Implicit Central Difference Approximations of Averaged Dynamics of Oscillatory Systems

Jakob Harzer, Jochem De Schutter, Per Rutquist and Moritz Diehl

Syscop Group Retreat Autumn 2023
September 18, 2023

~~Control~~

~~Optimization~~

Simulation

Implicit Central Difference Approximations of Averaged Dynamics of Oscillatory Systems

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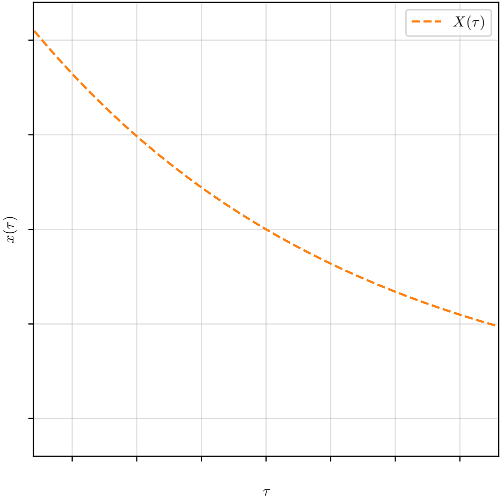
Setting



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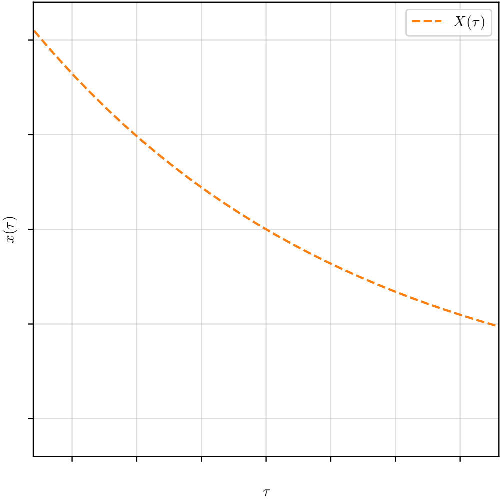

- $X(\tau)$ is a trajectory of

$$\frac{d}{d\tau}X = F(X) \tag{1}$$

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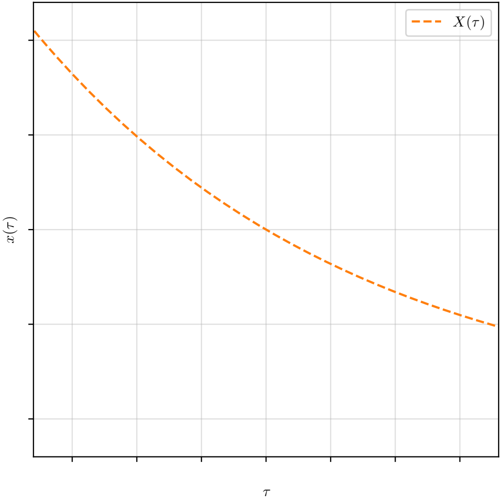

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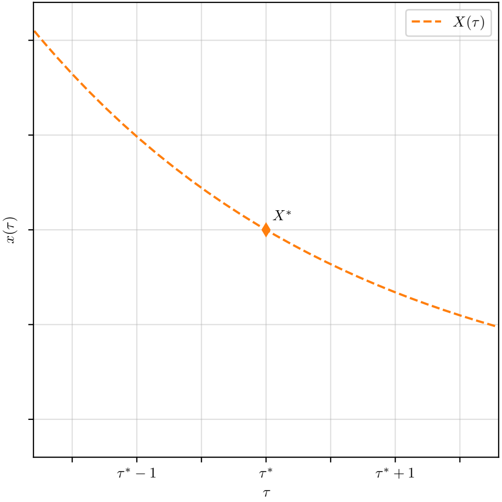

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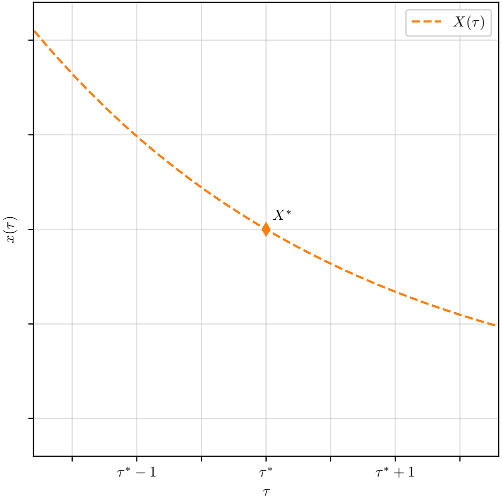

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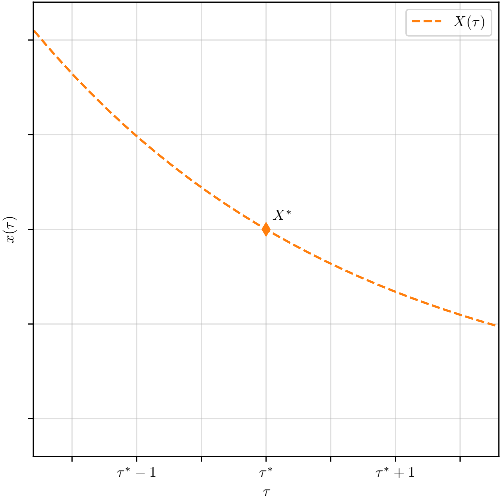

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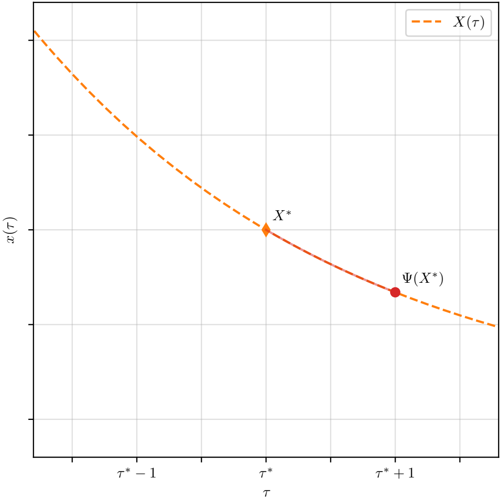

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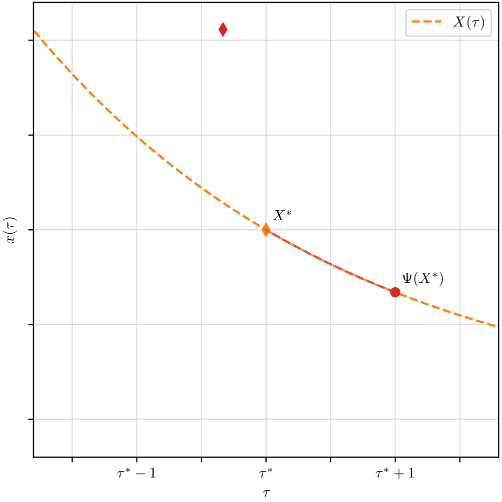

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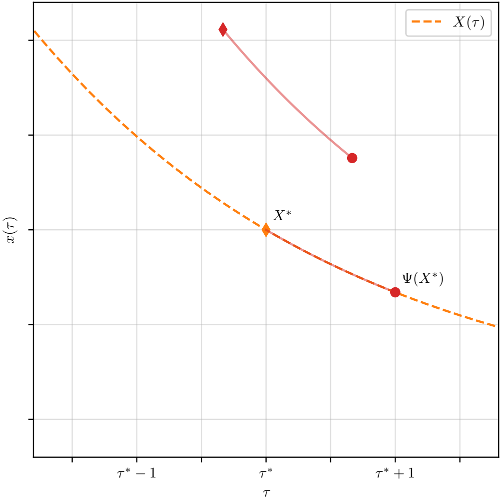

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Dynamics Approximations

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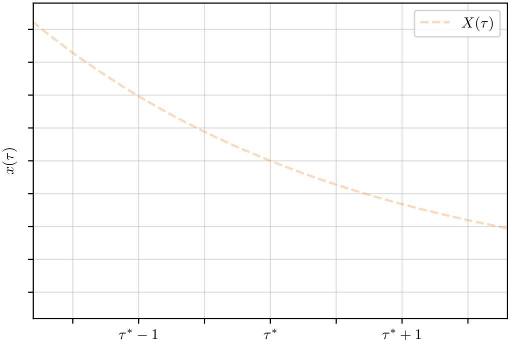

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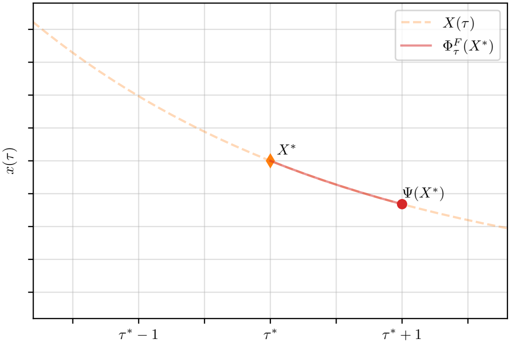

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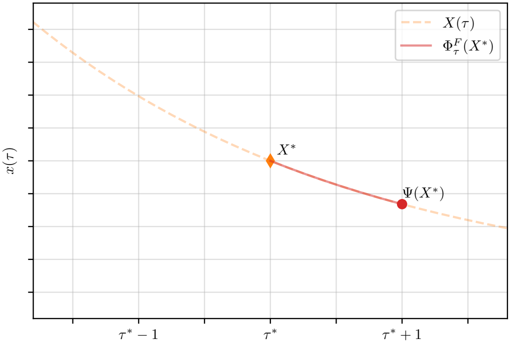

$$F(X^*) \approx \frac{\Psi^1(X^*) - X^*}{1} \tag{4}$$

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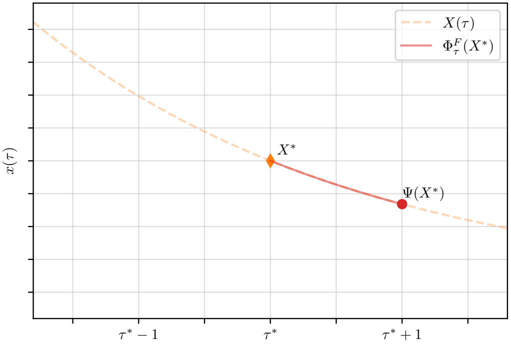

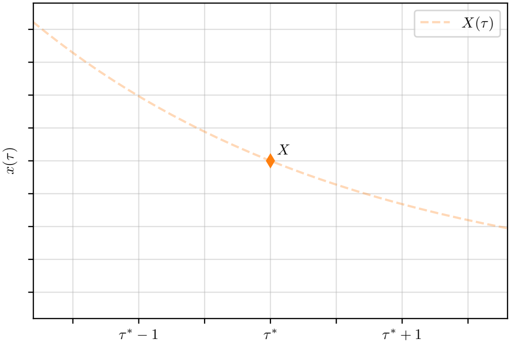

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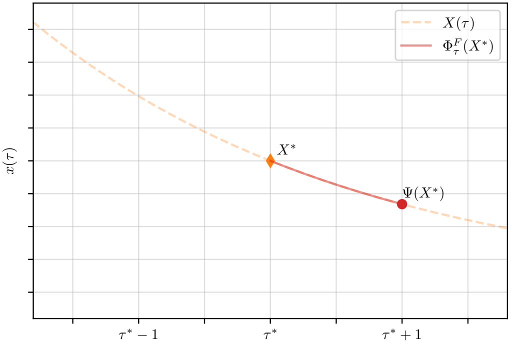

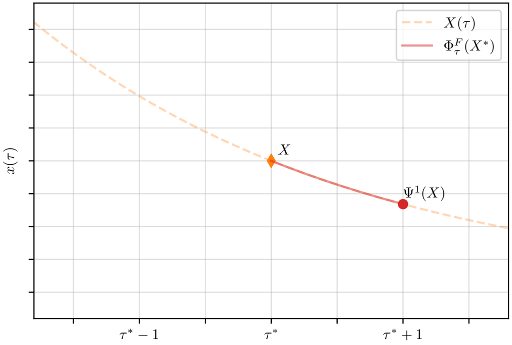

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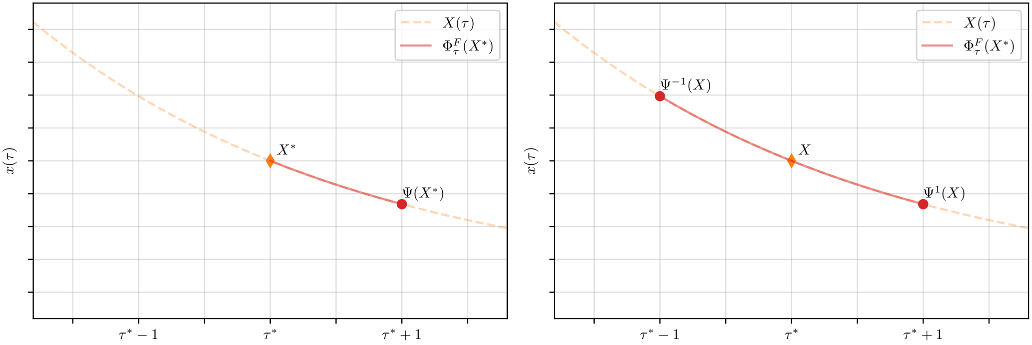

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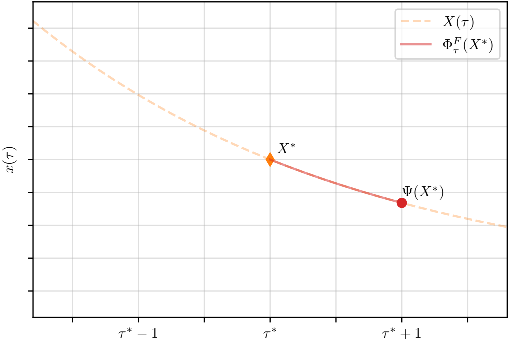

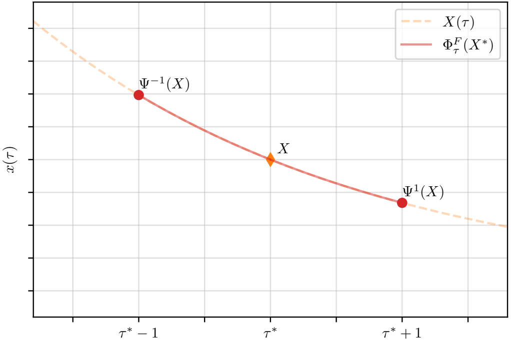

$$F(X^*) \approx \frac{\Psi^1(X^*) - X^*}{1} \tag{4}$$

$$F(X^*) \approx \frac{\Psi^1(X^*) - \Psi^{-1}(X^*)}{2} \tag{5}$$

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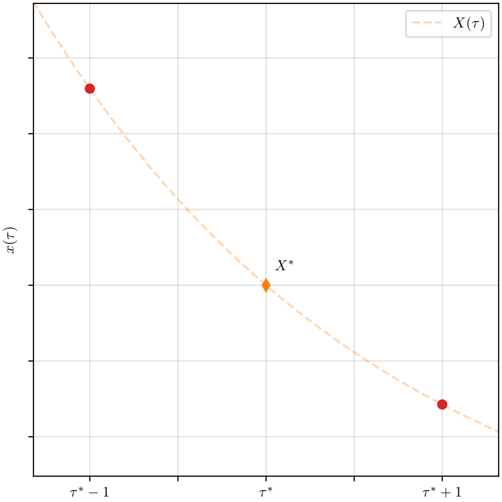
Implicit Dynamics Approximation - 2 Points



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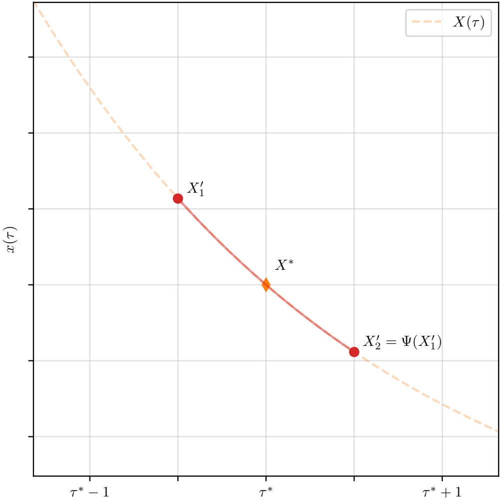

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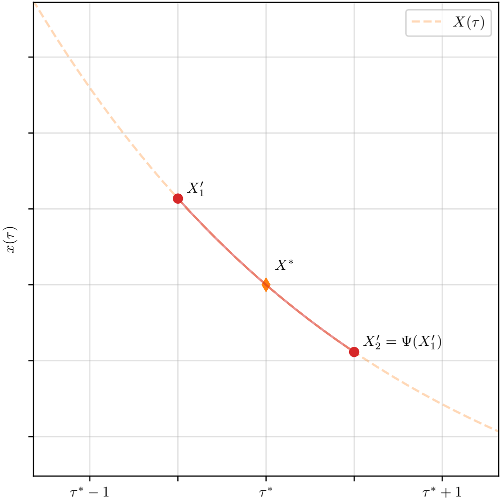

- Two points (X_1', X_2') at times $(\tau^* - 0.5, \tau^* + 0.5)$.

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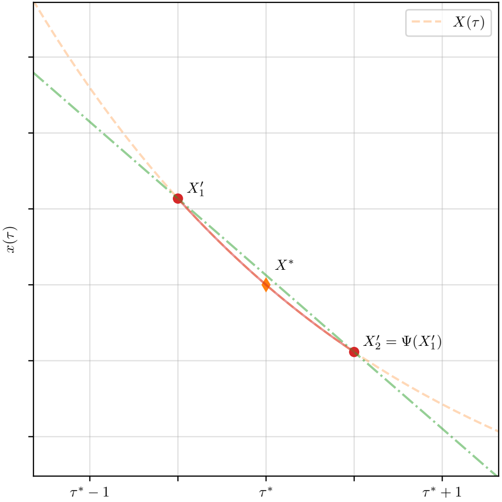

- Two points (X_1', X_2') at times $(\tau^* - 0.5, \tau^* + 0.5)$.
- Interpolating polynomial P.

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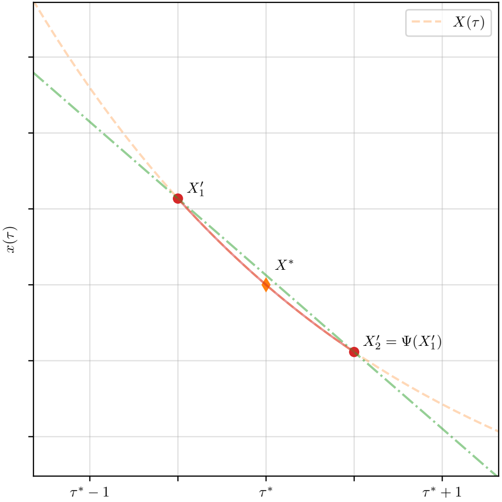

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$$0 = X_2' - \Psi(X_1') \tag{6a}$$
$$0 = P(\tau^*) - X^* \tag{6b}$$

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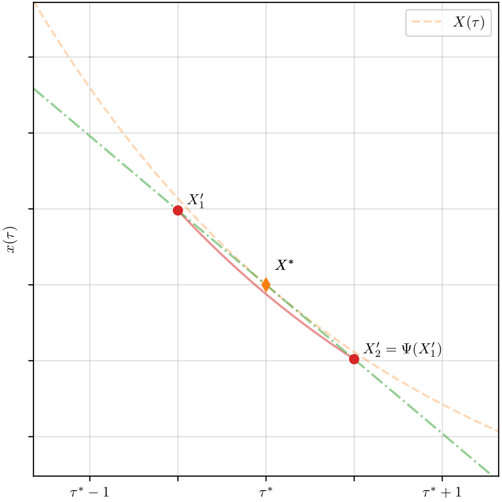

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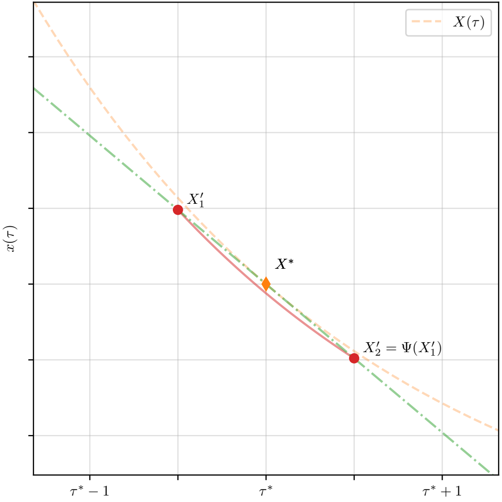
- Approximate the dynamics as

$$F(X^*) \approx \frac{X_2' - X_1'}{1} \quad (7)$$

Implicit Dynamics Approximation - Observations



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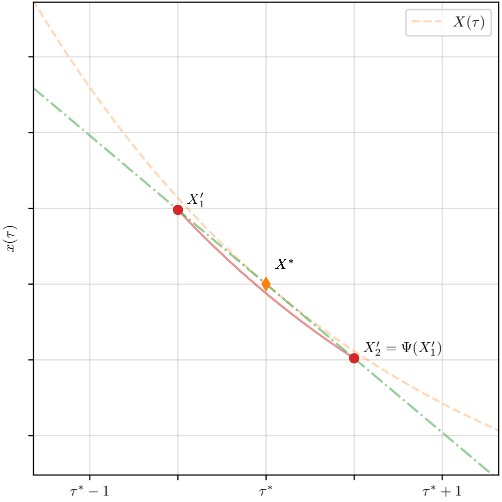

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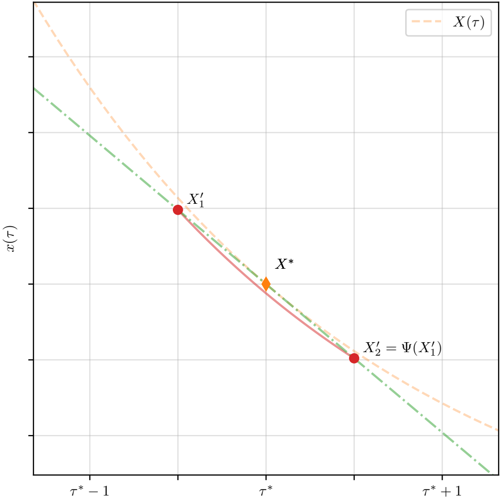

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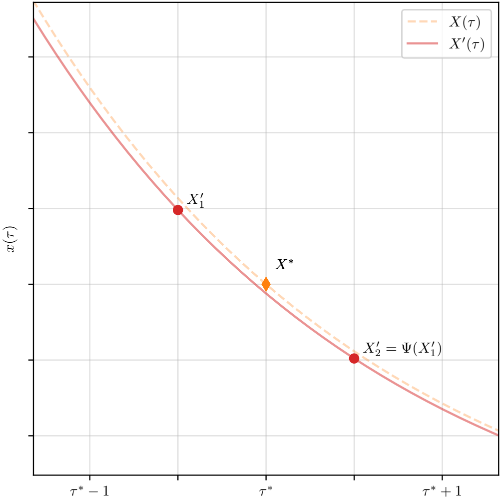

- We need to solve a nonlinear system of equations.
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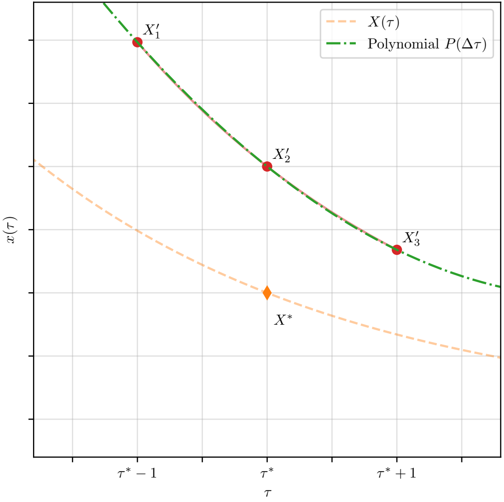
- Points X'_1, X'_2 lie on a solution of the system $X'(\tau)$, that is not $X(\tau)$.

Implicit Dynamics Approximation - 3 Points

- *Three* points (X_1', X_2', X_3') at times $(\tau^* - 1, \tau^*, \tau^* + 1)$.

Implicit Dynamics Approximation - 3 Points



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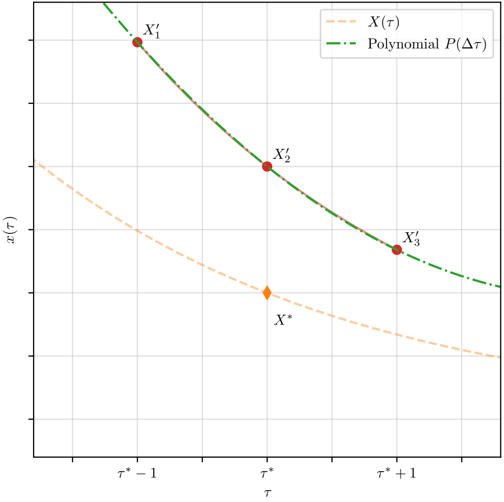

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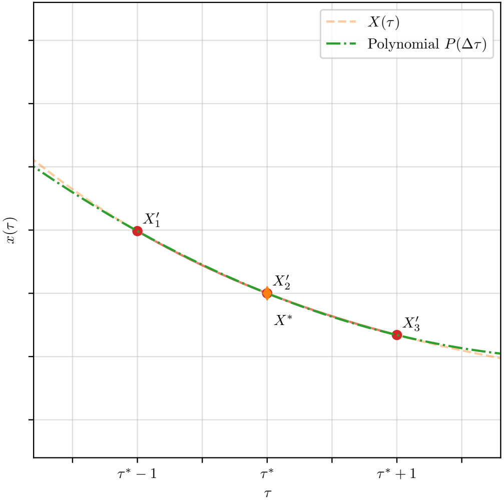
$$X'_2 = \Psi(X'_1) \tag{8}$$

$$X'_3 = \Psi(X'_2) \tag{9}$$

$$P(\tau^*) = X(\tau^*) \tag{10}$$

Implicit Dynamics Approximation - 3 Points



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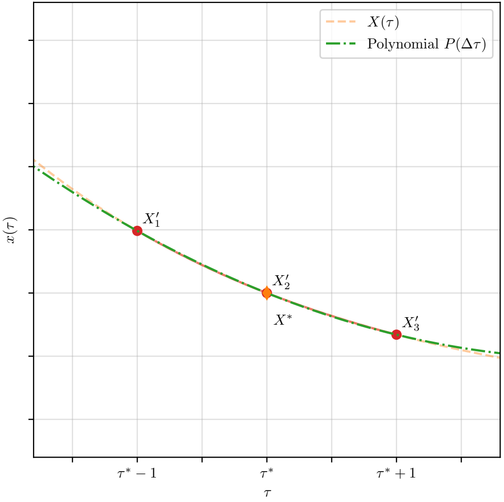

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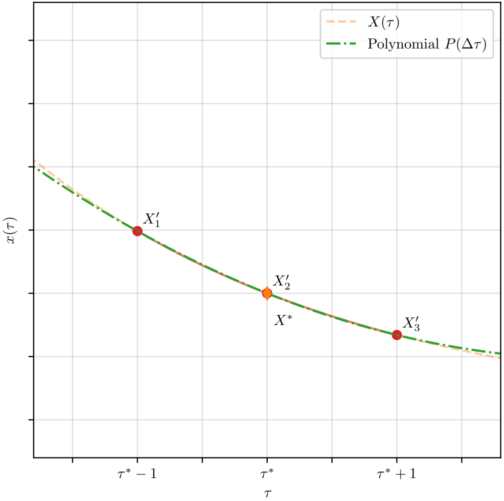
- Approximate the dynamics as

$$F(X^*) \approx \frac{X_3' - X_1'}{2} \quad (11)$$

Implicit Dynamics Approximation - 3 Points



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- We recover the explicit central difference scheme from before!

Implicit Dynamics Approximation - K Points

- Let $\tau = \tau^* + \Delta\tau$, K stroboscopic points $X'_1, \ldots, X'_K$ at equidistant times

$$\Delta\tau_k = k - \frac{K+1}{2}, \qquad k = 1, \ldots, K \tag{12}$$

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- i.e.

$$\Delta\tau_k \in \begin{cases} \{\ldots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots\} & \text{for } K \text{ even} \\ \{\ldots, -1, 0, 1, \ldots\} & \text{for } K \text{ odd} \end{cases} \tag{13}$$

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- Interpolating polynomial

$$P(\Delta\tau) = \sum_{k=1}^{K} \ell_k(\Delta\tau) X'_k, \qquad \text{where} \quad \ell_k(\Delta\tau) = \prod_{n=1, k\neq n}^{K} \frac{(\Delta\tau - \Delta\tau_n)}{(\Delta\tau_k - \Delta\tau_n)} \tag{14}$$

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- with

$$P(0) = \sum_{k=1}^{K} \ell_k(0) X'_k, \qquad \dot{P}(0) = \sum_{k=1}^{K} \dot{\ell}_k(0) X'_k. \tag{15}$$

Implicit Dynamics Approximation - K Points

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Implicit Dynamics Approximation - K Points

- Solve

$$0 = X'_2 - \Psi(X'_1) \tag{16a}$$

$$\vdots \tag{16b}$$

$$0 = X'_K - \Psi(X'_{K-1}) \tag{16c}$$

$$0 = X^* - \sum_{k=1}^{K} b_k X'_k, \tag{16d}$$

Implicit Dynamics Approximation - K Points

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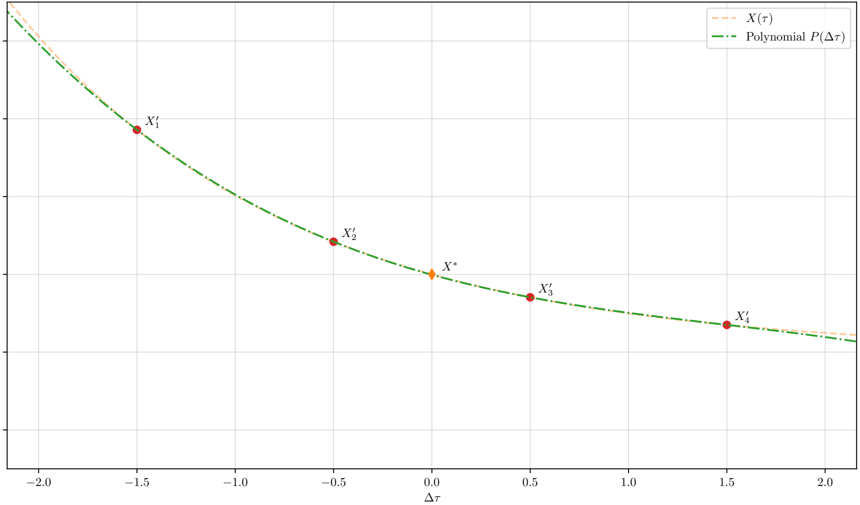
- Approximate the dynamics as

$$F(X^*) \approx \sum_{k=1}^{K} c_k X_k' \tag{17}$$

Example: $K = 4$



$X(\tau)$
Polynomial $P(\Delta\tau)$
X_1'
X_2'
X^*
X_3'
X_4'
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
$\Delta\tau$

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Averaged Dynamics of Oscillatory Systems

Highly Oscillatory Systems

- Highly Oscillatory Systems with $\epsilon \ll 1$

$$\dot{x} = f_0(x) + \epsilon f_1(x, \tau)$$

Highly Oscillatory Systems

- Highly Oscillatory Systems with $\epsilon \ll 1$

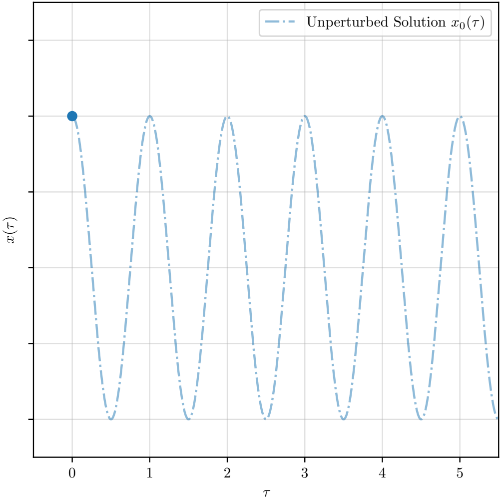
$$\dot{x} = f_0(x) + \epsilon f_1(x, \tau)$$

- Oscillatory Dynamics

$$\dot{x} = f_0(x)$$

with 1-periodic solution $x_0(\tau)$.

Highly Oscillatory Systems



- Highly Oscillatory Systems with $\epsilon \ll 1$

$$\dot{x} = f_0(x) + \epsilon f_1(x, \tau)$$

- Oscillatory Dynamics

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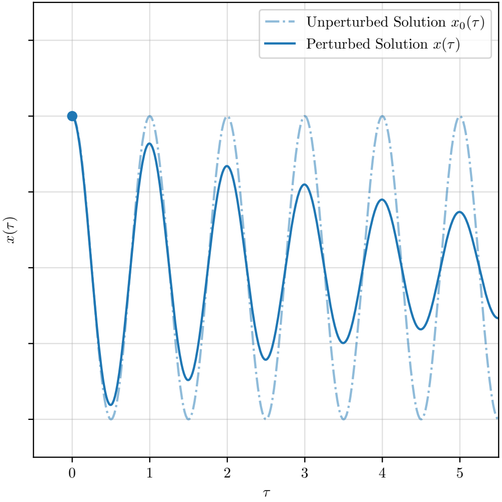
Highly Oscillatory Systems



- Highly Oscillatory Systems with $\epsilon \ll 1$

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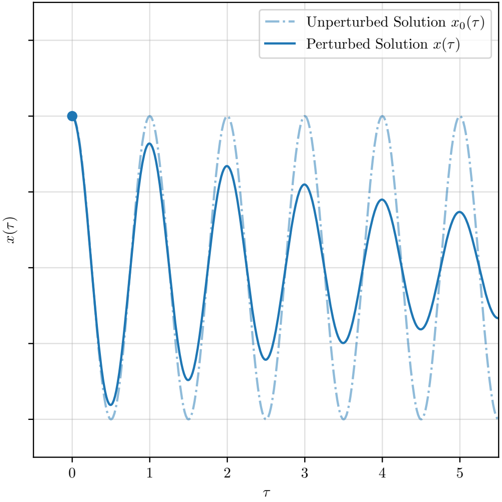
with 1-periodic solution $x_0(\tau)$.

Highly Oscillatory Systems



- Highly Oscillatory Systems with $\epsilon \ll 1$

$$\dot{x} = f_0(x) + \epsilon f_1(x, \tau)$$

- Oscillatory Dynamics

$$\dot{x} = f_0(x)$$

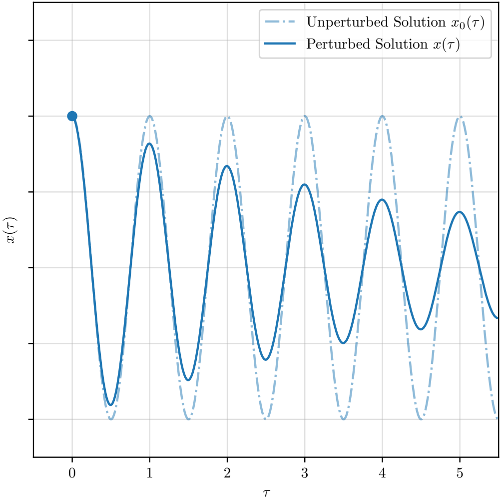
with 1-periodic solution $x_0(\tau)$.

- The perturbed solution $x(\tau)$ and unperturbed $x_0(\tau)$ differ by

$$\|x_0(\tau) - x(\tau)\| = \mathcal{O}(\epsilon)$$

on a timescale of 1.

Averaging Methods for Highly Oscillatory Systems



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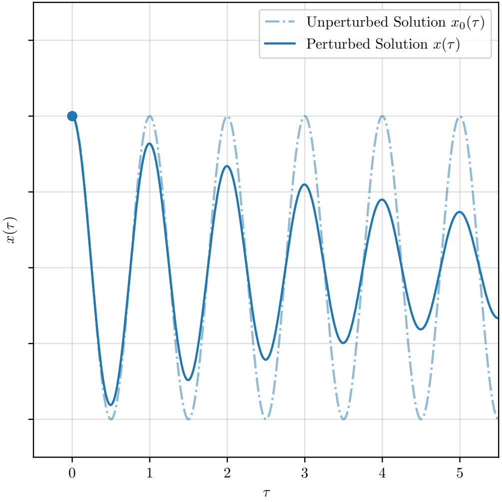

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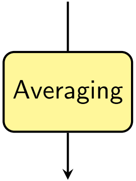

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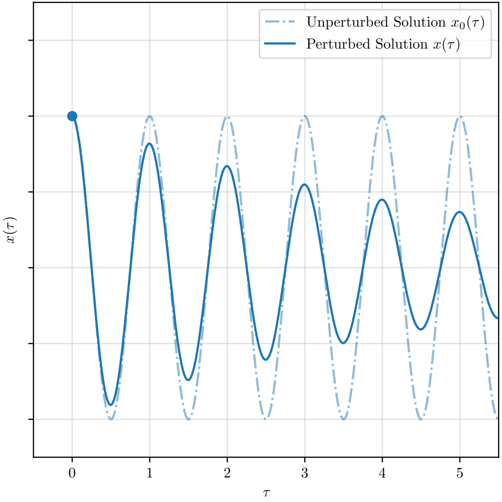
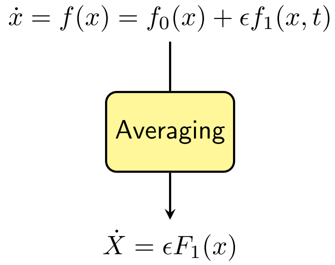
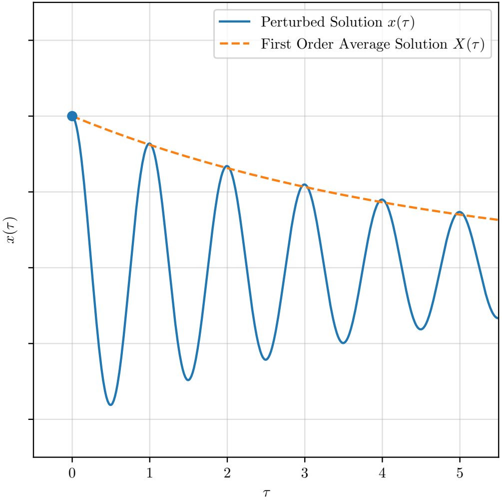


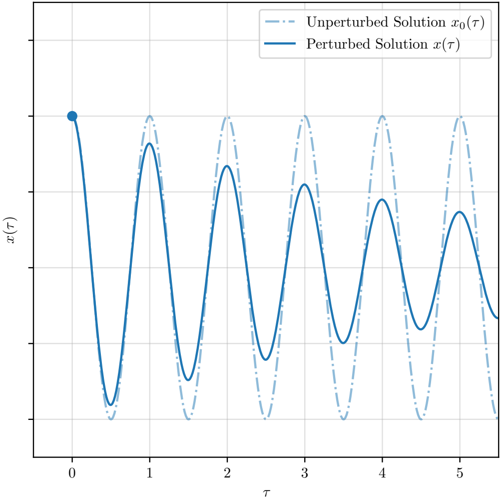




$$\dot{x} = f(x) = f_0(x) + \epsilon f_1(x, t)$$

Averaging

$$\dot{X} = \epsilon F_1(x)$$



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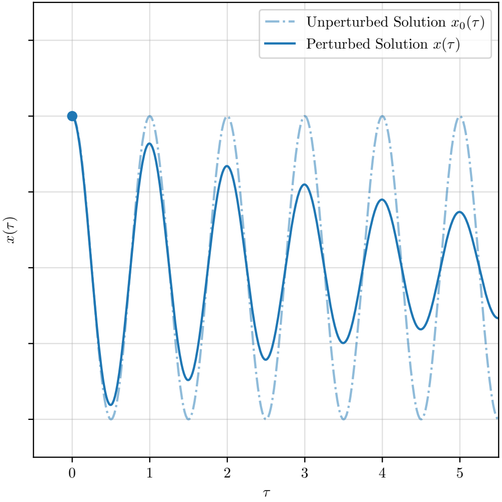
High Order Stroboscopic Averaging

Averaging Methods for Highly Oscillatory Systems



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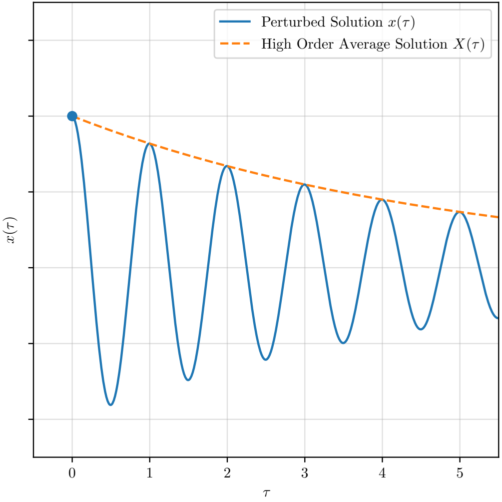
High Order Stroboscopic Averaging

$$\dot{X} = F(x) = \epsilon F_1(x) + \epsilon^2 F_2(x) + \dots$$



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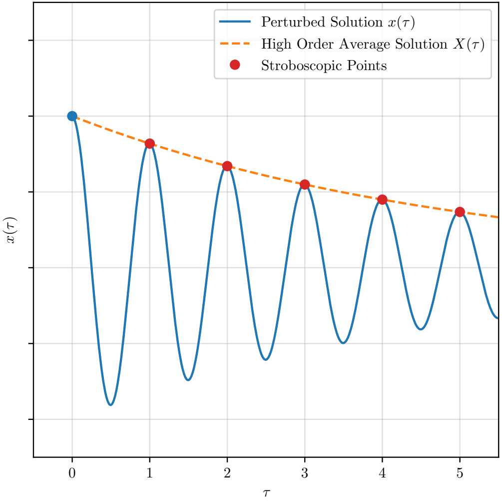

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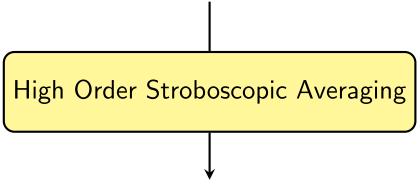
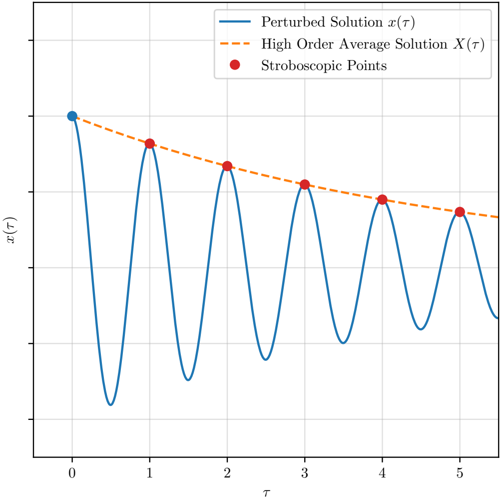


$$\dot{X} = F(x) = \epsilon F_1(x) + \epsilon^2 F_2(x) + \ldots$$

- If $x(0) = X(0)$ then the solution to averaged system satisfies

$$x(k) = X(k), \quad k \in \mathbb{Z}$$

Averaging Methods for Highly Oscillatory Systems



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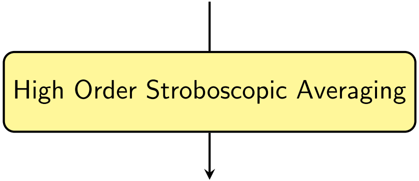

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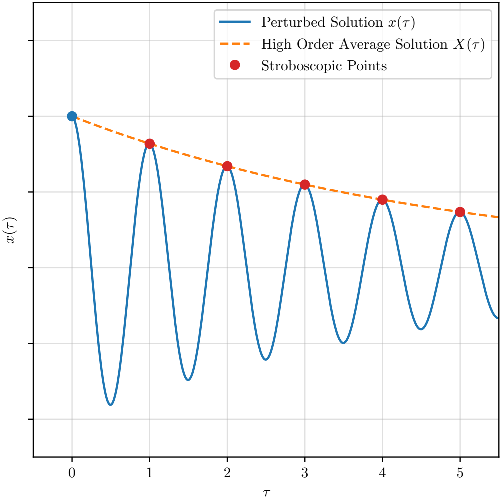
- Original system f on timescale $\mathcal{O}(1)$

Averaging Methods for Highly Oscillatory Systems



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- If $x(0) = X(0)$ then the solution to averaged system satisfies

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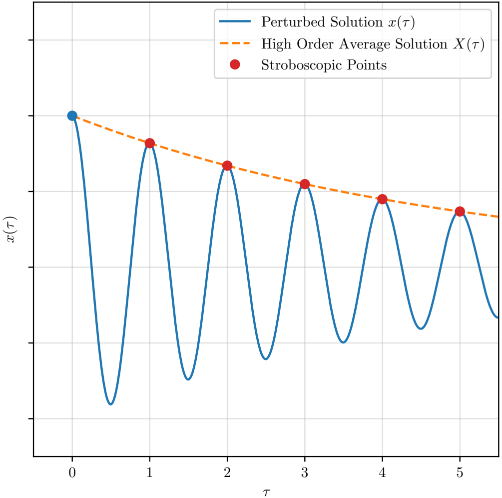
- Averaged system F on timescale $\mathcal{O}(1/\epsilon)$

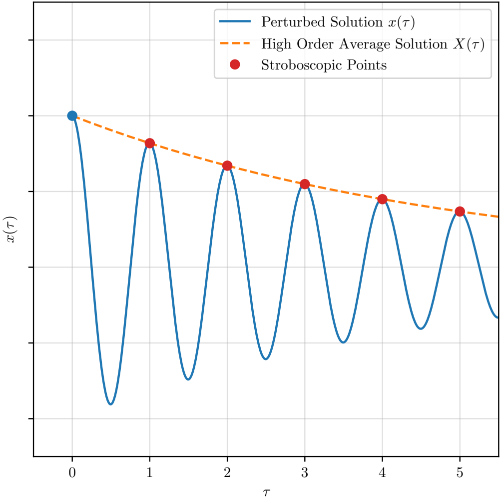
One-Cycle Map



- From before:

$$\Psi(X) = \Phi_1^F(X)$$

One-Cycle Map



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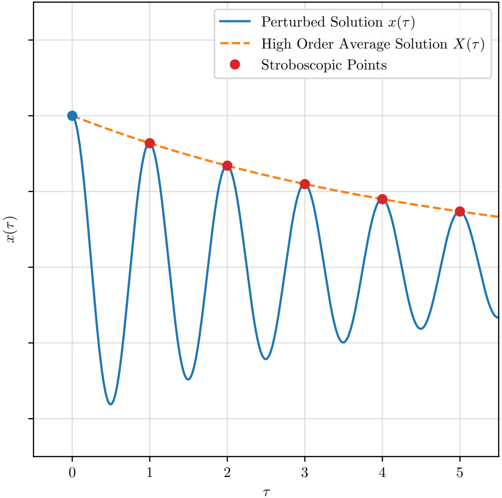


- From before:

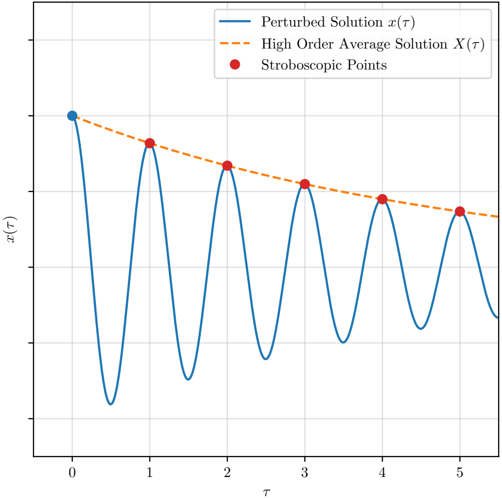
$$\Psi(X) = \Phi_1^F(X)$$
$$= \Phi_1^f(X)$$

- Micro-integration

$$\Psi(X) \approx \tilde{\Phi}_1^f(X)$$

by f.e. multiple RK steps.

One-Cycle Map



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by f.e. multiple RK steps.

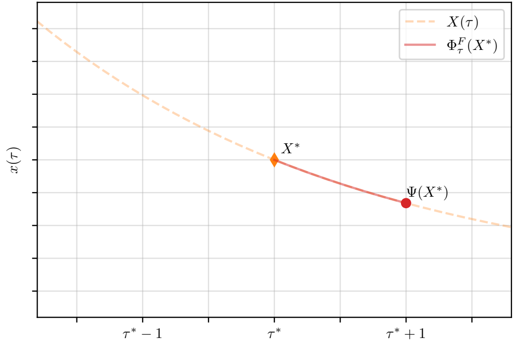
- We can use this 'one-cycle' map to approximate the average dynamics!

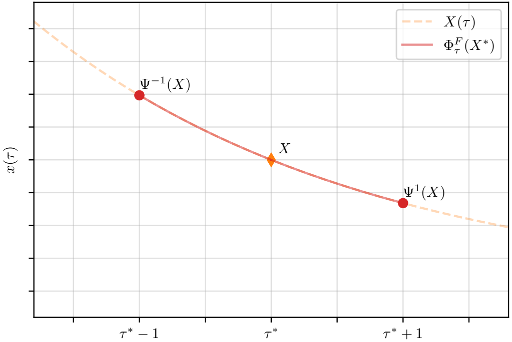
Implicit Central Difference Approximations of

Averaged Dynamics of Oscillatory Systems

Implicit Central Difference Approximations of Averaged Dynamics of Oscillatory Systems

Average Dynamics Approximation [1, 3]

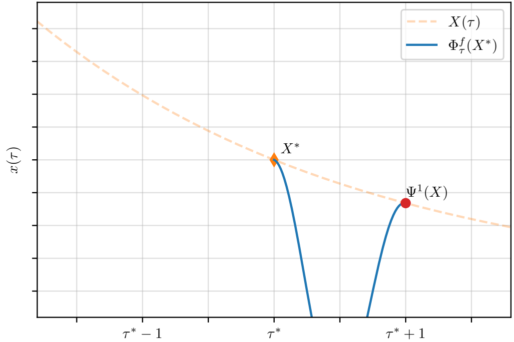


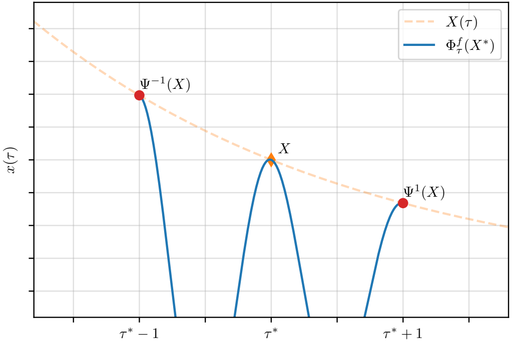


$$F(X^*) \approx \frac{\Psi^1(X^*) - X^*}{1}$$

$$F(X^*) \approx \frac{\Psi^1(X^*) - \Psi^{-1}(X^*)}{2}$$

Average Dynamics Approximation [1, 3]





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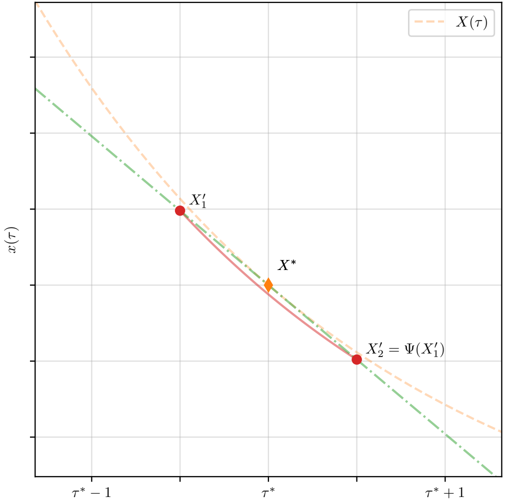
Implicit Averaged Dynamics Approximation



- Solve

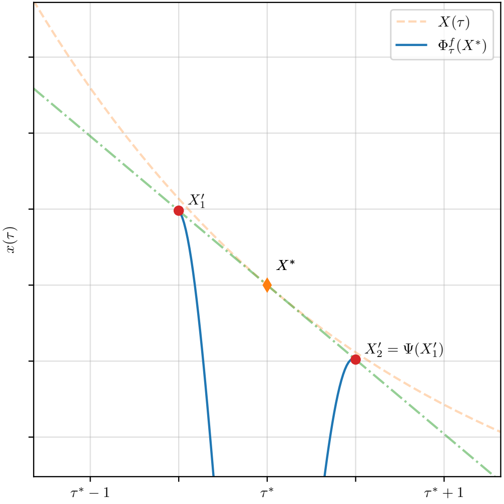
$$0 = X'_{k+1} - \Psi(X'_k), \quad k = 1, \dots, N-1$$

$$0 = X^* - \sum_{k=1}^{K} b_k X'_k,$$

- Approximate the dynamics as

$$F(X^*) \approx \sum_{k=1}^{K} c_k X'_k$$

Implicit Averaged Dynamics Approximation



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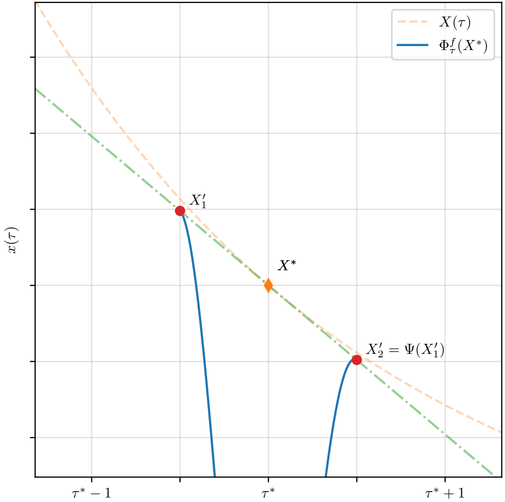

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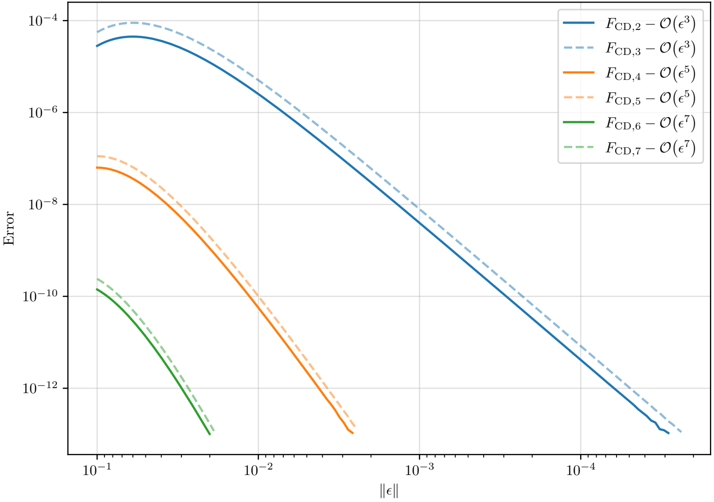
- Approximate the dynamics as

$$F(X^*) \approx \sum_{k=1}^{K} c_k X'_k$$

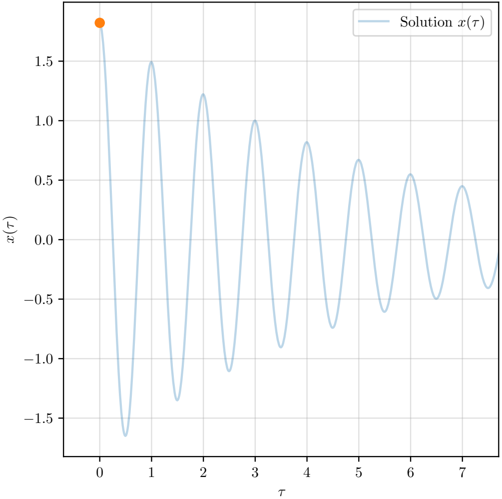
- Approximation Error:

$$\|F(X) - F_{\mathrm{CD},K}(X)\| = \begin{cases} \mathcal{O}\left(\epsilon^{K+1}\right) & \text{for } K \text{ even} \\ \mathcal{O}\left(\epsilon^{K}\right) & \text{for } K \text{ odd} \end{cases}$$

Approximation Error



Numerical Method for Efficient Simulation [1]



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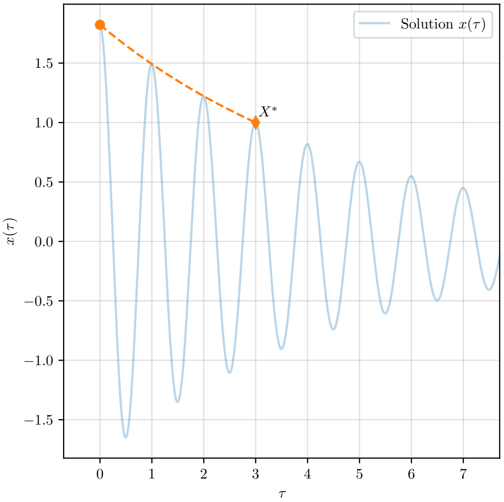

- At some point (τ^*, X^*):

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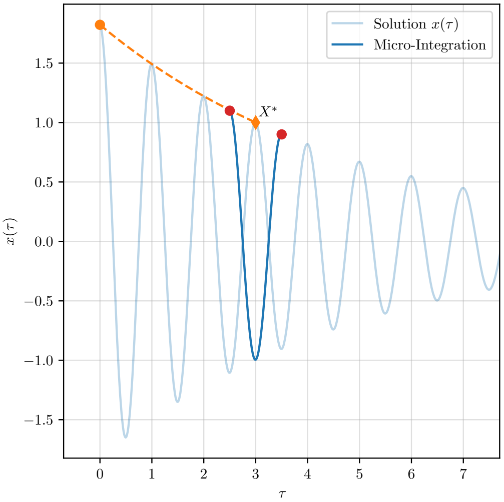

- At some point (τ^*, X^*):
  - (a) perform one or more micro-integrations to evaluate the one-cycle map

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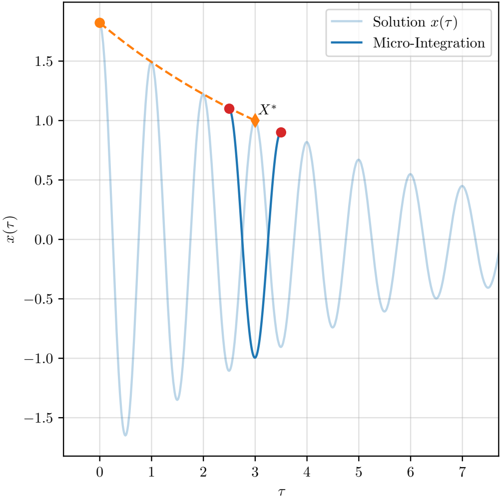

- At some point (τ^*, X^*):
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  - (b) approximate the averaged dynamics

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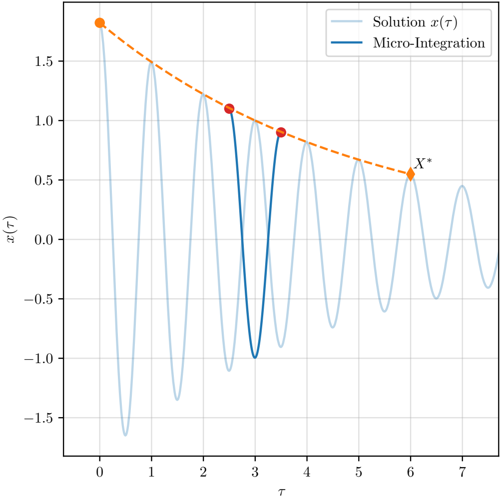

- At some point (τ^*, X^*):
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- Macro-integrate the averaged dynamics

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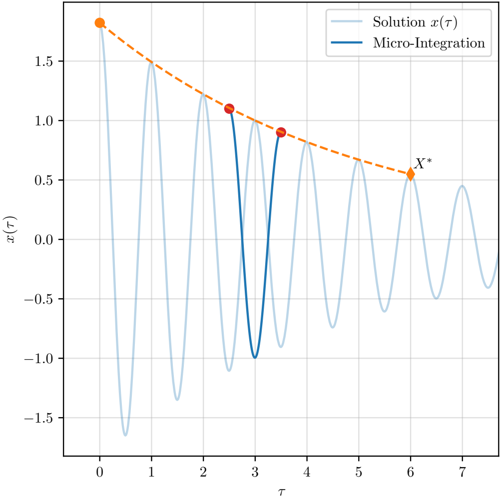

- At some point (τ^*, X^*):
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- Integration horizon of integer size N cycles since

$$x(N) = X(N)$$

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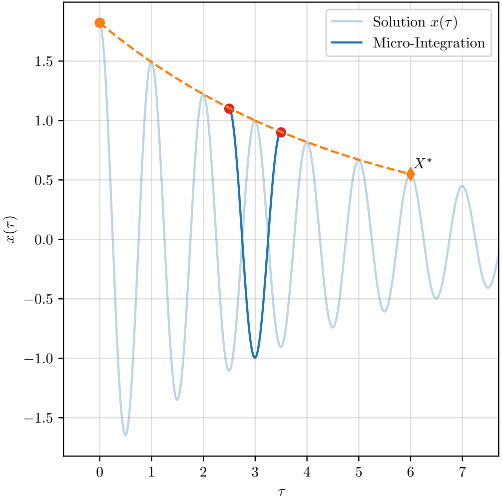

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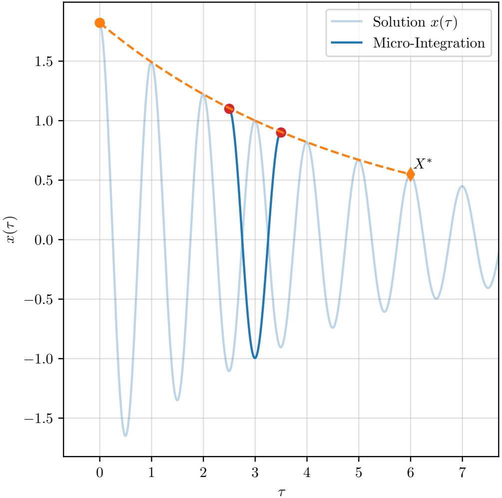

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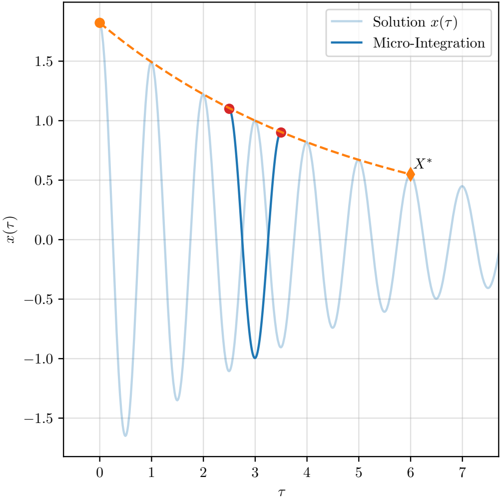

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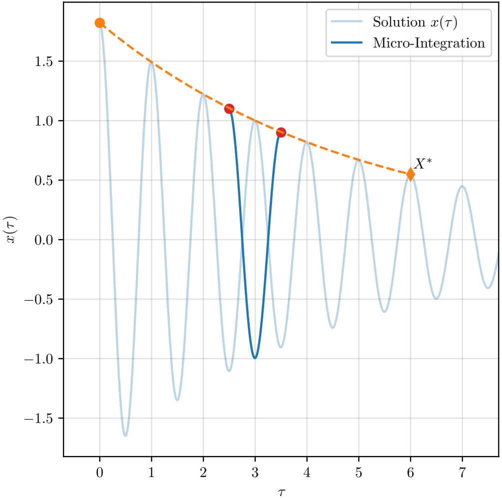

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  - (b) errors in the approximation of the dynamics
  - (c) errors in the macro-integration

Integration Experiment

- Linear Oscillator, $\epsilon = -10^{-3}$

$$\frac{d}{d\tau}x = \begin{bmatrix} \epsilon & -2\pi \\ 2\pi & \epsilon \end{bmatrix} x$$

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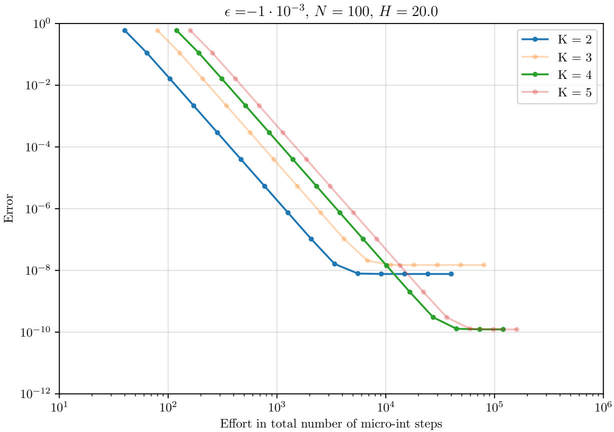

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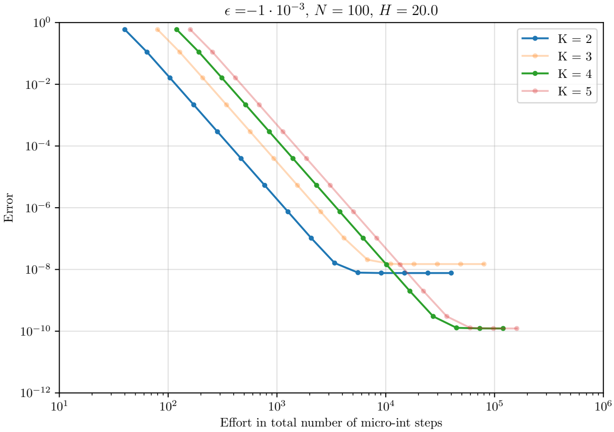
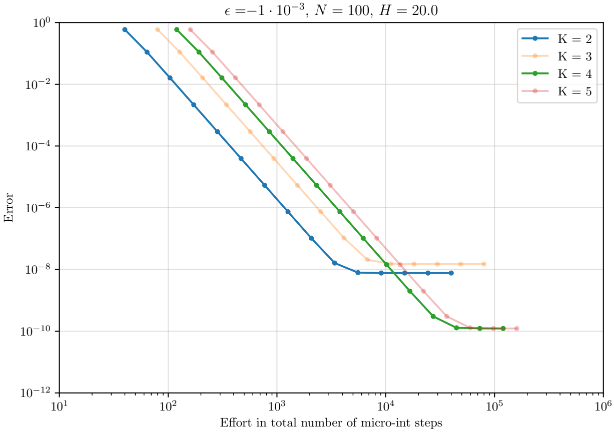


- Micro integration error is very dominant

Integration Experiment



- Micro integration error is very dominant
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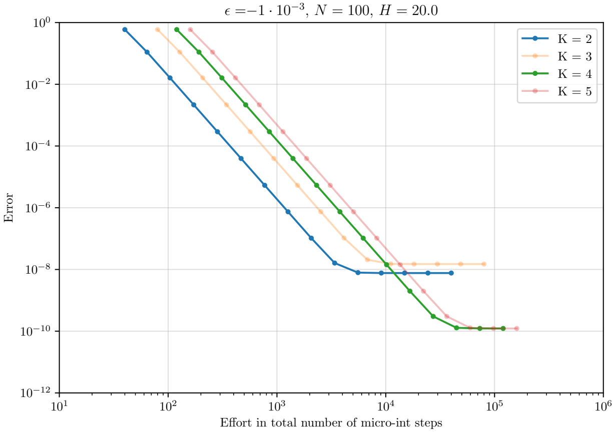
Integration Experiment



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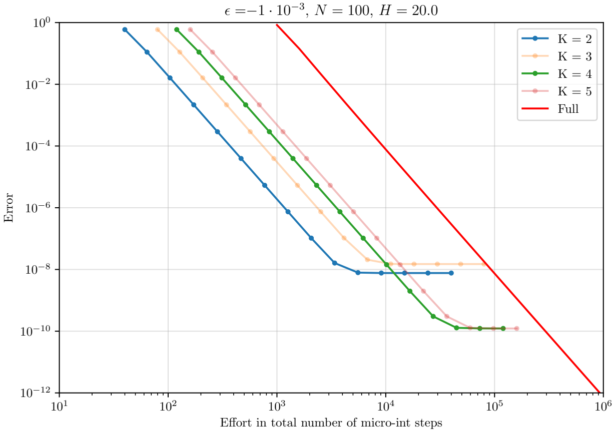

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- We derived implicit K-point methods to approximate the average dynamics
- The implicit methods (K even) are just as good as the explicit ones (K odd), but require less effort
- We can integrate highly oscillatory systems very efficiently.

$$\frac{d}{d\tau}x = f_0(x) + \epsilon f_1(x, \mathbf{u}, \tau) \tag{19}$$

Thank you for your attention!

Useful Sources

Mari Paz Calvo, Philippe Chartier, Ander Murua, and Jesús María Sanz-Serna.
A stroboscopic numerical method for highly oscillatory problems.
In Björn Engquist, Olof Runborg, and Yen-Hsi R. Tsai, editors, Numerical Analysis of Multiscale Computations, pages 71–85, Berlin, Heidelberg, 2012. Springer Berlin Heidelberg.

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Mathematics of Computation, 51:699–706, 1988.

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Numerische Mathematik, 53:621–652, 1988.

Jan Sanders, Ferdinand Verhulst, and J.B. Murdoch.
Averaging methods in nonlinear dynamical systems, 2d ed.
01 2007.

Coefficients Implicit Approximation

$\Delta\tau$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$K = 2$				$\frac{1}{2}$		$\frac{1}{2}$			
$K = 3$			0		1		0		
$K = 4$		$-\frac{1}{16}$		$\frac{9}{16}$		$\frac{9}{16}$		$-\frac{1}{16}$	
$K = 5$	0		0		1		0		0

Table: Coefficients b_k to relate the stroboscopic points X'_k to the integration point $X(\tau^*)$ via the interpolating polynomial. The lighter rows correspond to the introduced implicit method, the darker rows correspond to the existing explicit method.

Coefficients Implicit Approximation

$\Delta\tau$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$K = 2$				-1		1			
$K = 3$			$-\frac{1}{2}$		0		$\frac{1}{2}$		
$K = 4$		$\frac{1}{24}$		$-\frac{9}{8}$		$\frac{9}{8}$		$-\frac{1}{24}$	
$K = 5$	$\frac{1}{12}$		$-\frac{2}{3}$		0		$\frac{2}{3}$		$-\frac{1}{12}$

Table: Coefficients c_k of the (implicit) central difference approximation, c.f. [2]. The lighter rows correspond to the introduced implicit method, the darker rows correspond to the existing explicit method.