Numerical Optimization (Numerische Optimierung) – Exam

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Page	0	1	2	3	4	5	6	7	8	9	Sum
Points on page (max)	3	8	9	8	8	9	8	7	0	0	60
Points obtained											
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Mark: Exam inspected on: Signature of examiner:

Surname: First Name: Matriculation number:

Subject: Programme: Bachelor ☐ Master ☐ Lehramt ☐ others ☐ Signature:

Please fill in your name above. For the multiple choice questions, which give exactly one point, tick exactly one box for the right answer. For the text questions, give a short formula or text answer just below the question in the space provided and, if necessary, write on the **backpage of the same sheet** where the question appears, and add a comment "see backpage". You can request additional empty pages for drafting. They have to be handed in at the end, but their content will not be graded. Only answers on the original exam sheets count. The exam is a closed book exam, i.e., no books or other material are allowed besides 2 sheets (with 4 pages) of hand-written notes and a non-programmable calculator. Some legal comments are found in a footnote[1].

1. Which of the following functions $f(x)$, $f : \mathbb{R}^n \to \mathbb{R}$, is NOT convex (parameters $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$)?

(a) ☐ $\exp(\|Ax+b\|_1)$	(b) ☐ $\exp(\|Ax+b\|_2^2)$	(c) ☐ $\min\left(\|x\|_2^2,\ \|x\|_1\right)$	(d) ☐ $\max\left(\|x\|_2^2,\ \|x\|_1\right)$

1

2. Which of the following sets is NOT convex ($c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ $A \in \mathbb{R}^{m\times n}$)?

(a) ☐ $\{x \in \mathbb{R}^n \mid \|Ax+b\|_2^2 \le 3\}$	(b) ☐ $\{x \in \mathbb{R}^n \mid \log(c^\top x) \ge 3\}$
(c) ☐ $\{x \in \mathbb{R}^n \mid \exp(c^\top x) \le 3\}$	(d) ☐ $\{x \in \mathbb{R}^n \mid \|Ax+b\|_2^2 \ge 3\}$

1

3. Given two convex functions $f(x)$ and $g(x)$, which of the following functions is NOT necessarily convex?

(a) ☐ composition: $h(x) = f(g(x))$	(b) ☐ point-wise maximum: $h(x) = \max\{f(x), g(x)\}$
(c) ☐ sum: $h(x) = f(x) + g(x)$	(d) ☐ affine input transformation: $h(x) = f(Ax+b)$

1

[1]WITHDRAWING FROM AN EXAMINATION: In case of illness, you must supply proof of your illness by submitting a medical report to the Examinations Office. Please note that the medical examination must be done at the latest on the same day of the missed exam. In case of illness while writing the exam please contact the supervisory staff, inform them about your illness and immediately see your doctor. The medical certificate must be submitted latest 3 days after the medical examination. More information's: http://www.tf.uni-freiburg.de/studies/exams/withdrawing_exam.html

CHEATING/DISTURBING IN EXAMINATIONS: A student who disrupts the orderly proceedings of an examination will be excluded from the remainder of the exam by the respective examiners or invigilators. In such a case, the written exam of the student in question will be graded as 'nicht bestanden' (5.0, fail) on the grounds of cheating. In severe cases, the Board of Examiners will exclude the student from further examinations.

Points on page (max. 3)

4. Why is “globalization” used in the context of optimization?

(a) ☐ to ensure convergence to a stationary point	(b) ☐ to accelerate convergence
(c) ☐ to ensure convergence to the global minimum	(d) ☐ to make the iterations cheaper

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5. What does “strong duality” imply?

(a) ☐ The primal and dual optimal values are identical.	(b) ☐ The primal problem is a quadratic program.
(c) ☐ The dual function is convex.	(d) ☐ The dual problem is infeasible.

1	

6. Consider the following optimization problem, with $Q \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{m\times n}$, $g \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Which is the corresponding KKT matrix?

$$\min_{x \in \mathbb{R}^n} \; x^\top Q x + g^\top x \quad \text{s.t.} \quad Ax + b = 0$$

(a) ☐ $\begin{bmatrix} A^\top A & Q \\ Q & 0 \end{bmatrix}$	(b) ☐ $\begin{bmatrix} Q & -A^\top \\ A & 0 \end{bmatrix}$	(c) ☐ $\begin{bmatrix} Q & g \\ A^\top & b \end{bmatrix}$	(d) ☐ $\begin{bmatrix} xx^\top & A^\top A \\ A^\top A & Q \end{bmatrix}$

1	

7. Which statement about the Gauss-Newton method is true?

(a) ☐ The Hessian approximation is always positive semidefinite.	(b) ☐ It always converges to a stationary point.
(c) ☐ It can only be used for Quadratic Programs.	(d) ☐ The convergence rate is quadratic.

1	

8. Line search: Regard an iterative algorithm for finding a minimizer of the function $f : \mathbb{R}^n \to \mathbb{R}, x \mapsto f(x)$. At the current iterate x_k we have the search direction p_k.

   (a) We consider a trial step $t \cdot p_k$ with step length $t \in \mathbb{R}$. When does it satisfy the Armijo condition?

2	

   (b) Given a search direction p_k, describe the backtracking algorithm to find a point that satisfies the Armijo-condition.

2	

Points on page (max. 8)	

9. Algorithmic Differentiation. Regard the following algorithm to evaluate the function $f : \mathbb{R}^2 \to \mathbb{R}$:

```
function [f] = myfunction(x1,x2)
   v1=sin(x2);
   v2=x1^2;
   v3=x1*v1;
   v4=v2*x2;
   f =v3+v4;
end
```

Write an algorithm that computes the directional derivative $\nabla f(x)^\top \dot{x}$ in the direction $\dot{x} \in \mathbb{R}^2$ using the forward mode of automatic differentiation. You can do this by completing the empty lines the following template. Use the variables `v1dot, v2dot, v3dot, v4dot, fdot` in the intermediate lines.

```
function [f,fdot] = forwardAD(x1,x2,x1dot,x2dot)
   v1=sin(x2);

   ...
   v2=x1^2;

   ...
   v3=x1*v1;

   ...
   v4=v2*x2;

   ...
   f =v3+v4;

   ...
end
```

5

10. Interior Point Methods: What difficulty of the KKT conditions do interior point methods address? Explain how do they do it. Also give the second interpretation of this approach and show its equivalence.

4

Points on page (max. 9)

11. Consider the following optimization problem:

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & x_1 \\ \text{s.t.} \quad & x_1^2 + x_2^2 - 1 = 0, \end{aligned}$$

with $x = [x_1 \quad x_2]^\top$.

(a) Is this problem convex? Justify.

2	

(b) We would like to solve this problem using Newton's method. We introduce $\lambda \in \mathbb{R}$ as Lagrange multiplier, collect all variables in $z = [x_1 \quad x_2 \quad \lambda]^\top$ and iterate by solving linear equation systems of the form

$$F(z_k) + J(z_k)(z_{k+1} - z_k) = 0$$

for z_{k+1}. Note that that $F(z_k) \in \mathbb{R}^3$ and $J(z_k) \in \mathbb{R}^{3\times 3}$. Define $F(z)$ and $J(z)$ to make the equation system concrete, and compute the values of $F(z_k)$ and $J(z_k)$ for $z_k = [0 \quad -\frac{1}{2} \quad 1]$. You do not have to solve the equation system.

4	

12. Algorithmic Differentiation. Give one advantage and one disadvantage of the backward mode of AD compared to the forward mode when evaluating the gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$ with $n \gg 1$.

2	

Points on page (max. 8)	

13. Duality. Consider the following optimization problem, where $Q \in \mathbb{R}^{n \times n}$, $Q \succ 0$ and symmetric, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^\top Q x \quad \text{subject to} \quad Ax - b \geq 0,$$

(a) Derive the dual function for this problem. Make it as explicit as possible.

3	

(b) State the corresponding dual problem. You do not have to solve it.

1	

(c) Does strong duality hold? Justify.

1	

14. Regard the non-smooth optimization problem

$$\min_{x \in \mathbb{R}^2} \quad |x_1 - 1| + |x_2^3 - 1| \quad \text{subject to} \quad x_1^2 + x_2^2 = 1.$$

Formulate an equivalent smooth nonlinear program.

3	

Points on page (max. 8)	

15. A three-page question. Regard the following optimization problem:

$$\min_{x \in \mathbb{R}^2} \; x_1 + x_2^2 \quad \text{subject to} \quad \begin{cases} x_1^2 + x_2^2 & \leq & 1 \\ x_2 & \geq & x_1 \end{cases}$$

(a) Sketch the feasible set Ω of this problem.

2	

(b) Bring this problem into the NLP standard form

$$\min_{x \in \mathbb{R}^n} \; f(x) \quad \text{subject to} \quad \begin{cases} g(x) & = & 0, \\ h(x) & \geq & 0, \end{cases}$$

by defining the dimension n and the functions f, g, h (if applicable).

2	

FROM NOW ON TREAT THE PROBLEM IN THIS STANDARD FORM.

(c) Is this optimization problem convex? Justify.

2	

(d) Write down the Lagrangian function of this optimization problem. Do not forget to define the dimensions of any multipliers you might introduce.

1	

(e) A solution candidate of the problem is $x^* = (-1, 0)$. Identify the active set $\mathcal{A}(x^*)$ at this point.

2	

Points on page (max. 9)	

(f) Does LICQ hold at x^*? Justify.

2	

(g) Describe the tangent cone $T_\Omega(x^*)$ (the set of feasible directions) to the feasible set at this point x^*, by a set definition formula with explicitly computed numbers.

2	

(h) Formulate the necessary optimality conditions of first order (also called Karush-Kuhn-Tucker (KKT) conditions) that a local minimizer $x^* \in \mathbb{R}^2$ of this problem must satisfy. Make it specific to the given problem.

2	

(i) For the given point x^*, find Lagrange multipliers such that the KKT conditions are satisfied.

2	

Points on page (max. 8)	

(j) Describe the critical cone $C(x^*, \mu^*)$ at the point (x^*, μ^*) <u>in a set definition using explicitly computed numbers.</u>

2	

(k) Test whether the second order sufficient conditions of optimality (SOSC) hold at x^*.

2	

(l) Is the point x^* a local minimizer? Justify.

2	

(m) Is the point x^* also a global minimizer? Justify.

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