Summer School on Direct Methods for Optimal Control of Nonsmooth Systems
Albert-Ludwigs-Universität Freiburg – September 12 - September 15, 2023

Exercise 3: The Time-Freezing Reformulation for Systems with State Jumps, Inelastic Impacts

Prof. Dr. Moritz Diehl, Armin Nurkanović, Jonathan Frey, Florian Messerer, Anton Pozharskiy

The goal of this exercise is to get more familiar with the Time-Freezing reformulation which is used to transform systems with state jumps into Filippov systems.

1. **Time-Freezing Reformulation.** Consider a particle in a 2D space whose position is described by $(q_x, q_y) \in \mathbb{R}^2$. There is a ground plane at $q_y = 0$. This particle is under the influence of gravity which acts in the negative q_y direction (gravitational acceleration is $g = 10$). The particle's mass is $m = 1$. We assume that all impacts are inelastic, i.e., the coefficient of restitution is $\epsilon_{\mathrm{r}} = 0$.

   (a) For now assume that there is no friction force. Write down the complemenentarty Lagrangian system describing the particle's dynamics and derive the equivalent time-freezing piecewise smooth system. In particular, define the expression for the switching functions c, define the regions R_1 and R_2, and the dynamics f_1 (free flight dynamics) and f_2 (auxiliary dynamics). Use an auxiliary dynamics constant of $a_n = 20$.
   Hint: Use the formulae from Lecture 7.

   (b) Given the initial conditions for the dynamics $x_0 = (0, 1, 3, 0)$ calculate the point at which the particle will hit the ground and calculate by hand the values of θ which will occur once the particle slides on the ground.

   > Using simple kinematics one can see that the impact will occur at $\sqrt(0.2) \approx 0.44$. Solving the 2 linear equations from the balance of forces and sum of θ_i you should get a steady state $\theta_1 = \frac{2}{3}$ and $\theta_2 = \frac{1}{3}$

   Hint: This corresponds to the sliding mode of the Filippov system.

   (c) Now suppose that couloumb friction acts on the particle with a constant $\mu = 0.2$ at the contact with $q_y = 0$ in the horizontal direction. Redefine the switching functions, the regions of the PSS, the auxiliary y dynamics, and the time-freezing Filippov system via step reformulation for this case.

   (d) Given the same initial conditions calculate at what point the particle will come to rest.
   Hint: do not forget about the frictional losses at the impact.

   > As before using simple kinematics one can see that the impact will occur at $\sqrt{0.2} \approx 0.44$. at this point the velocity of the particle in the $-y$ direction is $10\sqrt{0.2}$ and as such the impact loss in the tangential direction is $\mu v = 2\sqrt{0.2}$. We then can do another simple kinematic calculation seeing that the constant friction force is simply proportional to the gravity constant $g\mu = 2$ and see that the particle will come to a stop at approximately $t \approx 0.44 + \frac{3-2\sqrt{2}}{2} \approx 1.49$ seconds.

2. **Time-Freezing Simulation, NOSNOC's expert mode:** NOSNOC enables automatic reformulation of Complementarity Lagrangian Systems into time-freezing systems. However, some advanced options also enable to manually manipulate the quantities defining the resulting Filippov system. In this task we will make use of this to verify our calculations from the first task. Please consult the additional hints provided in the template in order to complete the following tasks:

   (a) Implement the expressions you derived in the previous section for the particle without friction and use the provided utilities to plot the trajectory. Does the numerical solution match the analytical computation?

   (b) Do the same but use the model that includes friction. Does the numerical solution match the analytical computation?

3. **Inelastic Time-Freezing in Optimal Control Problems** In this task we will solve our first optimal control problem using time-freezing and FESD. Consider two rigid discs with masses m_1, m_2, that translate in 2D space. The discs undergo aerodynamic drag, the equation for which is provided for you in the example. The position of the discs is represented as $q = (q_{x_1}, q_{y_1}, q_{x_2}, q_{y_2}) \in \mathbb{R}^4$, and the full state also includes the disc velocities $v = (v_1, v_2) \in \mathbb{R}^4$. The system is controlled via forces applied to disc 1 in the x and y directions: $u \in \mathbb{R}^2$. There are no other external forces that act on either disc and the inertia matrix of the system is $M = \mathrm{diag}(m_1, m_1, m_2, m_2)$. The impacts between the two discs are inelastic and have a coefficient of restitution $\epsilon_\mathrm{r} = 0$. The free space dynamics of this system are defined by

$$\dot{x} = \begin{pmatrix} v \\ M^{-1} f_v(x, u) \end{pmatrix}$$

   with

$$f_v(x, u) = (u, 0, 0) - c_v \left(\frac{v_1}{\|v_1\| + \epsilon}, \frac{v_2}{\|v_2\| + \epsilon} \right)$$

   (a) As in section 1 of this exercise write down the equivalent Time-Freezing PSS for this system. *Hint:* This should not be much more difficult than section 1.

   (b) Implement the model you derived and solve the OCP for which the target is to have the discs switch places.

   Hint: This problem should converge within a minute or two depending on your hardware. If the solver takes longer than that to converge you may have made a mistake in your implementation.

4. **Elastic Time-Freezing in Optimal Control Problems** As discussed in the lectures Time-Freezing can also be applied to systems with elastic collisions. As such, consider a particle as in section 1 of this exercise except the collision with ground has a constant of restitution $\epsilon_\mathrm{r} > 0$.

   (a) Again write the Complemenentarity Lagrangian System describing the particle's dynamics and derive the equivalent time-freezing piecewise smooth system. In particular, define the expression for the switching functions c, define the regions R_1 and R_2, and the dynamics f_1 (free flight dynamics) and f_2 (auxiliary dynamics).

   (b) Now try running the example OCP of an underactuated ball in a box. Note how each collision is handled separately.

5. **Bonus: Hopping robot OCP** Consider a single legged rigid robot which is controlled via a reaction wheel (for rotation) and a force controlled joint in the leg. The free space of the model is as as follows: The configuration $q = (q_1, q_2, \psi, l)$ consists of the 2D position, orientation, and

leg length, respectively. It is assumed that the inertia matrix is a constant diagonal matrix $M = \mathrm{diag}(m_\mathrm{b} + m_\mathrm{l}, m_\mathrm{b} + m_\mathrm{l}, I_\mathrm{b} + I_\mathrm{l}, m_\mathrm{l})$. The free space dynamics of the robot are

$$\dot{x} = f(x, u) = (q, M^{-1} f_v(q, u), 1)$$

with

$$f_v(q, u) = (-\sin(\psi)u_2, (m_\mathrm{b}+m_\mathrm{l})g + \cos(\psi)u_2, u_1, u_2)$$

and for simplicity we assume that only the bottom of the foot makes contact with the ground at $q_2 - l\cos\psi = 0$.

(a) Implement the equivalent Time-Freezing PSS for the hopper. You can optionally also implement the lifting algorithm to improve the nonlinearity of the equality constraints. Some hints for this exercise:

   i) If the solver begins to take longer than several minutes or so to converge to a solution you may want to check your implementation for correctness.

   ii) This example can show you the difference between different linear solvers and how influential they can be. If you have HSL solvers installed you can use the MA27 routine in IPOPT to improve performance significantly.