Exercises for Lecture Course on Numerical Optimal Control (NOC)
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Exercise 8: Continuous-Time Optimal Control

Prof. Dr. Moritz Diehl, Andrea Zanelli, Dimitris Kouzoupis, Florian Messerer, Yizhen Wang

Consider the following continuous-time optimal control problem:

$$\begin{aligned} \min_{x(t),\, u(t)} \quad & \int_{t=0}^{T} L(x(t), u(t))\mathrm{d}t + E(x(T)) \\ \text{s.t.} \quad & x(0) = \bar{x}_0 \\ & \dot{x}(t) = f(x(t), u(t)), \quad t \in [0, T]. \end{aligned} \tag{1}$$

1. (a) Discretize problem (1) using the explicit Euler integrator with step-size h over N intervals. Write on paper the obtained discrete-time optimal control problem.

   (b) Write the first-order optimality conditions for the discretized problem obtained in (a). Use the Hamiltonian function defined as

   $$H(x, u, \lambda) := L(x, u) + \lambda^T f(x, u) \tag{2}$$

   to simplify these conditions.

   (c) Now let $N \to \infty$ and $h \to 0$. What type of problem do the conditions derived in (b) converge to?

   (d) Fix $N = 2$ and apply the Newton method to the first-order optimality conditions for the discretized optimal control obtained in (b). Derive the form of the linear systems associated with the Newton steps. Order the variables as $z = (\lambda_0,\, x_0,\, u_0, \lambda_1, x_1, u_1,\, \lambda_2,\, x_2)$ and the KKT conditions accordingly as $F(z) = 0$ with $F(z) := \nabla_z \mathcal{L}(z)$, where $\mathcal{L}(z)$ is the Lagrangian of the NLP.

   For notational simplicity we suggest you use the abbreviations $Q_k := h\nabla_x^2 H(x_k, u_k, \lambda_k)$, $R_k := h\nabla_u^2 H(x_k, u_k, \lambda_k)$, $S_k := h\nabla_{ux}^2 H(x_k, u_k, \lambda_k)$, $A_k := I + h\nabla_x f(x_k, u_k)^\top$, $B_k := h\nabla_u f(x_k, u_k)^\top$ for $k \in \{0, \dots, N-1\}$ and $Q_N := \nabla_x^2 E(x_N)$

   (e) **[Bonus]** The linear systems associated with the Newton steps in (d) can be solved exploiting the Riccati Difference Equation (equation 8.5 in the course's script). Derive this equation.

   (f) **[Bonus]** What kind of matrix ODE does the difference equation derived in (e) converge to for $N \to \infty$ and $h \to 0$?

   Hint: if you have not solved the bonus point (e) you can refer to equation 8.5 from the course's script.