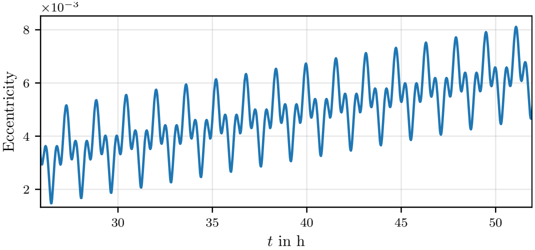
Efficient Numerical Optimal Control for Highly Oscillatory Systems

Jakob Harzer, Jochem De Schutter, Moritz Diehl

CDC 2022 Presentation
December 6th, 2022

Introduction



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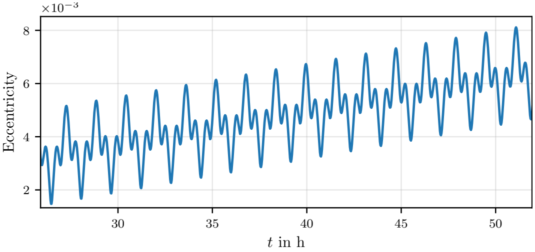

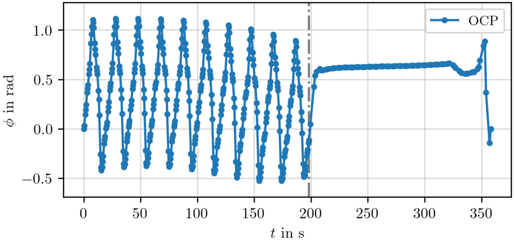

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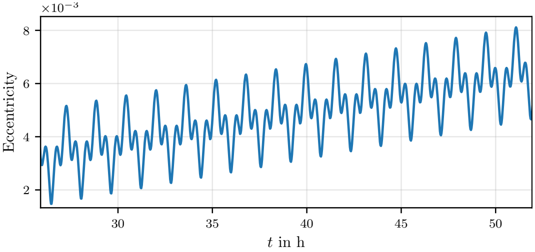

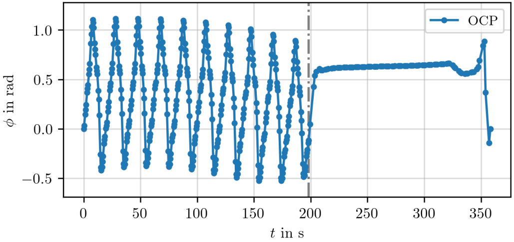

Can we make use of the fact that the expected trajectory of an OCP is highly oscillatory to speed up the optimization process?

The Envelope

- Highly oscillatory initial value problem

$$\dot{x}(t) = f(x(t))$$
$$x(0) = x_0$$

with solution $x(t) \in \mathbb{R}^{n_x}, t \in [0, t_\mathrm{f}]$ and known period duration T.

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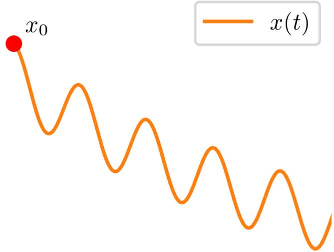

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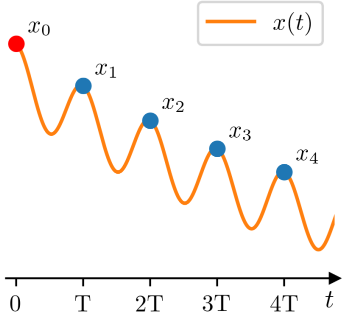

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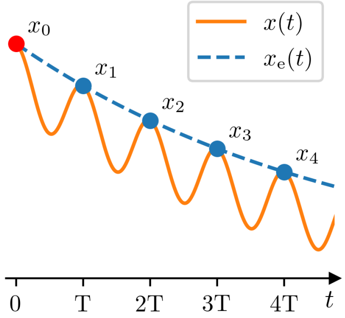
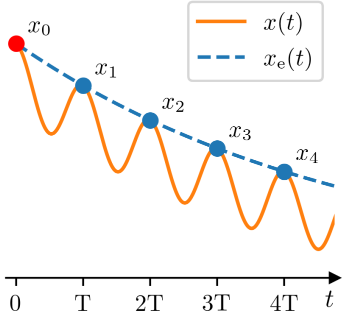
- Construct envelope $x_\mathrm{e}(t) \in \mathbb{R}^{n_x}$ that smoothly interpolates a series of 'periodic' points $x_1, x_2, \ldots$.



Envelope Dynamics

- Assumption: Envelope x_e stems from envelope dynamics

$$\frac{\mathrm{d}x_\mathrm{e}}{\mathrm{d}t}(t) = f_\mathrm{e}(x_\mathrm{e}(t))$$



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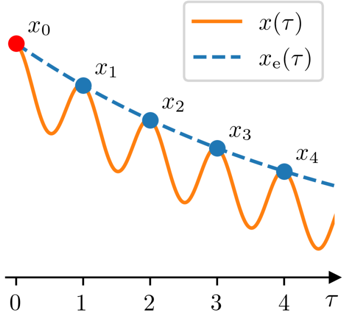

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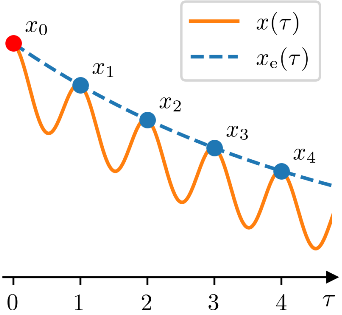

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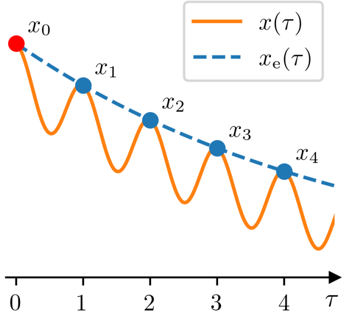
  1. Micro-integrate one period of the oscillations



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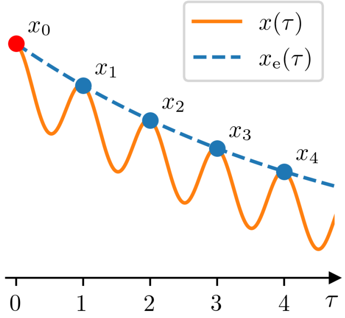
  1. Micro-integrate one period of the oscillations
  2. Approximate the envelope dynamics using a DAE

$$\begin{aligned}\frac{\mathrm{d}x_\mathrm{e}}{\mathrm{d}\tau}(\tau) &= f_\mathrm{e}(x_\mathrm{e}(\tau), z_\mathrm{e}(\tau))\\ 0 &= g_\mathrm{e}(x_\mathrm{e}(\tau), z_\mathrm{e}(\tau))\end{aligned}$$



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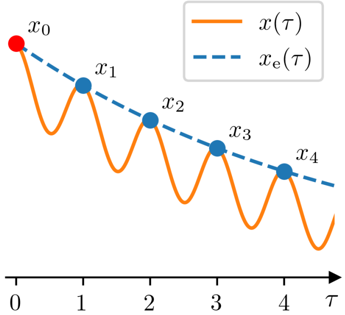
3. Macro-integrate the approximated slow envelope dynamics instead of the fast oscillating dynamics!



Related Work

"Basic underlying idea has appeared several times in the literature over the last fifty years"[3]

- Multirevolution Methods by Mace and Thomas[5], Graf[4]
- Envelope Following Methods by Petzold[6]
- Stroboscopic Averaging Methods by Calvo[2]
- Heterogeneous Multiscale Methods, Averaging Methods etc.

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New: Fully implicit DAE formulation, Use for Optimal Control Problems

Toy Example

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- Perturbed Predator-Prey Model with rabbits r and snow leopards s:

$$x = \begin{bmatrix} r \\ s \end{bmatrix} \in \mathbb{R}^2$$

$$\dot{x} = \begin{bmatrix} \alpha r - \beta r s \\ \delta r s - \gamma s \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}$$

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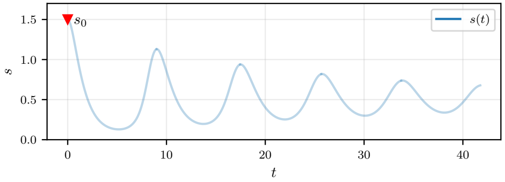
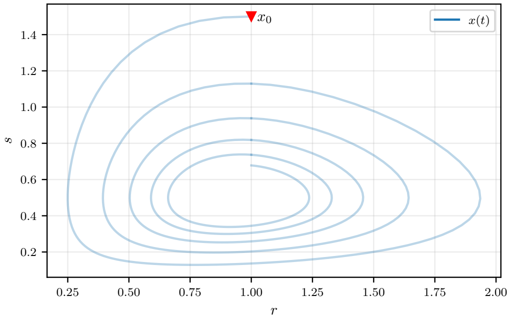
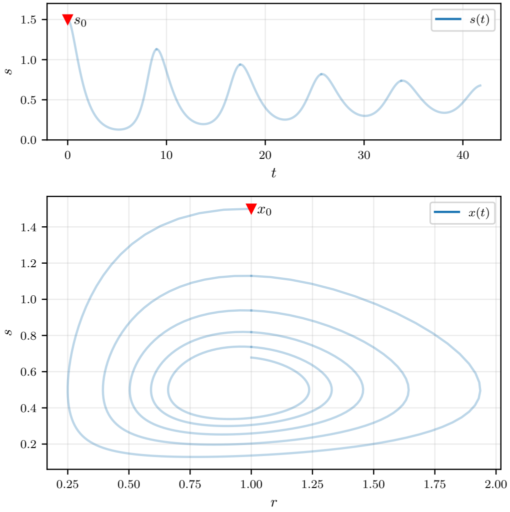




Cycle Discretization



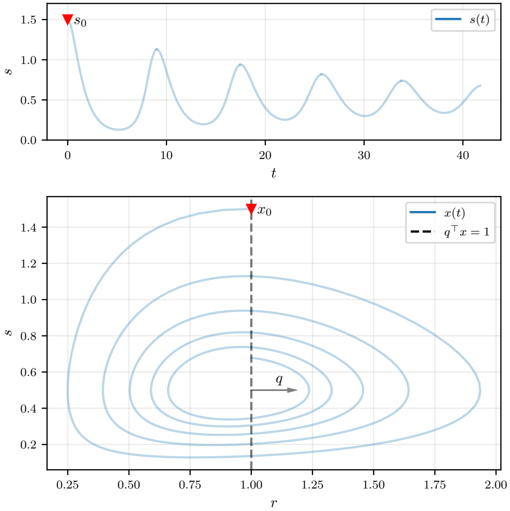
Cycle Discretization

- Affine phase conditions

$$q^\top x^- = b^- \quad (1a)$$

$$q^\top x^+ = b^+ \quad (1b)$$

for the start point $x^- \in \mathbb{R}^{n_x}$ and end point $x^+ \in \mathbb{R}^{n_x}$ of a cycle, respectively.



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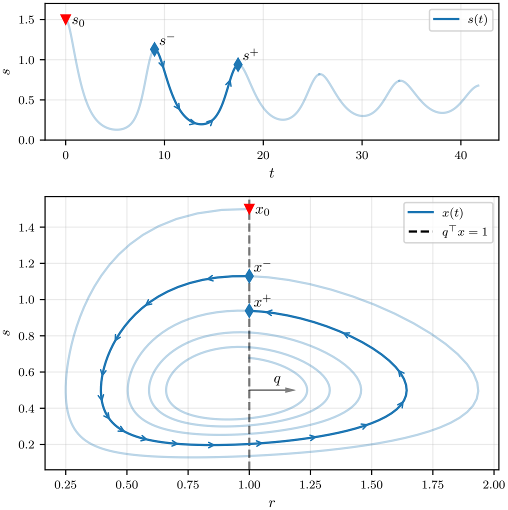

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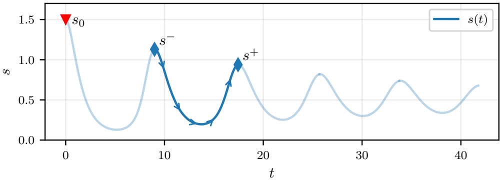
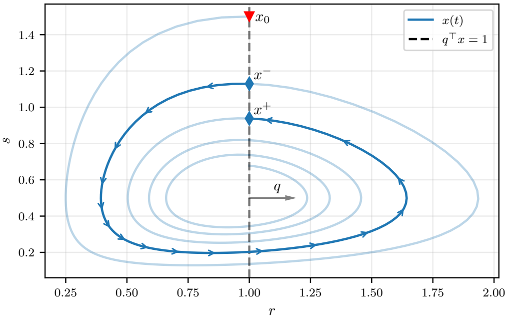
  - $b^- = b^+$
  - Similar to a Poincaré section





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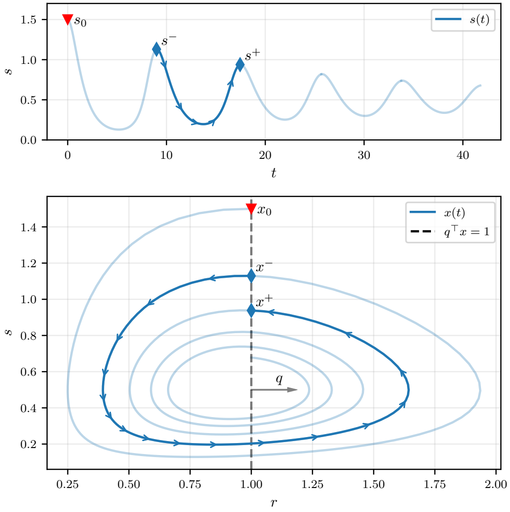
- $[Q|q]$ is an orthonormal basis of the statespace



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- Cycle Conditions

$$C(z_k, U_k) := \begin{cases} 0 & = q^\top x_k^- - b^- \\ 0 & = x_k^+ - F(x_k^-, U_k, T_k) \\ 0 & = q^\top x_k^+ - b^+ \end{cases} \in \mathbb{R}^{n_x+2} \tag{2}$$

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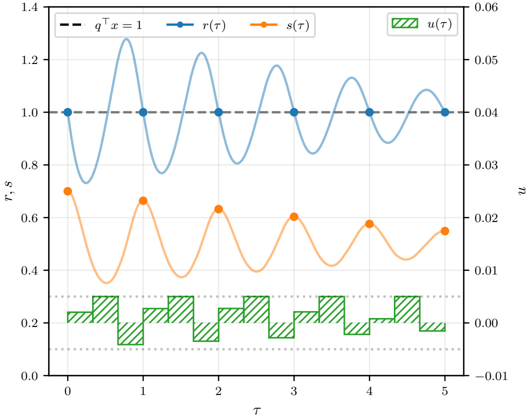
  that define one cycle.
- Algebraic cycle variables

$$z_k = (x_k^-, x_k^+, T_k)$$

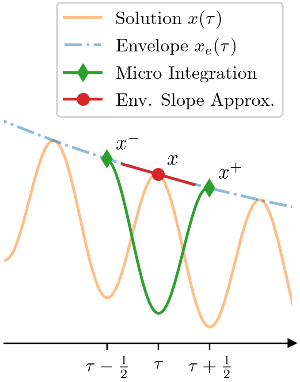
N-Cycle OCP

$$\min_{w} \quad \sum_{k=0}^{N-1} L_{\mathrm{c}}(z_k, U_k) + E(x_{N-1}^{+})$$

$$\begin{aligned} \text{s.t.} \quad & 0 = Q^\top (x_0^- - x_0), \\ & 0 = C(z_k, U_k) \qquad k = 0, .., N-1, \\ & 0 = Q^\top (x_k^- - x_{k-1}^+) \quad k = 1, .., N-1 \end{aligned}$$



Central Difference Envelope-DAE



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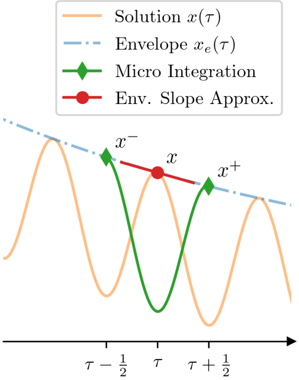
$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = x^+ - x^- \qquad \text{(slope approx.)}$$

$$0 = Q^\top \left(\frac{x^+ + x^-}{2} - x \right) \qquad \text{(connecting cond.)}$$

$$0 = C(z, U) \qquad \text{(cycle cond.).}$$



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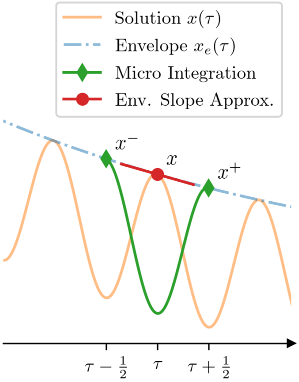
- summarize by $f_\mathrm{e}(x, z) := x^+ - x^-$ and $0 = g_\mathrm{e}(x, z)$ with $x = x$ and $z = (x^+, x^-, T)$



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- Macro-Integration of the Envelope-DAE over $\tau \in [0, N]$

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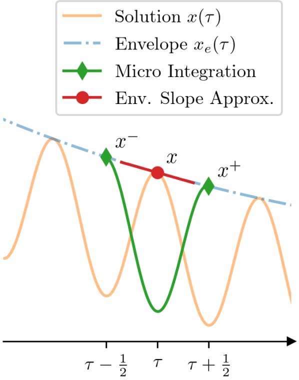

Example Integration: Predator-Prey IVP

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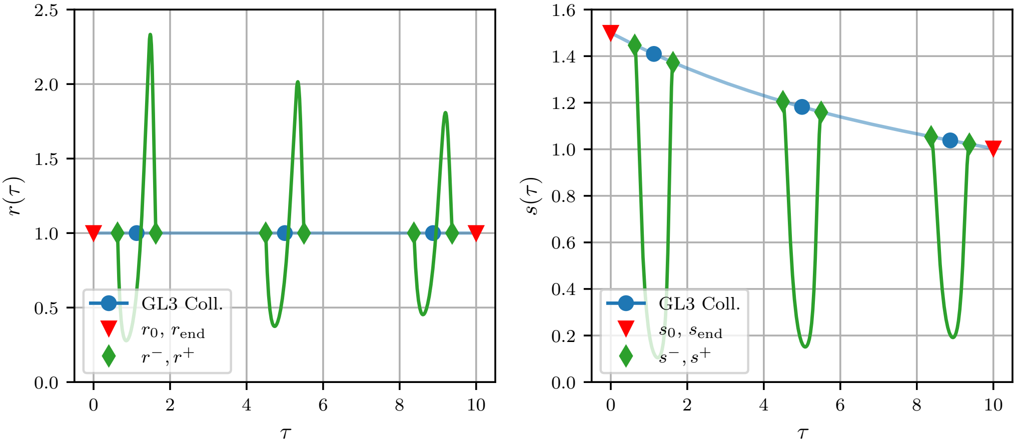

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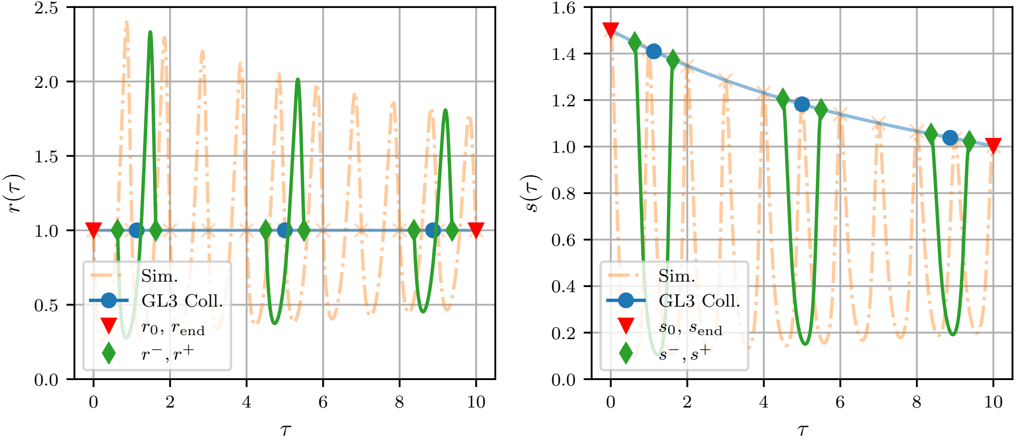

Control Parametrization

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- Control matrix

$$U(\tau) = [u_0(\tau), u_1(\tau), \ldots, u_{N_{\text{ctr}}-1}(\tau)] \in \mathbb{R}^{n_u \times N_{\text{ctr}}}$$

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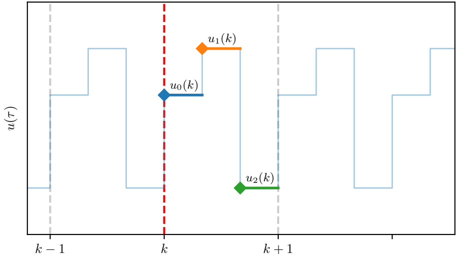

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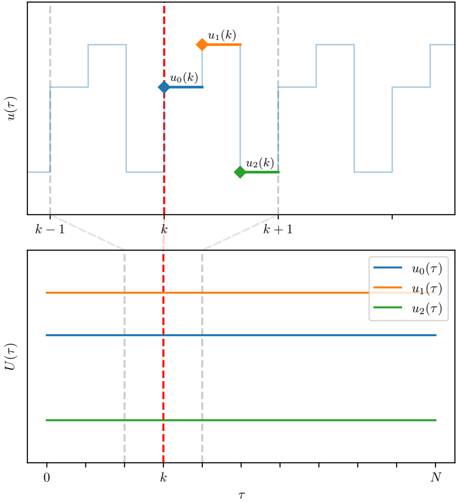
- Simplest case: constant periodic control

$$U(\tau) = U_{\mathrm{const}}$$



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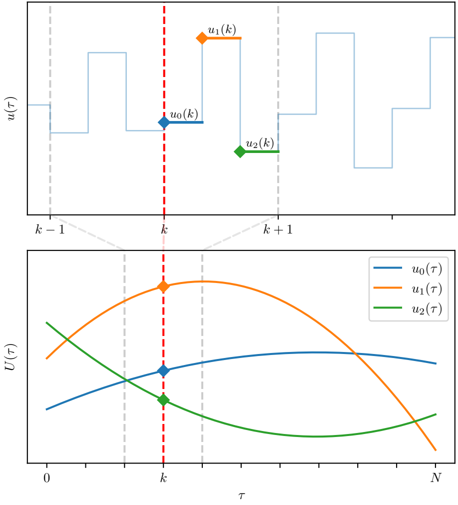

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- DAE:

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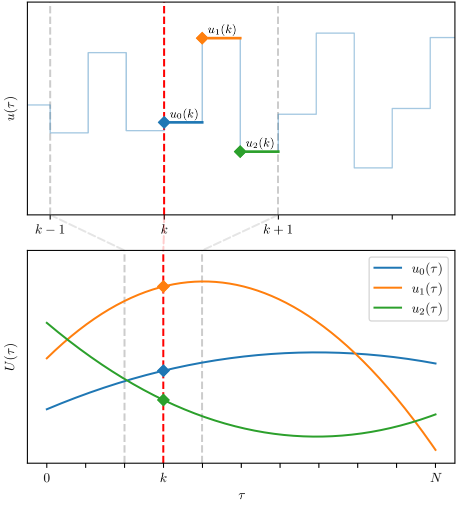

Example: Predator Prey OCP

$$
\begin{aligned}
\min_{x_\mathrm{e}(\cdot), z_\mathrm{e}(\cdot), U(\cdot)} \quad & \|x_\mathrm{e}(N) - \bar{x}_N\|_Q^2 \\
\text{s.t.} \quad & 0 = x_\mathrm{e}(0) - \bar{x}_0, \\
& \dot{x}_\mathrm{e} = f_\mathrm{e}(x_\mathrm{e}(\tau), z_\mathrm{e}(\tau), U(\tau)), \qquad \tau \in [0, N], \\
& 0 = g_\mathrm{e}(x_\mathrm{e}(\tau), z_\mathrm{e}(\tau), U(\tau)), \qquad \tau \in [0, N], \\
& 0 \leq h(x_\mathrm{e}(\tau), z_\mathrm{e}(\tau), U(\tau)), \qquad \tau \in [0, N]
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- Drive populations to target state $\bar{x}_N = [1, 0.7]^\top$.

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\end{aligned}
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- Drive populations to target state $\bar{x}_N = [1, 0.7]^\top$.
- Discretize with 3 stage Gauss-Legendre collocation scheme over $N = 20$ periods.

Example: Predator Prey OCP

$$\min_{x_\mathrm{e}(\cdot), z_\mathrm{e}(\cdot), U(\cdot)} \quad \|x_\mathrm{e}(N) - \bar{x}_N\|_Q^2$$

$$\begin{aligned} \text{s.t.} \quad 0 &= x_\mathrm{e}(0) - \bar{x}_0, \\ \dot{x}_\mathrm{e} &= f_\mathrm{e}(x_\mathrm{e}(\tau), z_\mathrm{e}(\tau), U(\tau)), && \tau \in [0, N], \\ 0 &= g_\mathrm{e}(x_\mathrm{e}(\tau), z_\mathrm{e}(\tau), U(\tau)), && \tau \in [0, N], \\ 0 &\leq h(x_\mathrm{e}(\tau), z_\mathrm{e}(\tau), U(\tau)), && \tau \in [0, N] \end{aligned}$$

- Drive populations to target state $\bar{x}_N = [1, 0.7]^\top$.
- Discretize with 3 stage Gauss-Legendre collocation scheme over $N = 20$ periods.
- The target state cannot be reached!

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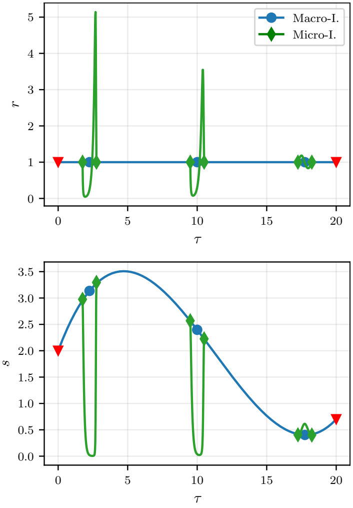

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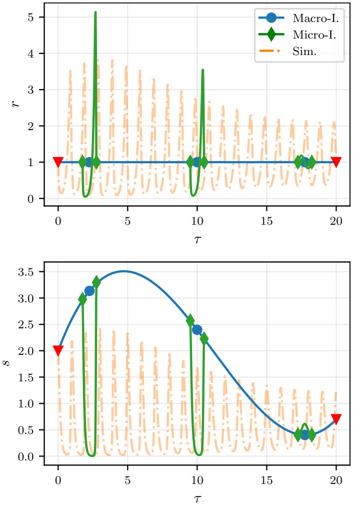

Highly Oscillatory Trajectories

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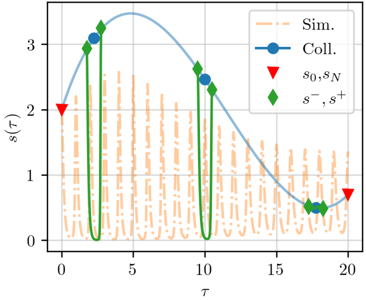

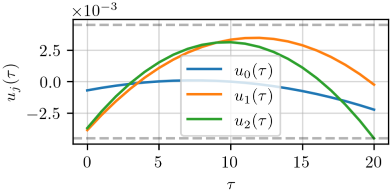

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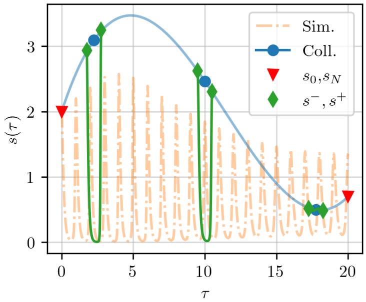
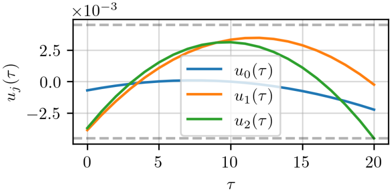
$$J = \|x_{\text{end}} - \bar{x}_{\text{end}}\|_Q^2 + \alpha \sum_{i=1}^{d} b_i w_i^2$$

where $w_i = \ddot{x}_{\text{e}}(\tau_i)$





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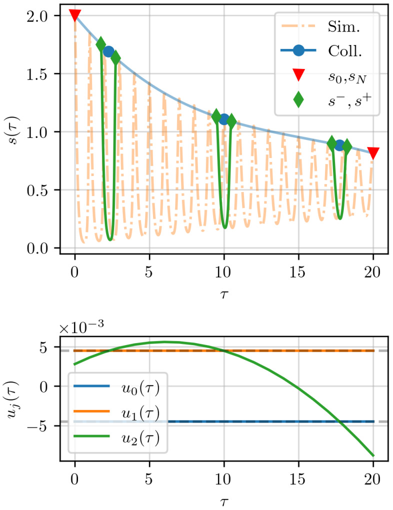
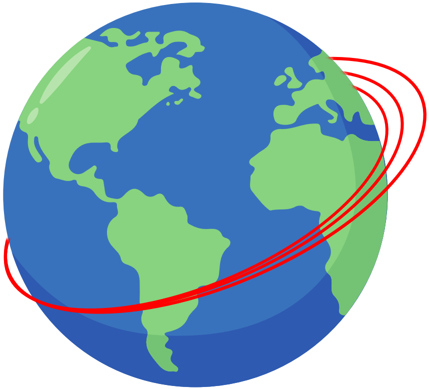


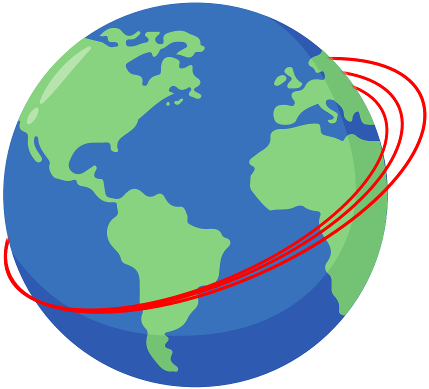
Example: Low Thrust Satellite



Graphics from Vecteezy.com

Example: Low Thrust Satellite

- Move satellite from circular park orbit at $500\,$km to mission orbit at $1400\,$km, problem of [1].



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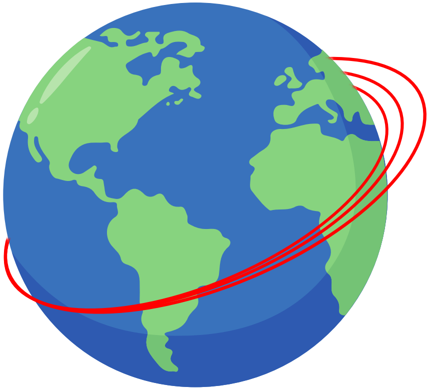

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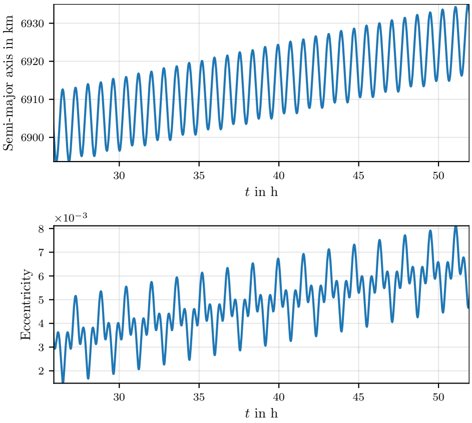

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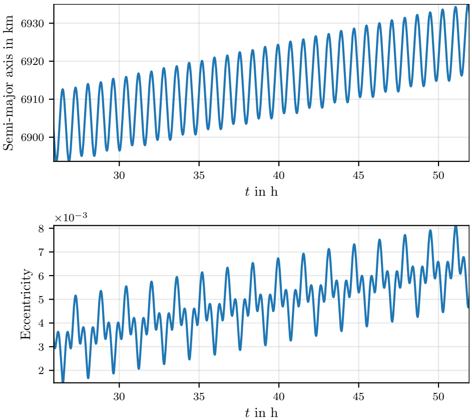

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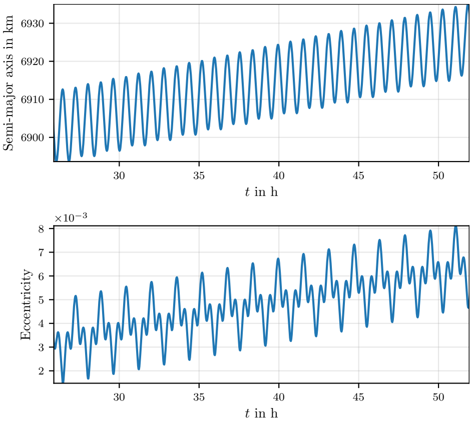
- Here: Two stages with $N = 289$ orbits, we search for an optimal periodic control scheme with $N_{\text{ctrl}} = 30$ controls for each orbit in both stages



Example: Low Thrust Satellite

- State: Modified Equinoctial Elements

$$x = [p, f, g, h, k, L, m, t]^\top \in \mathbb{R}^8$$

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\min_{\substack{x_\mathrm{e}(\cdot), z_\mathrm{e}(\cdot),\\ U_1, U_2}} \quad & m(2N) \\
\text{s.t.} \quad & 0 = x(0) - \bar{x}_0, \\
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& 0 \le h(x, z, U_1), && \tau \in [0, N], \\
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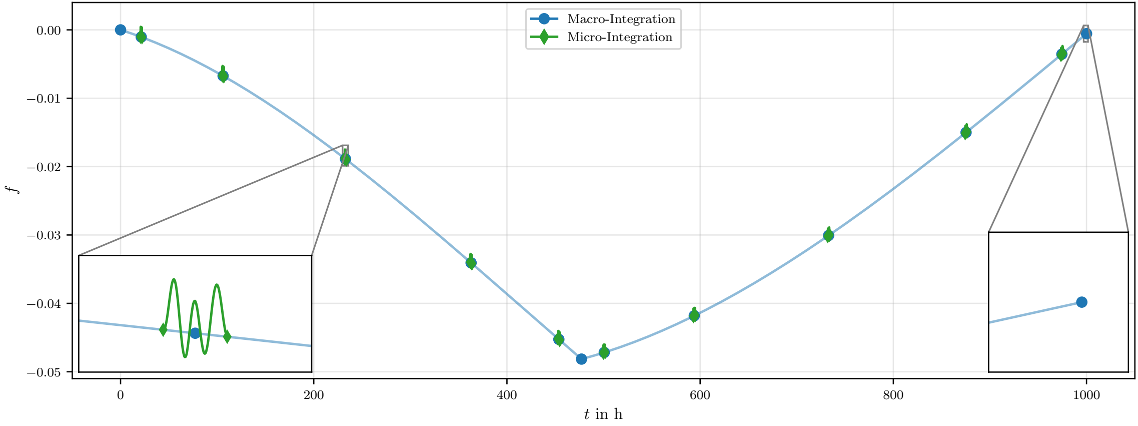
& \underline{x_\mathrm{end}} \le x_\mathrm{e}(2N) \le \overline{x_\mathrm{end}}
\end{aligned}$$

Example: Low Thrust Satellite - Solution



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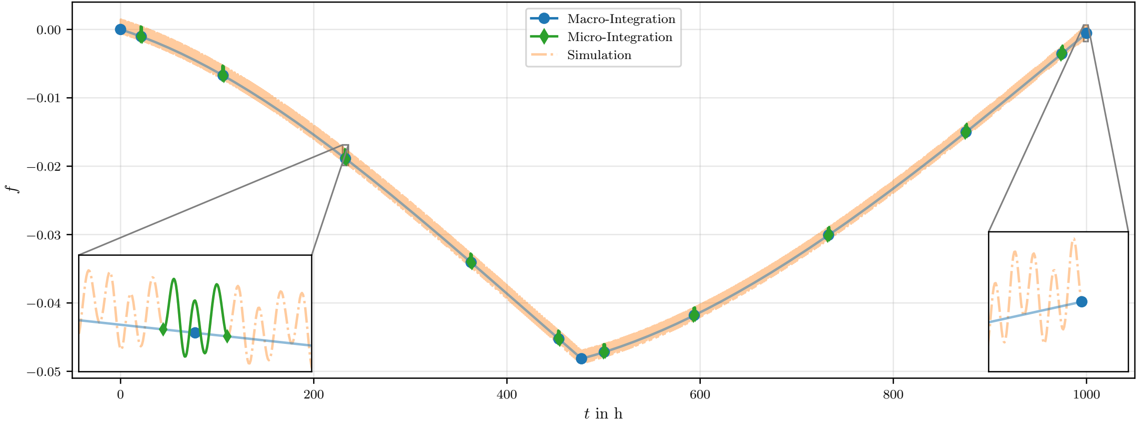
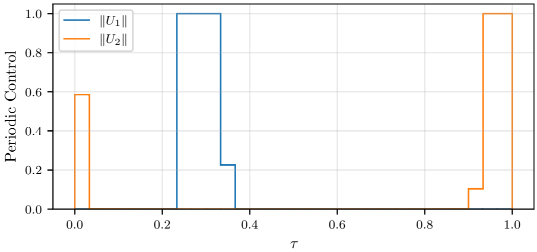


Example: Low Thrust Satellite - Results

- Effectively simulate only 10 out of 578 orbits



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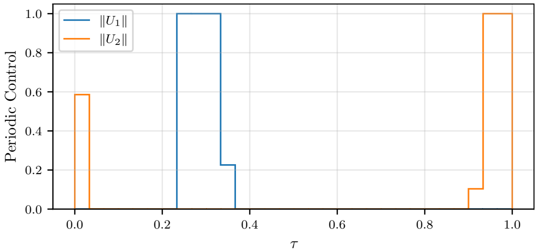

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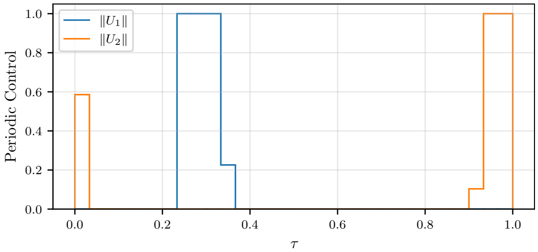

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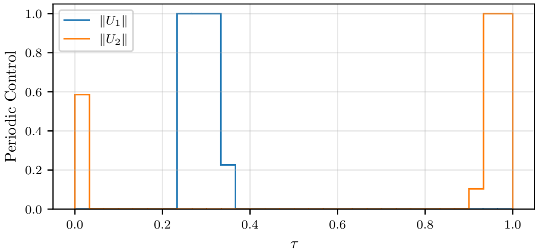
- Optimal final mass $m = 155.48\,$kg almost coincides with result of [1]



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Thank you for your attention!

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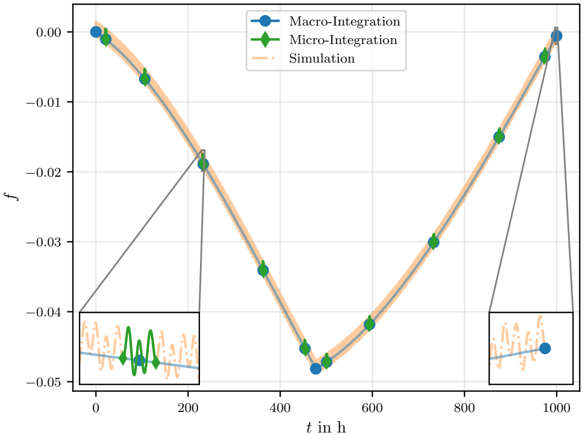
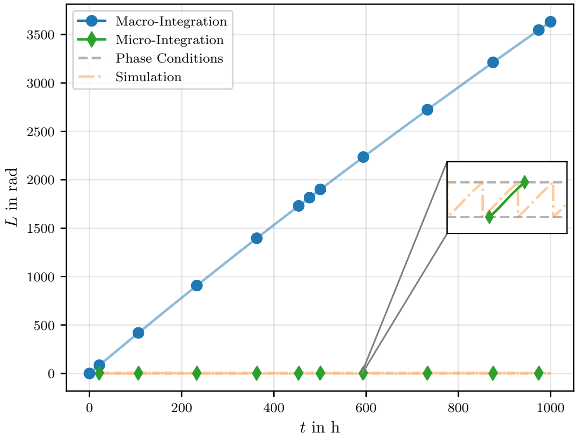
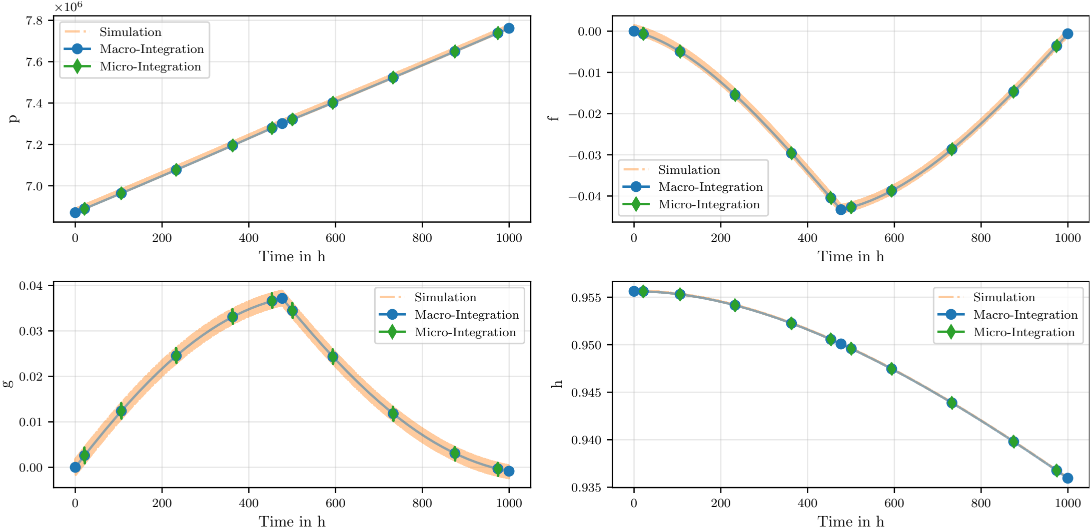
Example: Low Thrust Satellite - Solution

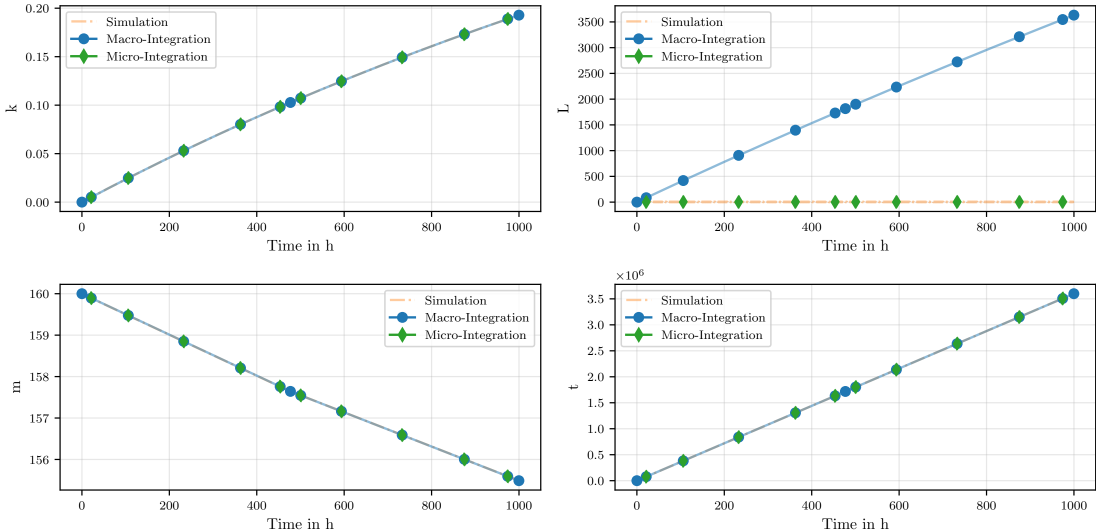




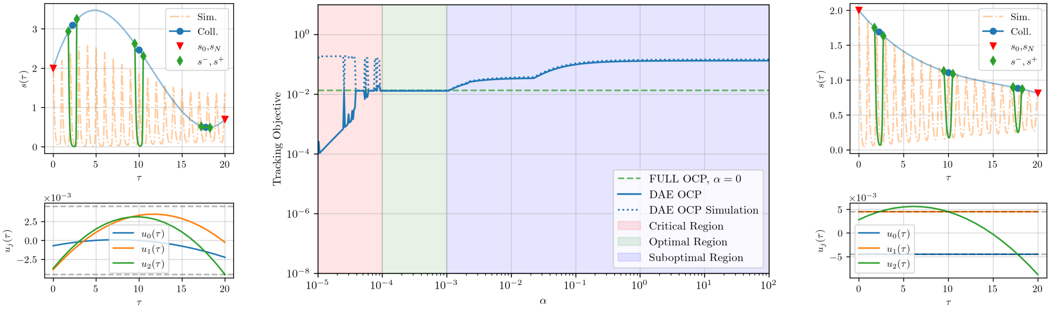
Example: Low Thrust Satellite - States 1



Example: Low Thrust Satellite - States 2



Regularization on Second State Derivative Solves the Problem



Left (red): too small regularization

Right (green): sufficiently high regularization

Note: "sweet spot" around $\alpha = 0.5 \cdot 10^{-3}$ where objective is unaffected by regularization

Macro-Integration of the DAE

- Integrate DAE from x_0 over N periods
- Single implicit d-stage Runge-Kutta step with butcher coefficients c, A, b:

$$
\begin{aligned}
0 &= x_0 - \bar{x}_0 \\
0 &= x_N - x_0 + N \sum_{j=1}^{d} b_j v_j \\
0 &= x_i - x_0 + N \sum_{j=1}^{d} a_{i,j} v_j && i = 1, \dots, d \\
0 &= v_i - f_{\mathrm{e}}(x_i, z_i) && i = 1, \dots, d \\
0 &= g_{\mathrm{e}}(x_i, z_i) && i = 1, \dots, d
\end{aligned}
$$

Butcher tableau:

$$
\begin{array}{c|cccc}
c_1 & a_{1,1} & a_{1,2} & \dots & a_{1,d} \\
c_2 & a_{2,1} & a_{2,2} & \dots & a_{2,d} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_d & a_{d,1} & a_{d,2} & \dots & a_{d,d} \\
\hline
 & b_1 & b_2 & \dots & b_d
\end{array}
$$

Forward Euler, $d = 1$:

$$
\begin{array}{c|c}
0 & 0 \\
\hline
 & 1
\end{array}
$$

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$$\begin{aligned}
\min_{w} \quad & \|x_N - \bar{x}_N\|_Q^2 \\
\text{s.t.} \quad & 0 = x_0 - \bar{x}_0, \\
& 0 = x_N - x_0 + N \sum_{j=1}^{d} b_j v_j, \\
& 0 = x_i - x_0 + N \sum_{j=1}^{d} a_{i,j} v_j \qquad i = 1, \dots, d, \\
& 0 = v_i - f_\mathrm{e}(x_i, z_i, U_\mathrm{const}) \qquad i = 1, \dots, d, \\
& 0 = g_\mathrm{e}(x_i, z_i, U_\mathrm{const}) \qquad i = 1, \dots, d, \\
& 0 \leq h(w)
\end{aligned}$$

- $w = (x_0, x_1, \dots, x_d, v_1, \dots, v_d, z_1, \dots, z_d, x_N, U_\mathrm{const})$

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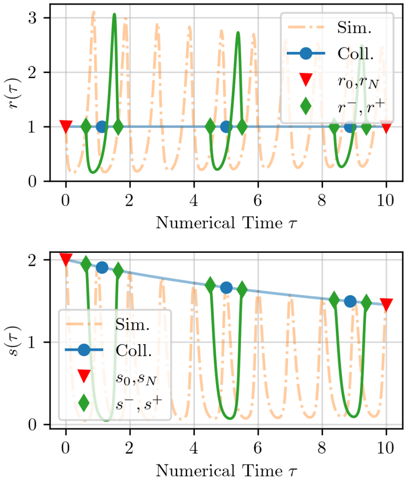
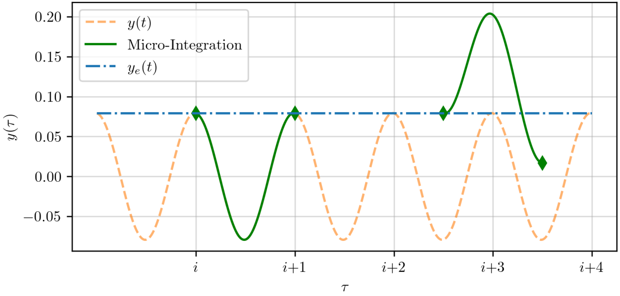
$$

- $U^*_\mathrm{const} = [-2,\ 2,\ 2] \cdot 10^{-3}$



Phase Issue



$$\dot{x} = \begin{bmatrix} \dot{y} \\ \dot{t} \end{bmatrix} = \begin{bmatrix} -y - \sin(\omega t) \\ 1 \end{bmatrix} \tag{3}$$

$$x(\tau) = \begin{bmatrix} x(\tau) \\ t(\tau) \end{bmatrix} = \begin{bmatrix} Ce^{-T\tau} - \frac{\sin(2\pi\tau)}{\omega^2+1} + \frac{\omega\cos(2\pi\tau)}{\omega^2+1} \\ T\tau \end{bmatrix} \tag{4}$$

Numerical Time

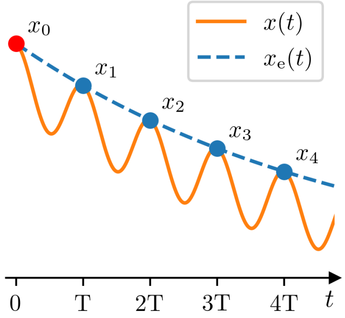
- Physical time: $t \in [0, t_\mathrm{f}] \to$ Numerical time: $\tau \in [0, N]$
- Scaling with period duration T:

$$\tau(t) = \frac{t}{T} \qquad\qquad t(\tau) = \tau T$$

- Trajectories now in numerical time as $x(\tau)$, dynamics read

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}(\tau) = T f(x(\tau))$$

- With numerical time, we can integrate over whole periods
- Physical time t can be included as a state



Numerical Time

- Physical time: $t \in [0, t_\mathrm{f}]$ $\rightarrow$ Numerical time: $\tau \in [0, N]$
- Scaling with period duration T:

$$\tau(t) = \frac{t}{T} \qquad\qquad t(\tau) = \tau T$$

- Trajectories now in numerical time as $x(\tau)$, dynamics read

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}(\tau) = T f(x(\tau))$$

- With numerical time, we can integrate over whole periods
- Physical time t can be included as a state