

## Exercise 2: Smoothing, Nonsmooth Systems and Finite Elements with Switch Detection

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The goal of this exercise is to get familiar with the limitations of time-stepping and smoothing used to solve optimal control problems subject to nonsmooth dynamical systems. Furthermore, the exercise provide a practical introduction to Filippov systems. Finally, we will get to use the Finite Elements with Switch Detection (FESD) for solving simulation and optimal control problems.

### 1. Smoothing nonsmooth systems. Continuation of Exercise 1.2

- (a) Solve the cart pole OCP from the previous exercise with a small smoothing parameter  $\sigma$ . How does the trajectory look like? Can you explain this behavior?
- (b) Solve the OCP with different  $\sigma$  values in the range  $[10^{-8}, 10]$ . Visualize how the obtained objective function value relates to  $\sigma$
- (c) Run the same experiment, "warm starting" the solver, i.e. providing an initial guess from a solution with more smoothing. What do you observe?

### 2. Simulation of a scalar nonsmooth systems – Crossing

Consider the initial value problem of a dynamic system with state  $x(t) \in \mathbb{R}$  which evolves over time as follows:

$$\dot{x}(t) = \begin{cases} 1, & \text{if } x < 0 \\ 3, & \text{if } x > 0 \end{cases} \quad (1)$$

for the initial state  $x(0) = -\sqrt{2}$  and a simulation time of  $T = 2$ .

- (a) qualitatively visualize the exact solution.

- (b) derive exact switch time  $t_s$ .

- (c) derive the solution  $x(2)$

- (d) Recall the definition of a Filippov system:

$$\dot{x} \in F_F(x) = \{ F(x)\theta \mid \sum_i^{n_f} \theta_i = 1 \text{ with } \theta_j \geq 0, j = 1, \dots, n_f, \text{ and } \theta_i = 0 \text{ if } x \notin \bar{R}_i \}$$

where  $R_i := \{ x \in \mathbb{R}^{n_x} \mid \text{diag}(S_{i,\bullet} c(x)) > 0 \}$

Derive the functions  $F : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x \times n_f}$  and  $c : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_c}$ , and the matrix  $S \in \mathbb{R}^{n_f \times n_c}$  for the example in Eq. (1).

## About NOSNOC and nosnoc\_py.

For the following exercises you will need to install NOSNOC, respectively `nosnoc_py`. These packages implement the Finite Elements with Switch Detection discretization (FESD) for optimal control and simulation problems with nonsmooth systems.

- **Matlab:** clone <https://github.com/nurkanovic/nosnoc> and follow the instructions
- **Python:** clone [https://github.com/FreyJo/nosnoc\\_py/](https://github.com/FreyJo/nosnoc_py/) and follow the instructions

### 3. Simulation of scalar nonsmooth systems – with nosnoc

- Formulate the problem in `nosnoc` by using your definitions of  $F, c, S$
- Solve the problem numerically without FESD, then with FESD. How do the numerical solutions compare to the exact one?

### 4. Sliding mode variation

- Modify the Filippov system corresponding to equation (1), such that  $\dot{x} = -1$  if  $x > 0$ .
- Simulate it as in the exercise before.
- Bonus: create a similar example with a spontaneous exit of a sliding mode. Initialize the `nosnoc` solver to get a result exiting on either side.

### 5. Optimal control with nonsmooth systems

- Solve the cart pole OCP without smoothing with FESD in `nosnoc`.  
You will need to provide the cart pole model in the Fillipov form.  
Use the following options to discretize the OCP with `nosnoc`: 20 control intervals, with 2 finite elements and Radau-IIA Butcher tableau of size  $n_s = 2$ .
- What do you observe compared to the solution in Task 1? Can you explain why a different behavior can be observed?