text{(5a)} \\
\text{s.t.} \quad & \bar{x}_0 = \bar{\bar{x}}_0, && \text{(5b)} \\
& \bar{x}_{k+1} = f_k(\bar{x}_k, \bar{u}_k, 0), \quad k = 0, \ldots, N-1, && \text{(5c)} \\
& 0 \geq h_k(\bar{x}_k, \bar{u}_k), \qquad k = 0, \ldots, N-1, && \text{(5d)} \\
& 0 \geq h_N(\bar{x}_N). && \text{(5e)}
\end{aligned}
$$

Approximate Robustification

- Model uncertainty tube by ellipsoids around nominal trajectory $\bar{x}$, $\bar{u}$

$$\bar{x}_0 = f_0(0) =: \bar{\bar{x}}_0, \qquad \bar{x}_{k+1} = f_k(\bar{x}_k, \bar{u}_k, 0), \qquad k = 0, \dots, N-1, \tag{6}$$

$$x_k \in \mathcal{E}(P_k, \bar{x}_k), \qquad k = 0, \dots, N. \tag{7}$$

- Plan with linear feedback law to reduce uncertainty

$$u_k = \kappa_k(x_k) = \bar{u}_k + K_k(x_k - \bar{x}_k), \quad k = 0, \dots, N-1, \quad K_0 = 0. \tag{8}$$

- Propagate ellipsoids according to dynamics linearized at $\bar{x}$, $\bar{u}$

$$P_0 = 0, \quad P_{k+1} = (A_k + B_k K_k) P_k (A_k + B_k K_k)^\top + \sigma^2 \Gamma_k \Gamma_k^\top \tag{9}$$

$$A_k = \frac{\partial f_k}{\partial x_k}(\bar{x}_k, \bar{u}_k, 0), \quad B_k = \frac{\partial f_k}{\partial u_k}(\bar{x}_k, \bar{u}_k, 0), \quad \Gamma_k = \frac{\partial f_k}{\partial w_k}(\bar{x}_k, \bar{u}_k, 0), \qquad k = 0, \dots, N-1.$$

Closed-loop Robustified NMPC problem

$$\min_{\bar{x}, \bar{u}, K, P, \beta} \quad \sum_{k=0}^{N-1} l(\bar{x}_k, \bar{u}_k) + E(\bar{x}_N) \tag{10a}$$

$$\text{s.t.} \quad \bar{x}_0 = \bar{\bar{x}}_0, \tag{10b}$$

$$\bar{x}_{k+1} = f(\bar{x}_k, \bar{u}_k, 0), \qquad k = 0, \dots, N-1, \tag{10c}$$

$$P_0 = 0, \tag{10d}$$

$$P_{k+1} = (A_k + B_k K_k) P_k (A_k + B_k K_k)^\top + \sigma^2 \Gamma_k \Gamma_k^\top, \qquad k = 0, \dots, N-1, \tag{10e}$$

$$0 \geq h_k(\bar{x}_k, \bar{u}_k) + \sqrt{\beta_k + \epsilon \mathbf{1}}, \qquad k = 0, \dots, N-1, \tag{10f}$$

$$0 \geq h_N(\bar{x}_k) + \sqrt{\beta_N + \epsilon \mathbf{1}}, \tag{10g}$$

$$\beta_k^i = \nabla h_k^i(\bar{x}_k, \bar{u}_k)^\top \begin{bmatrix} I \\ K_k \end{bmatrix} P_k \begin{bmatrix} I \\ K_k \end{bmatrix}^\top \nabla h_k^i(\bar{x}_k, \bar{u}_k), \qquad \begin{matrix} i = 1, \dots, n_{h_k}, \\ k = 0, \dots, N-1, \end{matrix} \tag{10h}$$

$$\beta_N^i = \nabla h_N^i(\bar{x}_N)^\top P_N \nabla h_N^i(\bar{x}_N), \qquad i = 1, \dots, n_{h_N}, \tag{10i}$$

where $K = (K_1, \dots, K_{N-1})$, $K_0 = 0$, $\nabla h_k^i(\bar{x}_k, \bar{u}_k) = \nabla_{(x,u)} h_k^i(\bar{x}_k, \bar{u}_k)$, $\nabla h_N^i(\bar{x}_N) = \nabla_x h_N^i(\bar{x}_N)$, $\beta_k \in \mathbb{R}^{n_{h_k}}$, $\epsilon > 0$, $\mathbf{1} = (1, \dots, 1)$, $\sqrt{\cdot}$ elementwise.

Summarized formulation

- Collect $y = (\bar{x}, \bar{u})$, $M = \mathrm{vec}(K)$, P eliminated.
- Summarize as

$$
\begin{aligned}
\min_{y, M, \beta} \quad & f(y) && \text{(11a)}\\
\text{s.t.} \quad & g(y) = 0, && \text{(11b)}\\
& h(y) + \sqrt{\beta + \epsilon \mathbf{1}} \leq 0, && \text{(11c)}\\
& \sigma^2 H(y, M) - \beta = 0. && \text{(11d)}
\end{aligned}
$$

- Lagrangian

$$
\mathcal{L}(y, M, \lambda, \mu, \eta) = f(y) + \lambda^\top g(y) + \mu^\top (h(y) + \sqrt{\beta + \epsilon \mathbf{1}}) + \eta^\top (\sigma^2 H(y, M) - \beta) \quad (12)
$$

KKT conditions

- KKT conditions

$$
\begin{aligned}
\nabla f(y) + \nabla g(y)\lambda + \nabla h(y)\mu + \sigma^2 \nabla_y H(y, M)\eta &= 0, && \text{(13a)}\\
\sigma^2 \nabla_M H(y, M)\eta &= 0, && \text{(13b)}\\
\tfrac{1}{2}\,\mathrm{diag}(\beta)^{-\frac{1}{2}}\mu - \eta &= 0, && \text{(13c)}\\
g(y) &= 0, && \text{(13d)}\\
0 \leq \mu \perp h(y) + \sqrt{\beta + \epsilon \mathbf{1}} &\leq 0, && \text{(13e)}\\
\sigma^2 H(y, M) - \beta &= 0. && \text{(13f)}
\end{aligned}
$$

- Interpret (13b) as FONC of $\min_M \; \eta^\top H(y, M)$
- Defines $m(y, \eta) = \arg\min_M \; \eta^\top H(y, M)$
- Eliminate $M = m(y, \eta)$

Eliminate ...

- Reduced KKT conditions (M eliminated)

$$
\begin{aligned}
\nabla f(y) + \nabla g(y)\lambda + \nabla h(y)\mu + \sigma^2 \nabla_y H(y, m(y,\eta))\eta &= 0, && \text{(14a)}\\
\tfrac{1}{2}\,\mathrm{diag}(\beta)^{-\frac{1}{2}}\mu - \eta &= 0, && \text{(14b)}\\
g(y) &= 0, && \text{(14c)}\\
0 \leq \mu \perp h(y) + \sqrt{\beta + \epsilon \mathbf{1}} &\leq 0, && \text{(14d)}\\
\sigma^2 H(y, m(y,\eta)) - \beta &= 0. && \text{(14e)}
\end{aligned}
$$

- Freeze uncertainty parts at some given $\bar{y}$, $\bar{\eta}$, $\bar{M} = m(\bar{y}, \bar{\eta})$

$$
\begin{aligned}
\nabla f(y) + \nabla g(y)\lambda + \nabla h(y)\mu + \sigma^2 \nabla_y H(\bar{y}, \bar{M})\bar{\eta} &= 0, && (15a)\\
\tfrac{1}{2}\operatorname{diag}(\beta)^{-\frac{1}{2}}\mu - \eta &= 0, && (15b)\\
g(y) &= 0, && (15c)\\
0 \le \mu \perp h(y) + \sqrt{\beta + \epsilon \mathbf{1}} &\le 0, && (15d)\\
\sigma^2 H(\bar{y}, \bar{M}) - \beta &= 0. && (15e)
\end{aligned}
$$

Perturbed nominal problem

- Define $\bar{c} = \nabla_y H(\bar{y}, \bar{M})\bar{\eta}$, $\bar{H} = H(\bar{y}, \bar{M})$

$$\nabla f(y) + \nabla g(y)\lambda + \nabla h(y)\mu + \sigma^2 \bar{c} = 0, \tag{16a}$$
$$g(y) = 0, \tag{16b}$$
$$0 \leq \mu \perp h(y) + \sqrt{\beta + \epsilon \mathbf{1}} \leq 0, \tag{16c}$$
$$\sigma^2 \bar{H} - \beta = 0. \tag{16d}$$

- View as KKT conditions of perturbed nominal problem

$$\min_{y} \quad f(y) + \sigma^2 \bar{c}^\top y \tag{17a}$$
$$\text{s.t.} \quad g(y) = 0, \tag{17b}$$
$$h(y) + \sqrt{\beta + \epsilon \mathbf{1}} \leq 0, \tag{17c}$$
$$\sigma^2 \bar{H} - \beta = 0. \tag{17d}$$

Algorithm sketch

1. Initialize y, η
2. Obtain M via

$$M \leftarrow \arg\min_{M} \ \eta^\top H(y, M). \tag{18}$$

3. Set $\quad \bar{c} \leftarrow \nabla_y H(y, M)\eta, \quad \bar{H} \leftarrow \sqrt{H(y, M)}.$
4. Obtain y, β, λ, μ, η from

$$\begin{aligned} \min_{y} \quad & f(y) + \sigma^2 \bar{c}^\top y && (19a) \\ \text{s.t.} \quad & g(y) = 0, && (19b) \\ & h(y) + \sqrt{\beta + \epsilon \mathbf{1}} \leq 0, && (19c) \\ & \sigma^2 \bar{H} - \beta = 0. && (19d) \end{aligned}$$

5. If converged: stop. Else: back to 2.

Convergence analysis sketch

- Stationary point of algorithm is KKT point of original NLP.
- Linear local convergence for sufficiently small uncertainty level σ.
  - Assumption: LICQ, SOSC of nominal problem ($\sigma = 0$)
  - Error in Jacobian of KKT residual map, evaluated at solution, scales as $\mathcal{O}(\sigma)$
  - For σ sufficiently small, error will be sufficiently small for locally linear covergence
  - slower convergence for larger σ

Back to the robustified OCP

- For robustified OCP, $\min_{M} \eta^\top H(y, M)$ corresponds to

$$\min_{K,P} \quad \sum_{k=0}^{N-1} \mathrm{Tr}\left(C_k \begin{bmatrix} I \\ K_k \end{bmatrix} P_k \begin{bmatrix} I \\ K_k \end{bmatrix}^\top \right) + \mathrm{Tr}(C_N P_N) \tag{20a}$$

$$\text{s.t.} \quad P_0 = 0, \tag{20b}$$

$$P_{k+1} = (A_k + B_k K_k) P_k (A_k + B_k K_k)^\top + \sigma^2 \Gamma_k \Gamma_k^\top, \quad k = 0, \dots, N-1. \tag{20c}$$

where $$C_k = \nabla_{(x,u)} h_k(\bar{x}_k, \bar{u}_k)\, \mathrm{diag}(\eta_k) \nabla_{(x,u)} h_k(\bar{x}_k, \bar{u}_k)^\top \tag{21}$$

$$C_N = \nabla h_N(\bar{x}_N)\, \mathrm{diag}(\eta_N) \nabla h_N(\bar{x}_N)^\top \tag{22}$$

- Equivalent in structure to finite-horizon stochastic LQR
- Analytic solution via Riccati recursion

Riccati recursion

$$\min_{K,P} \quad \sum_{k=0}^{N-1} \mathrm{Tr}\left(C_k \begin{bmatrix} I \\ K_k \end{bmatrix} P_k \begin{bmatrix} I \\ K_k \end{bmatrix}^\top \right) + \mathrm{Tr}(C_N P_N) \tag{23a}$$

$$\text{s.t.} \qquad P_0 = 0, \tag{23b}$$

$$P_{k+1} = (A_k + B_k K_k) P_k (A_k + B_k K_k)^\top + \sigma^2 \Gamma_k \Gamma_k^\top, \quad k = 0, \dots, N-1. \tag{23c}$$

solved by Riccati recursion

$$S_N = C_N, \tag{24a}$$

$$K_k^* = -(C_k^{uu} + B_k^\top S_{k+1} B_k)^{-1} (C_k^{ux} + B_k^\top S_{k+1} A_k), \tag{24b}$$

$$S_k = C_k^{xx} + A_k^\top S_{k+1} A_k + (C_k^{xu} + A_k^\top S_{k+1} B_k) K_k^*, \quad k = N-1, \dots, 1 \tag{24c}$$

$K_0^* = K_0 = 0$, followed by a linear Lyapunov matrix forward simulation

$$P_0^* = 0, \qquad P_{k+1}^* = (A_k + B_k K_k^*) P_k^* (A_k + B_k K_k^*)^\top + \sigma^2 \Gamma_k \Gamma_k^\top, \quad k = 0, \dots, N-1. \tag{25}$$

- Perturbed nominal OCP

$$
\begin{aligned}
\min_{\bar{x},\bar{u}} \quad & \sum_{k=0}^{N-1} l(\bar{x}_k,\bar{u}_k) + E(\bar{x}_N) + \sum_{k=0}^{N-1} \bar{c}_k^\top \begin{bmatrix}\bar{x}_k\\ \bar{u}_k\end{bmatrix} + \bar{c}_N^\top \bar{x}_N && \text{(26a)}\\
\text{s.t.} \quad & \bar{x}_0 = \bar{\bar{x}}_0, && \text{(26b)}\\
& \bar{x}_{k+1} = f(\bar{x}_k,\bar{u}_k,0), \qquad k = 0,\dots,N-1, && \text{(26c)}\\
& 0 \geq h_k(\bar{x}_k,\bar{u}_k) + \bar{b}_k, \quad k = 0,\dots,N-1, && \text{(26d)}\\
& 0 \geq h_N(\bar{x}_N) + \bar{b}_N. && \text{(26e)}
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{b}_k &= \sqrt{H_k(\bar{x}_k,\bar{u}_k,\bar{P}_k,\bar{K}_k) + \epsilon\mathbf{1}}, \; k = 0,\dots,N-1, && \text{(27a)}\\
\bar{b}_N &= \sqrt{H_N(\bar{x}_k,\bar{P}_k) + \epsilon\mathbf{1}}. && \text{(27b)}
\end{aligned}
$$

Algorithm

Algorithm 1 Sequential inexact robust optimization for tube-based robust optimal control

Input: Initial guess $\bar{x}$, $\bar{u}$, η

repeat

    $K \leftarrow \texttt{riccatiRecursion}(\bar{x}, \bar{u}, \eta)$

    $P \leftarrow \texttt{lyapunovForward}(\bar{x}, \bar{u}, K)$

    if $\texttt{Stationarity}(\bar{x}, \bar{u}, K, P, \eta) \leq \texttt{TOL}$

        break

    end if

    $\bar{b}, \bar{c} \leftarrow \texttt{getPerturbation}(\bar{x}, \bar{u}, \eta, P, K)$

    $\bar{x}, \bar{u}, \lambda, \mu \leftarrow \texttt{solvePerturbedOCP}(\bar{c}, \bar{b})$

    $\eta \leftarrow \frac{1}{2}\,\mathrm{diag}(\bar{b})^{-1}\mu$

return: $\bar{x}, \bar{u}, P, K$

Some notes

- Setting $K = 0$ (instead of Riccati recursion) gives algorithm for solving open-loop robust OCP
  - Variation: Setting K_k to precomputed stabilizing feedback gains
  - cf. Feng, Di Cairano, Quirynen. Inexact Adjoint-based SQP Algorithm for Real-Time Stochastic Nonlinear MPC. *Proceedings of the IFAC World Congress*, 2020.
  - Zanelli, Frey, Messerer, Diehl. Zero-Order Robust Nonlinear Model Predictive Control with Ellipsoidal Uncertainty Sets. *Proceedings of the IFAC Conference on Nonlinear Model Predictive Control (NMPC)*, 2021.

Test problem: control of a towing kite

- $x = (\theta, \phi, \psi) \in \mathbb{R}^3$, $u \in \mathbb{R}$, $w \in \mathbb{R}$, $w \in [-2, 2]$

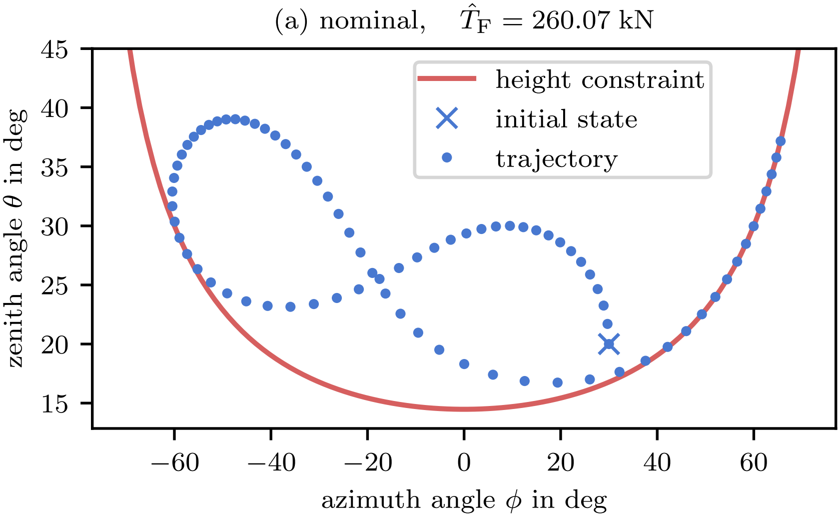
$$\begin{aligned} \dot\theta &= \frac{v_{\mathrm{a}}(\theta,u,w)}{L}\left(\cos\psi - \frac{\tan\theta}{E(u)}\right) \\ \dot\phi &= \frac{-v_{\mathrm{a}}(\theta,u,w)\sin\psi}{L\sin\theta} \\ \dot\psi &= \frac{u v_{\mathrm{a}}(\theta,u,w)}{L} + \dot\phi\cos\theta \end{aligned} \qquad \text{where} \quad \begin{aligned} v_{\mathrm{a}}(\theta,u,w) &= (v_0 + w)E(u)\cos\theta, \\ E(u) &= E_0 - \tilde{c}u^2 \end{aligned}$$

- Integration with one step RK4, $N = 80$, $T/N = 0.3$
- Stage cost (negative tether thrust):

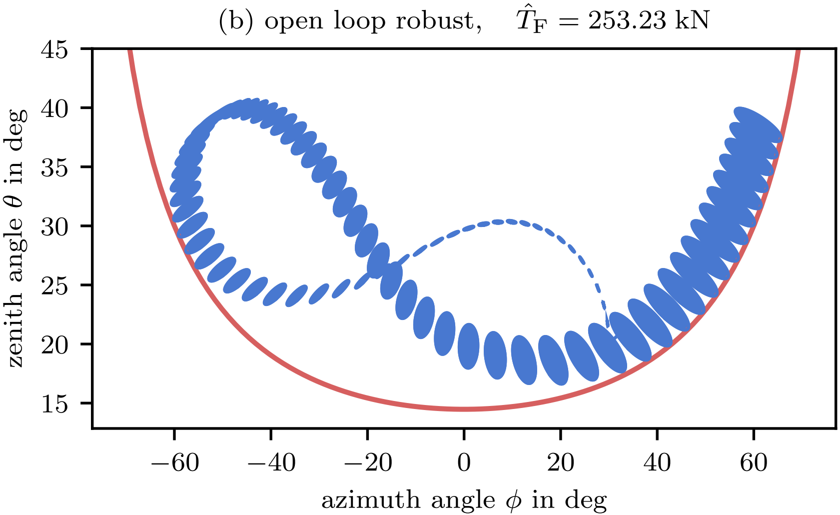
$$l_k(x,u) = -0.5\rho v_0^2 A \cos^3\theta (E(u)+1)\sqrt{E^2(u)+1}$$

- Control constraints: $-10 \le u \le 10$, minimal heigth: $L\sin\theta\cos\phi \ge h_{\min}$

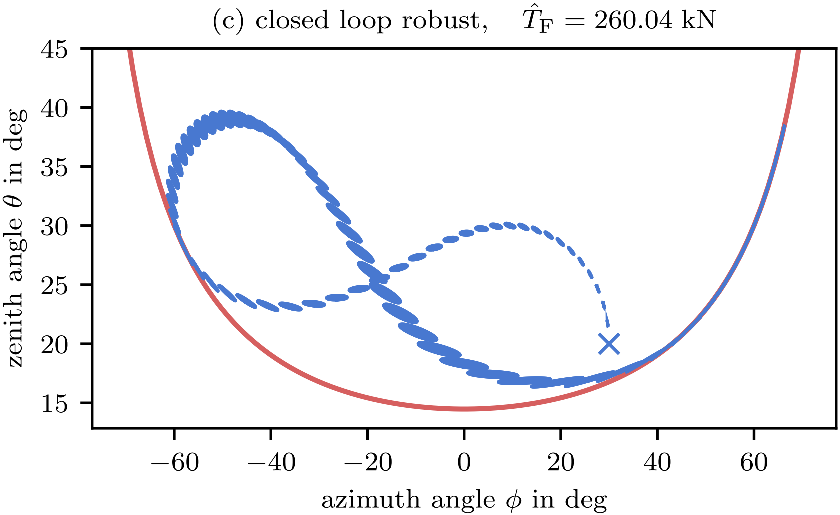
Nominal solution



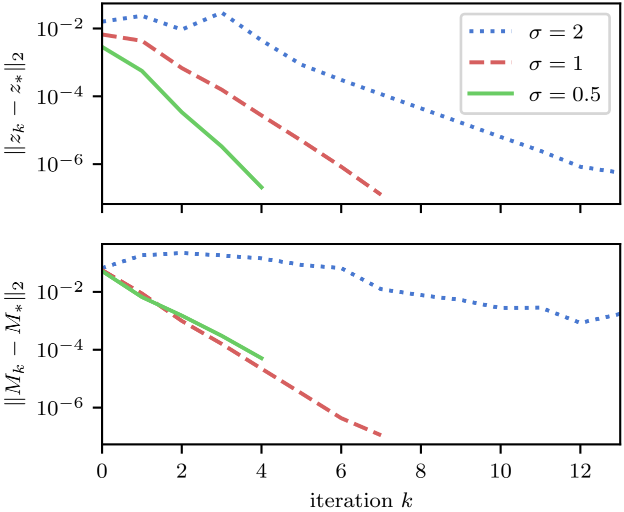
Open loop robust solution ($K = 0$)



Closed loop robust solution



Convergence behavior



Summary

- Representation of uncertainty tube by ellipsoids
- Approximate uncertainty propagation and constraint satisfaction based on linearization
- Include linear feedback law to reduce conservativness
- **Optimize** feedback gains to minimize backoffs (automatically weighted by impact on cost)
- Alternatingly solve perturbed nominal OCP and back-off minimization (Riccati) until convergence