Exercises for Lecture Course on Numerical Optimal Control (NOC)
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Exercise 8: Continuous-Time Optimal Control

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Consider the following continuous-time optimal control problem:

$$\begin{aligned} \min_{x(t),\, u(t)} \quad & \int_{t=0}^{T} L(x(t), u(t)) \mathrm{d}t + E(x(T)) \\ \text{s.t.} \quad & x(0) = \bar{x}_0 \\ & \dot{x}(t) = f(x(t), u(t)), \quad t \in [0, T]. \end{aligned} \tag{1}$$

1. (a) Discretize problem (1) using the explicit Euler integrator with step-size h over N intervals. Write on paper the obtained discrete-time optimal control problem.

$$\begin{aligned} \min_{x,\, u} \quad & h \sum_{i=0}^{N-1} L(x_i, u_i) + E(x_N) \\ \text{s.t.} \quad & x_0 = \bar{x}_0 \\ & x_{i+1} = x_i + hf(x_i, u_i),\ i = 0, ..., N-1 \end{aligned}$$

   (b) Write the first-order optimality conditions for the discretized problem obtained in (a). Use the Hamiltonian function defined as

$$H(x, u, \lambda) := L(x, u) + \lambda^T f(x, u) \tag{2}$$

   to simplify these conditions.

$$\begin{aligned} r_{E_0} &:= \bar{x}_0 - x_0 &&= 0 \\ r_{Sx_0} &:= h\nabla_{x_0} H(x_0, u_0, \lambda_1) - \lambda_0 + \lambda_1 &&= 0 \\ r_{Su_0} &:= h\nabla_{u_0} H(x_0, u_0, \lambda_1) &&= 0 \\ r_{E_1} &:= x_0 + hf(x_0, u_0) - x_1 &&= 0 \\ r_{Sx_1} &:= h\nabla_{x_1} H(x_1, u_1, \lambda_2) - \lambda_1 + \lambda_2 &&= 0 \\ r_{Su_1} &:= h\nabla_{u_1} H(x_1, u_1, \lambda_2) &&= 0 \\ \vdots &:= \vdots &&= \vdots \\ r_{E_N} &:= x_{N-1} + hf(x_{N-1}, u_{N-1}) - x_N &&= 0 \\ r_{Sx_N} &:= \nabla_{x_N} E(x_N) - \lambda_N &&= 0 \end{aligned}$$

   (c) Now let $N \to \infty$ and $h \to 0$. What type of problem do the conditions derived in (b) converge to?

$$\begin{aligned} x(0) &= \bar{x}_0 \\ \dot{\lambda} &= -\nabla_x H(x, u, \lambda) \\ \dot{x} &= f(x, u) \\ 0 &= \nabla_u H(x, u, \lambda) \\ \lambda(T) &= \nabla_x E(x(T)) \end{aligned}$$

(d) Fix $N = 2$ and apply the Newton method to the first-order optimality conditions for the discretized optimal control obtained in (b). Derive the form of the linear systems associated with the Newton steps. Order the variables as $z = (\lambda_0,\, x_0,\, u_0, \lambda_1, x_1, u_1,\, \lambda_2,\, x_2)$ and the KKT conditions accordingly as $F(z) = 0$ with $F(z) := \nabla_z \mathcal{L}(z)$, where $\mathcal{L}(z)$ is the Lagrangian of the NLP.

For notational simplicity we suggest you use the abbreviations $Q_k := h\nabla_x^2 H(x_k, u_k, \lambda_k)$, $R_k := h\nabla_u^2 H(x_k, u_k, \lambda_k)$, $S_k := h\nabla_{ux}^2 H(x_k, u_k, \lambda_k)$, $A_k := I + h\nabla_x f(x_k, u_k)^\top$, $B_k := h\nabla_u f(x_k, u_k)^\top$ for $k \in \{0, \ldots, N-1\}$ and $Q_N := \nabla_x^2 E(x_N)$

$$\frac{\partial F(z)}{\partial z}\Delta z = -F(z)$$

$$\begin{bmatrix} & -I & & & & & & \\ -I & Q_0 & S_0^T & A_0^T & & & & \\ & S_0 & R_0 & B_0^T & & & & \\ & A_0 & B_0 & & -I & & & \\ & & & -I & Q_1 & S_1^T & A_1^T & \\ & & & & S_1 & R_1 & B_1^T & \\ & & & & A_1 & B_1 & & -I \\ & & & & & & -I & Q_2 \end{bmatrix} \begin{bmatrix} \Delta\lambda_0 \\ \Delta x_0 \\ \Delta u_0 \\ \Delta\lambda_1 \\ \Delta x_1 \\ \Delta u_1 \\ \Delta\lambda_2 \\ \Delta x_2 \end{bmatrix} = - \begin{bmatrix} r_{E_0} \\ r_{Sx_0} \\ r_{Su_0} \\ r_{E_1} \\ r_{Sx_1} \\ r_{Su_1} \\ r_{E_2} \\ r_{Sx_2} \end{bmatrix}$$

(e) **[Bonus]** The linear systems associated with the Newton steps in (d) can be solved exploiting the Riccati Difference Equation (equation 8.5 in the course's script). Derive this equation.

Consider the last block coupling stage 1 and 2:

$$\begin{bmatrix} Q_1 & S_1^T & A_1^T & \\ S_1 & R_1 & B_1^T & \\ A_1 & B_1 & & -I \\ & & -I & Q_2 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta u_1 \\ \Delta\lambda_2 \\ \Delta x_2 \end{bmatrix} = - \begin{bmatrix} r_{Sx_1} \\ r_{Su_1} \\ r_{E_2} \\ r_{Sx_2} \end{bmatrix}.$$

Assuming that Q_2 is invertible, we can eliminate Δx_2, in order to obtain the following reduced system:

$$\begin{bmatrix} Q_1 & S_1^T & A_1^T \\ S_1 & R_1 & B_1^T \\ A_1 & B_1 & -Q_2^{-1} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta u_1 \\ \Delta\lambda_2 \end{bmatrix} = - \begin{bmatrix} r_{Sx_1} \\ r_{Su_1} \\ r_{E_2} + Q_2^{-1} r_{S_{x_2}} \end{bmatrix}.,$$

where the fact that $\Delta x_2 = Q_2^{-1}(\Delta\lambda_2 - r_{S_{x_2}})$ has been used. We can further reduce the system by eliminating $\Delta\lambda_2$:

$$\Delta\lambda_2 = Q_2 \left([A_1\; B_1] \begin{bmatrix} \Delta x_1 \\ \Delta u_1 \end{bmatrix} + \tilde{r}_{E_2} \right),$$

where $\tilde{r}_{E_2} := r_{E_2} + Q_2^{-1} r_{S_{x_2}}$, obtaining the system

$$\begin{bmatrix} Q_1 + A_1^T Q_2 A_1 & S_1^T + A_1^T Q_2 B_1 \\ S_1 + B_1^T Q_2 A_1 & R_1 + B_1^T Q_2 B_1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta u_1 \end{bmatrix} = - \begin{bmatrix} r_{Sx_1} + A_1^T Q_2 \tilde{r}_{E_2} \\ r_{Su_1} + B_1^T Q_2 \tilde{r}_{E_2} \end{bmatrix}.$$

Finally, eliminating u_1 using a Schur complement, the block associated with stages 0 and 1 takes the form

$$\begin{bmatrix} Q_0 & S_0^T & A_0^T & \\ S_0 & R_0 & B_0^T & \\ A_0 & B_0 & & -I \\ & & -I & P_1 \end{bmatrix} \begin{bmatrix} \Delta x_0 \\ \Delta u_0 \\ \Delta\lambda_1 \\ \Delta x_1 \end{bmatrix} = - \begin{bmatrix} r_{Sx_0} \\ r_{Su_0} \\ r_{E_1} \\ \tilde{r}_{Sx_1} \end{bmatrix},$$

where

$$P_1 := Q_1 + A_1^T Q_2 A_1 - (S_1^T + A_1^T Q_2 B_1)(R_1 + B_1^T Q_2 B_1)^{-1}(S_1 + B_1^T Q_2 A_1).$$

Noting that the structure of (1e) is the same in (1e), a recursion can be defined that can be used to progressively reduce the system for an arbitrary number of stages N:

$$P_k := Q_k + A_k^T P_{k+1} A_k - (S_k^T + A_k^T P_{k+1} B_k)(R_k + B_k^T P_{k+1} B_k)^{-1}(S_k + B_k^T P_{k+1} A_k).$$

(f) **[Bonus]** What kind of matrix ODE does the difference equation derived in (e) converge to for $N \to \infty$ and $h \to 0$?

Hint: if you have not solved the bonus point (e) you can refer to equation 8.5 from the course's script.

The difference equation has the form

$$\begin{aligned} P_k := \;& hQ_c + (I + hA_c)^T P_{k+1}(I + hA_c) - \\ & (hS_c^T + (I + hA_c)^T P_{k+1} hB_c)(hR_c + hB_c^T P_{k+1} hB_c)^{-1}(hS_c + hB_c^T P_{k+1}(I + hA_c)). \end{aligned}$$

Expanding and eliminating the terms of order 2 or higher, we obtain

$$\begin{aligned} P_k := \;& hQ_c + P_{k+1} + hA_c^T P_{k+1} + hP_{k+1} A_c - \\ & (hS_c^T + P_{k+1} hB_c)\frac{1}{h}(R_c)^{-1}(hS_c + hB_c^T P_{k+1}). \end{aligned}$$

dividing by h and for $h \to 0$ we obtain:

$$-\dot{P} := \; Q_c + A_c^T P + P A_c - (S_c^T + P B_c) R_c^{-1} (S_c + B_c^T P).$$