Convexity Exploiting Newton-Type Optimization for Learning and Control

Moritz Diehl
Systems Control and Optimization Laboratory
Department of Microsystems Engineering and Department of Mathematics
University of Freiburg

joint work with Florian Messerer and Katrin Baumgärtner

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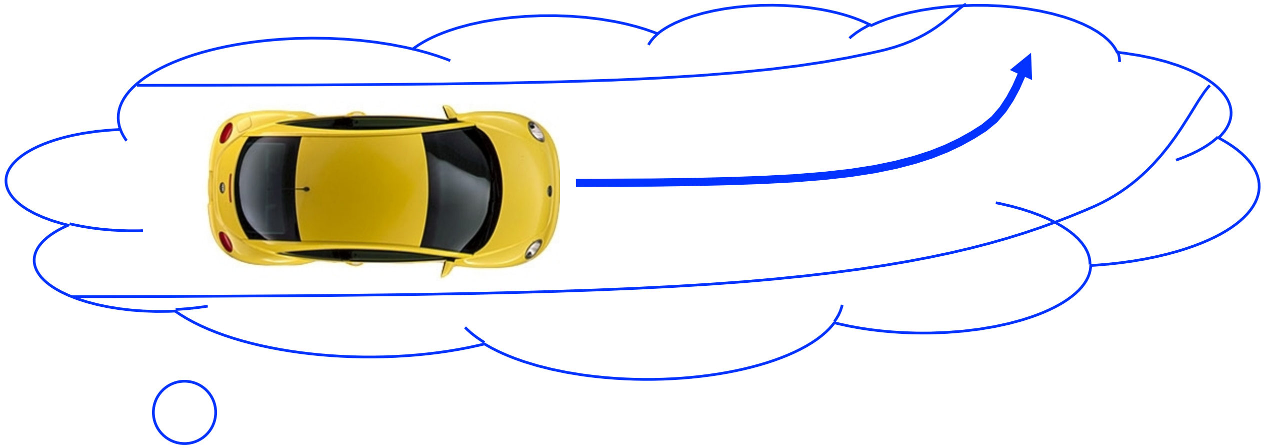
Overview

- Model Predictive Control and two Applications
- Convexity Exploiting Newton-Type Optimization
  - Sequential Convex Programming (SCP)
  - Generalized Gauss-Newton (GGN)
  - Sequential Convex Quadratic Programming (SCQP)
  - Local Convergence Analysis and Desirable Divergence
- Zero-Order Optimization-based Iterative Learning Control
  - Tutorial Example
  - Bounding the Loss of Optimality
  - Local Convergence Analysis

Model Predictive Control (MPC)

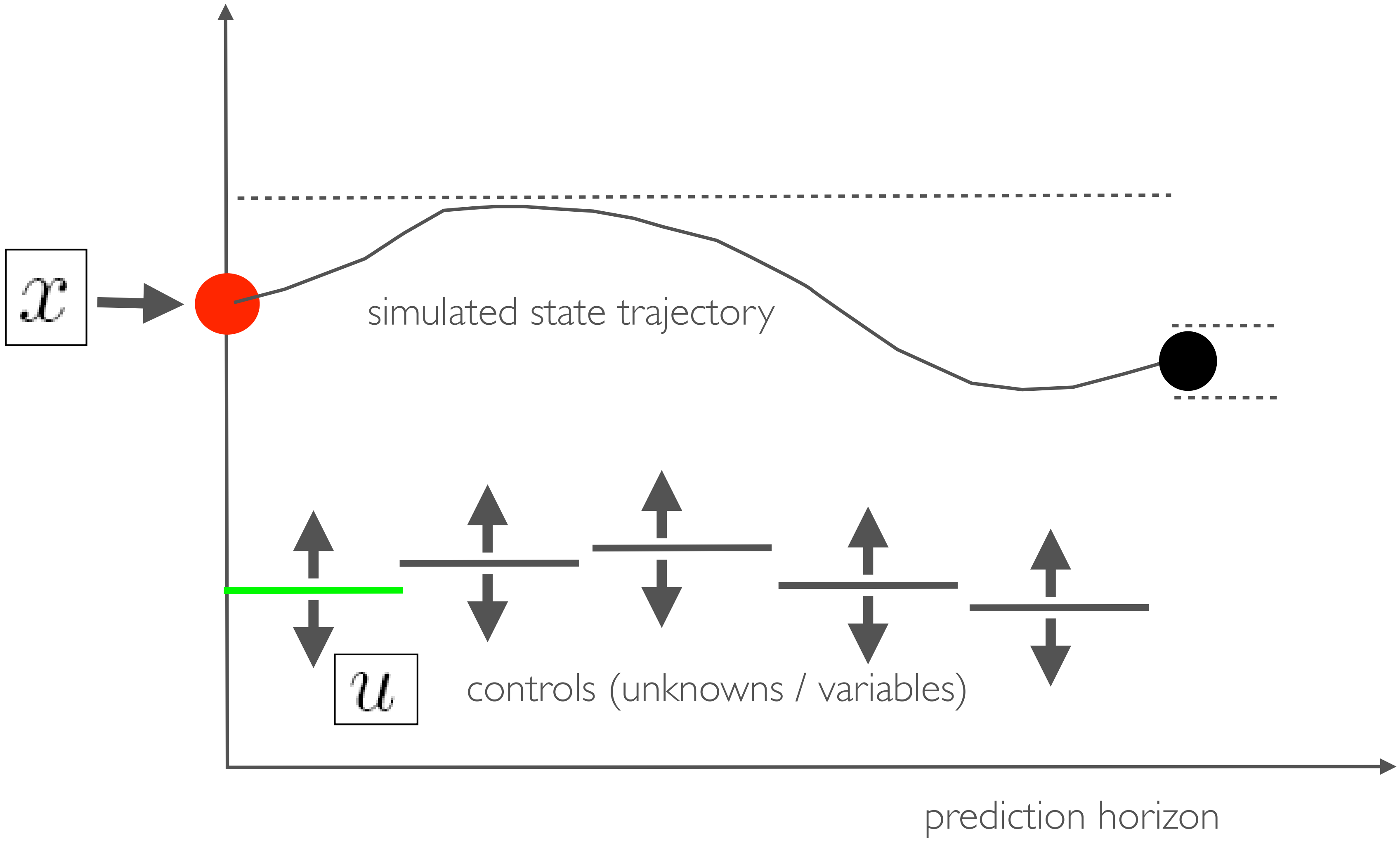
Always look a bit into the future



Example: driver predicts and optimizes, and therefore slows down before a curve

Optimal Control Problem in MPC

For given system state $\boldsymbol{x}$, which controls $\boldsymbol{u}$ lead to the best objective value without violation of constraints ?



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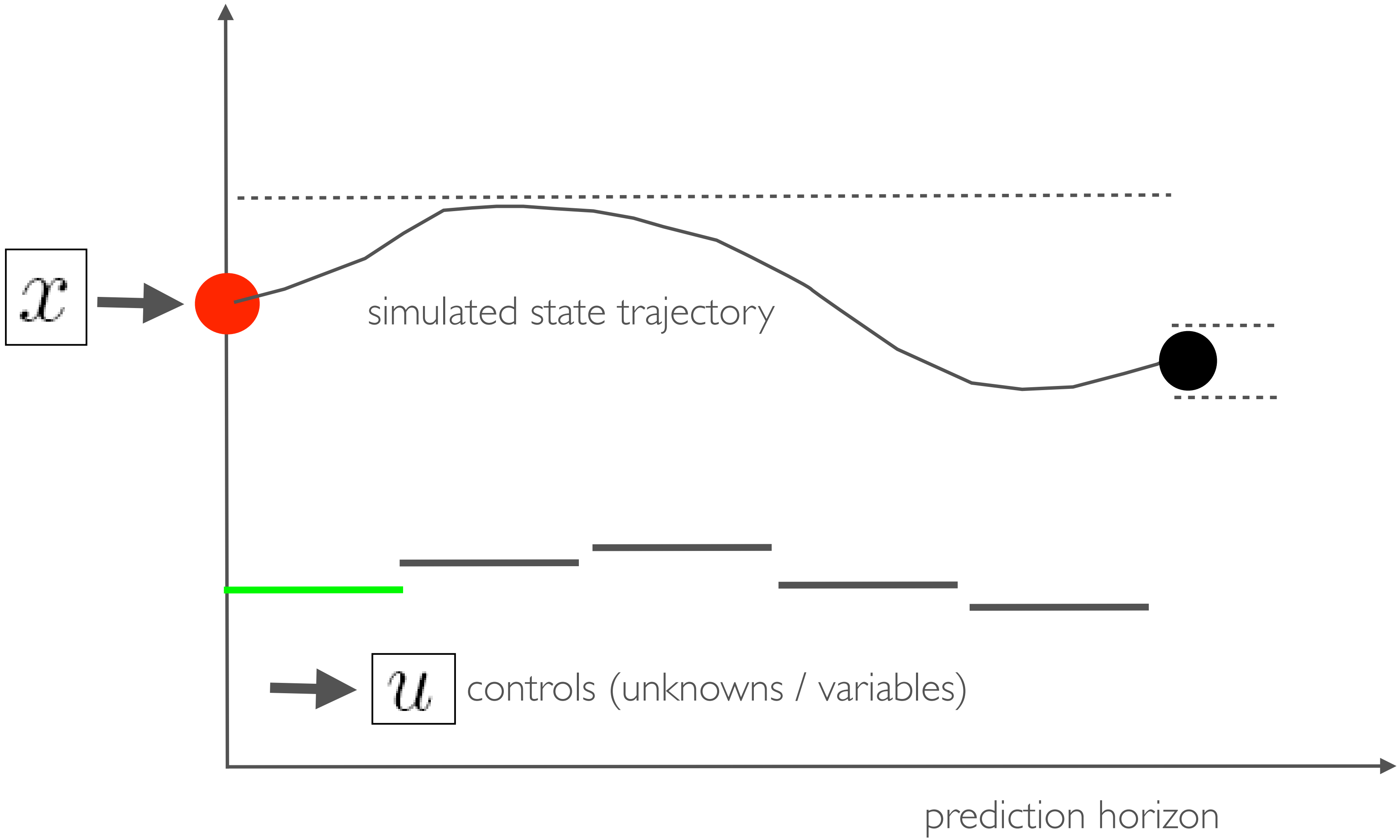
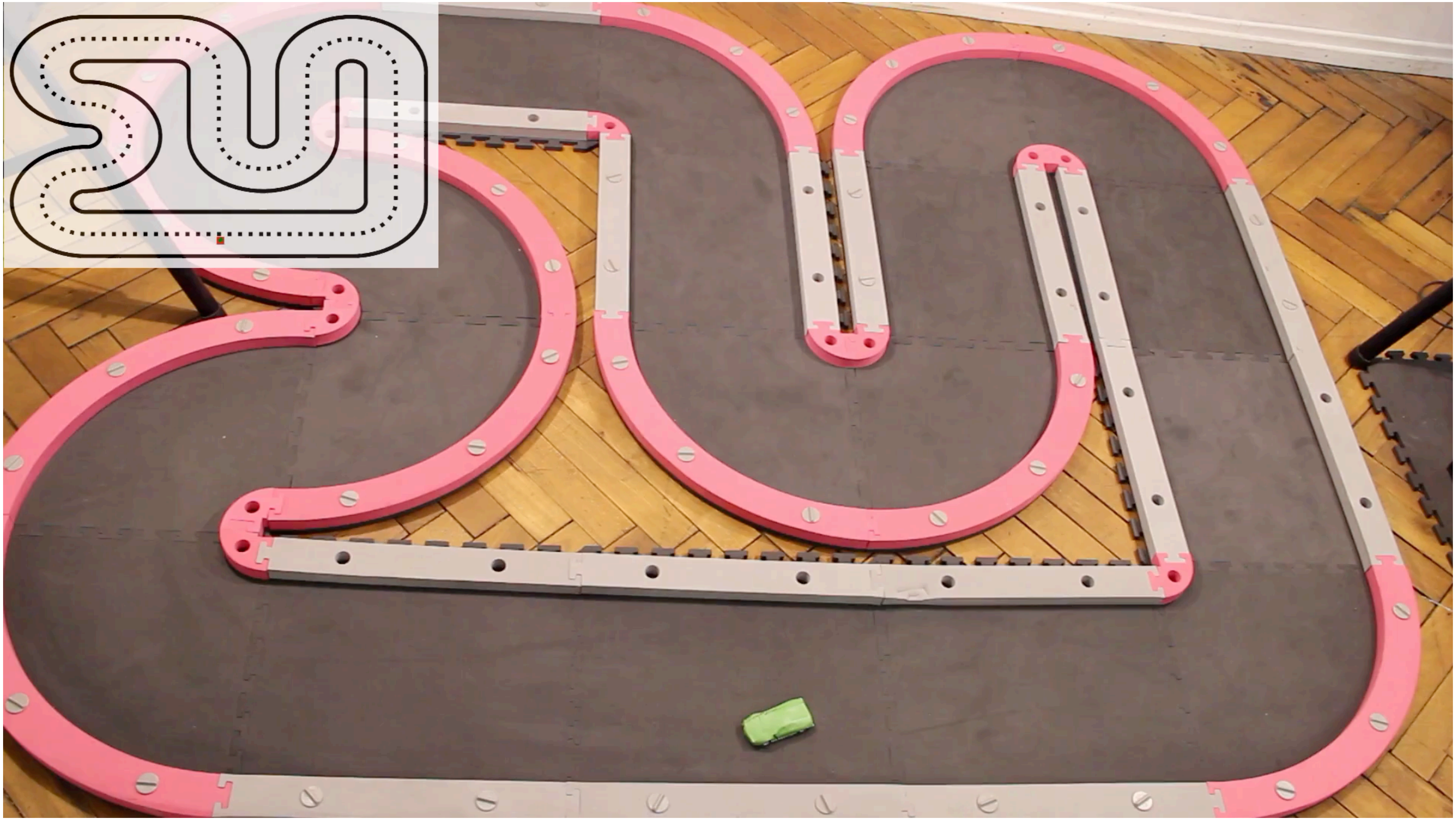


Model Predictive Control of RC Race Cars (in Freiburg)



Minimize least squares distance to centerline, respect constraints. Use nonlinear embedded optimization software *acados* coupled to ROS, sample at 100 Hz.

[Kloeser et al., submitted]

eco4wind: MPC for wind turbine control

Industrial partners: IAV, SENVION (now bankrupt)

Aim: minimise fatigue and oscillations, respect constraints.

Nonlinear MPC with about 40 states based on ACADO/acados with QP solver HPIPM running on industrial hardware at IAV.

Optimization Problem in Nonlinear Model Predictive Control

$$
\begin{aligned}
\underset{w \in \mathbb{R}^{n_w}}{\text{minimize}} \quad & \sum_{i=0}^{N} \varphi_i(F_i(s_i, u_i)) + \varphi_N(F_N(s_N)) \\
\text{subject to} \quad & s_0 = x, \\
& s_{i+1} = S_i(s_i, u_i), \quad i = 0, \ldots, N-1, \\
& H_i(s_i, u_i) \in \Omega_i, \quad i = 0, \ldots, N-1, \\
& H_N(s_N) \in \Omega_N
\end{aligned}
$$

- variables $w = (s, u)$ with $s = (s_0, \ldots, s_N)$ and $u = (u_0, \ldots, u_{N-1})$
- convexities in φ_i (e.g. quadratic) and Ω_i (e.g. polyhedral, ellipsoidal)
- nonlinearities in dynamic system S_i and constraint functions F_i, H_i
- often: S_i result of time integration (direct multiple shooting)

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Nonlinear optimization with convex substructure

$$\begin{aligned} \underset{w \in \mathbb{R}^{n_w}}{\text{minimize}} \quad & \phi_0(F_0(w)) \\ \text{subject to} \quad & F_i(w) \in \Omega_i \quad i = 1, \dots, m, \\ & G(w) = 0 \end{aligned}$$

Assumptions:

- twice continuously differentiable functions $G : \mathbb{R}^{n_w} \to \mathbb{R}^{n_g}$ and $F_i : \mathbb{R}^{n_w} \to \mathbb{R}^{n_{F_i}}$ for $i = 0, 1, \dots, m$.
- outer function $\phi_0 : \mathbb{R}^{n_{F_0}} \to \mathbb{R}$ convex.
- sets $\Omega_i \subset \mathbb{R}^{n_{F_i}}$ convex for $i = 1, \dots, m$,
  (possibly $z \in \Omega_i \Leftrightarrow \phi_i(z) \leq 0$ with smooth convex ϕ_i)

Idea:
exploit convex substructure via *iterative convex approximations*.

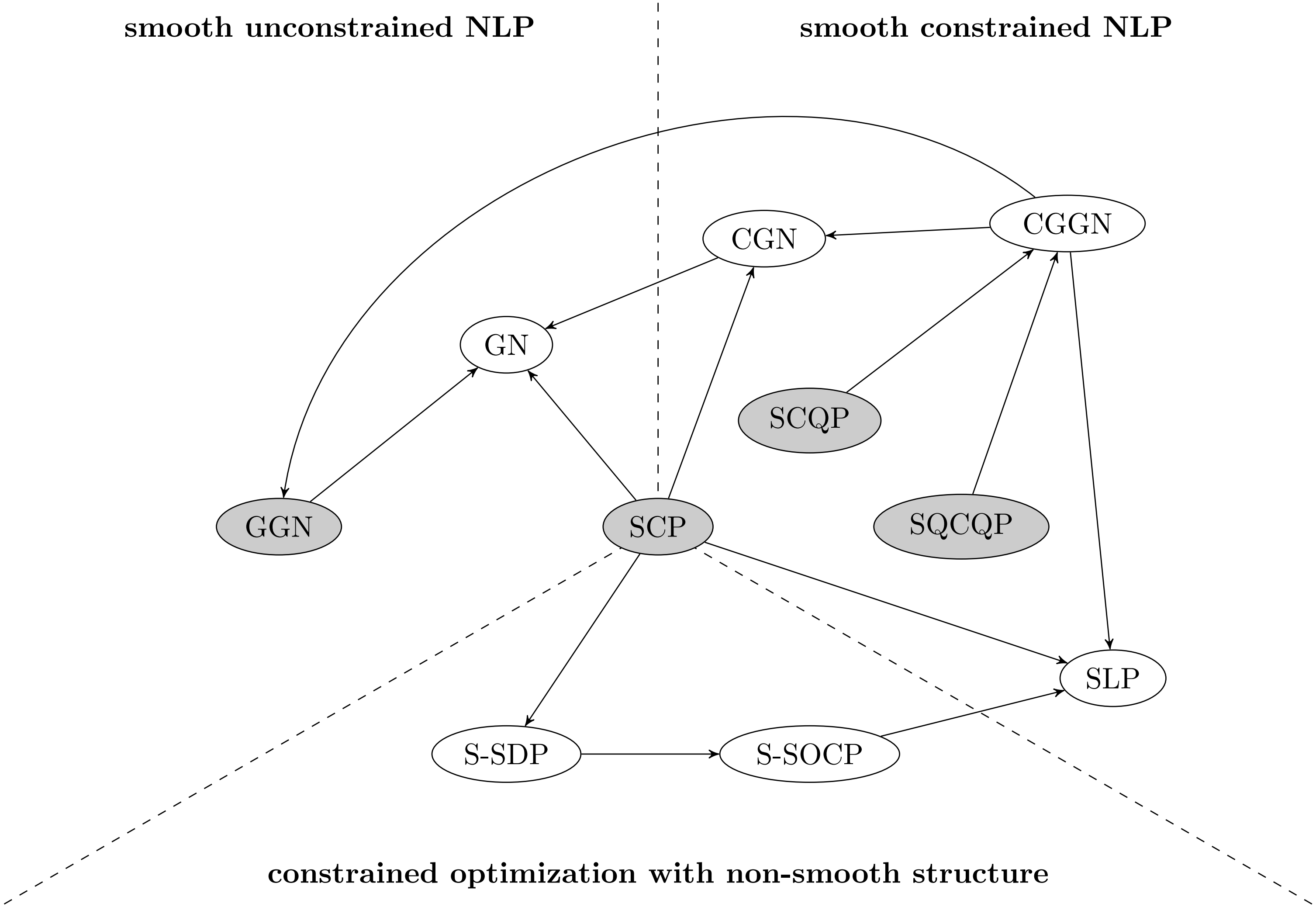
Why is this class of problems and algorithms interesting ?

- many optimization problems have "convex-over-nonlinear" structure
- standard NLP solvers cannot address all non-smooth convex constraints
- there exist many mature and efficient convex optimization solvers

Some application areas:

- nonlinear least squares for estimation and tracking
  [Gauss 1809; Bock 1983; Li and Biegler 1989; Sideris and Bobrow 2004]
- nonlinear matrix inequalities for reduced order controller design
  [Fares, Noll, Apkarian 2002; Tran-Dinh et al. 2012]
- ellipsoidal terminal regions in nonlinear model predictive control
  [Chen and Allgöwer 1998; Verschueren 2016]
- robustified inequalities in nonlinear optimization
  [Nagy and Braatz 2003; D., Bock, Kostina 2006]
- tube-following optimal control problems [Van Duijkeren 2019]
- non-smooth composite minimization [Apkarian et al. 2008; Lewis and Wright 2016]
- deep neural network training with convex loss functions
  [Schraudolph 2002; Martens 2016]

Class Picture of Iterative Convex Approximation Methods



[Messerer and D., in preparation]

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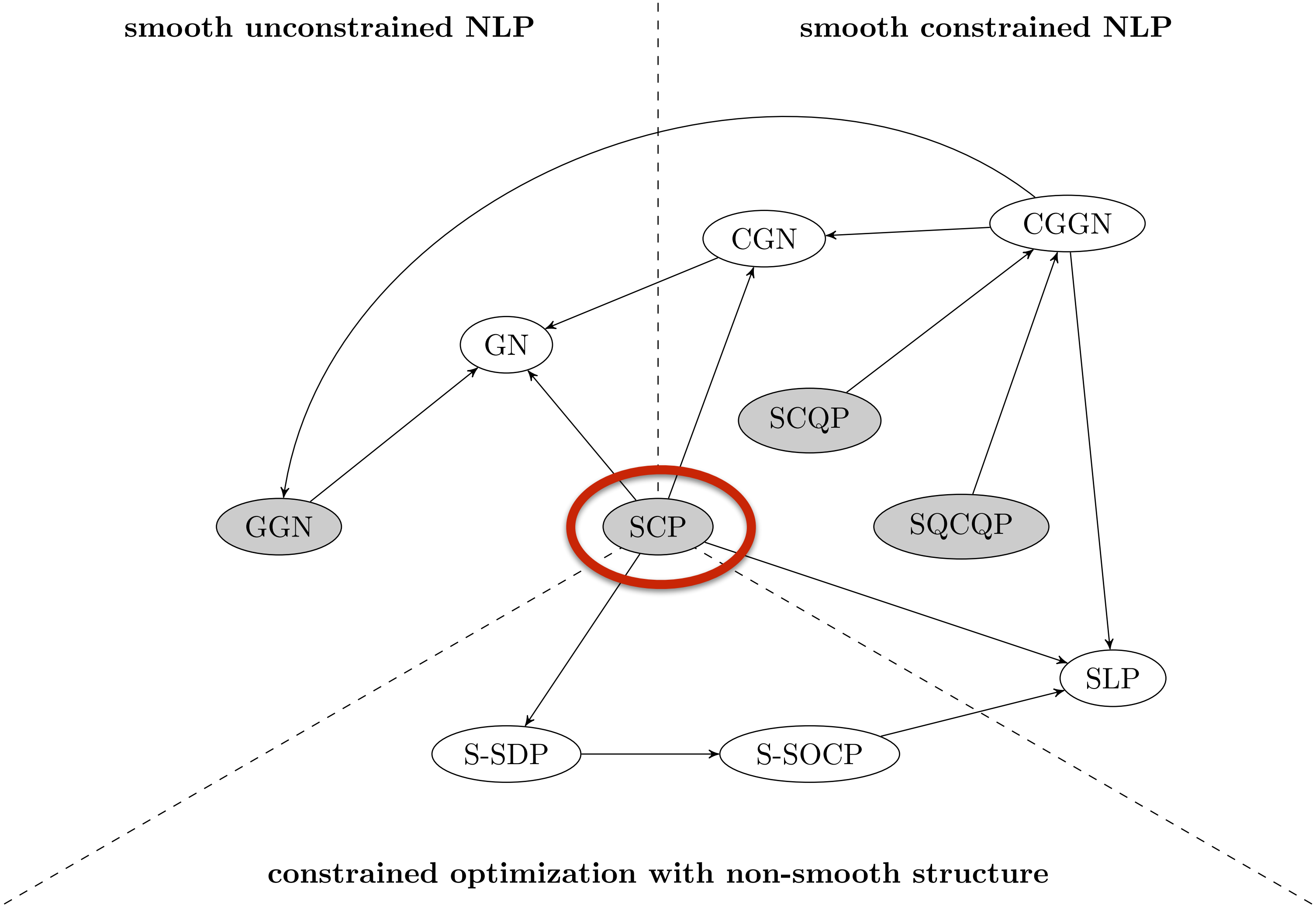

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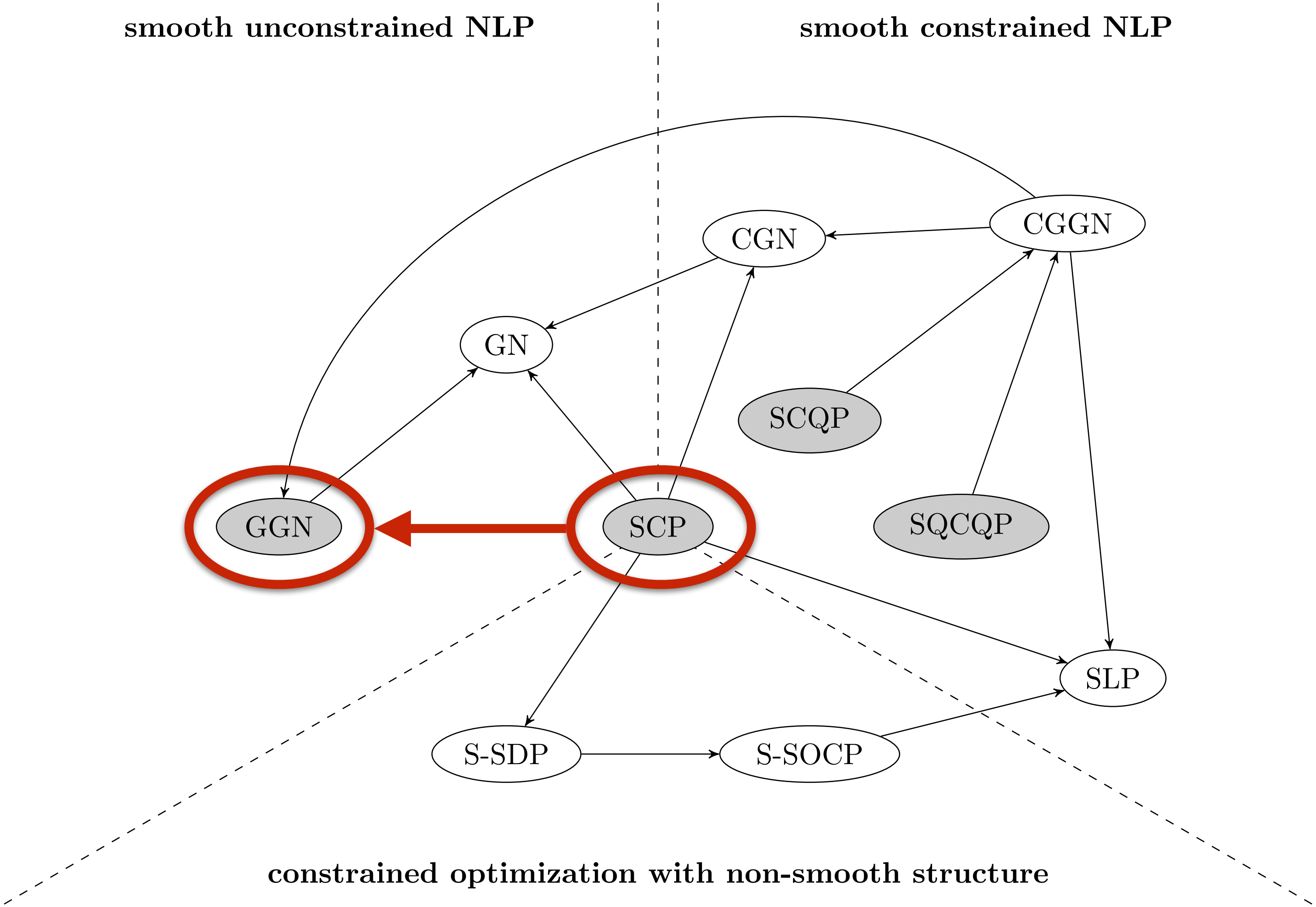

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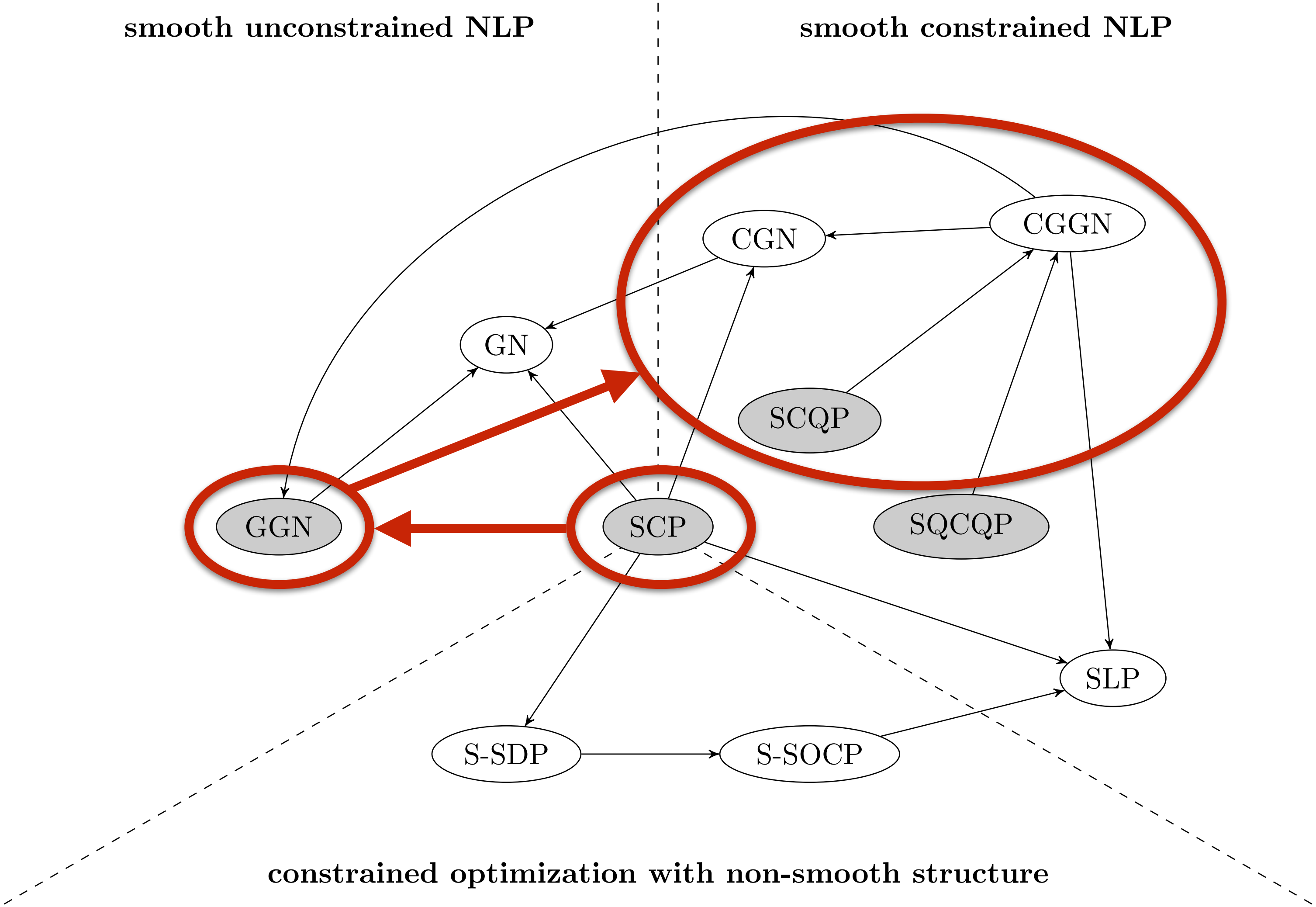

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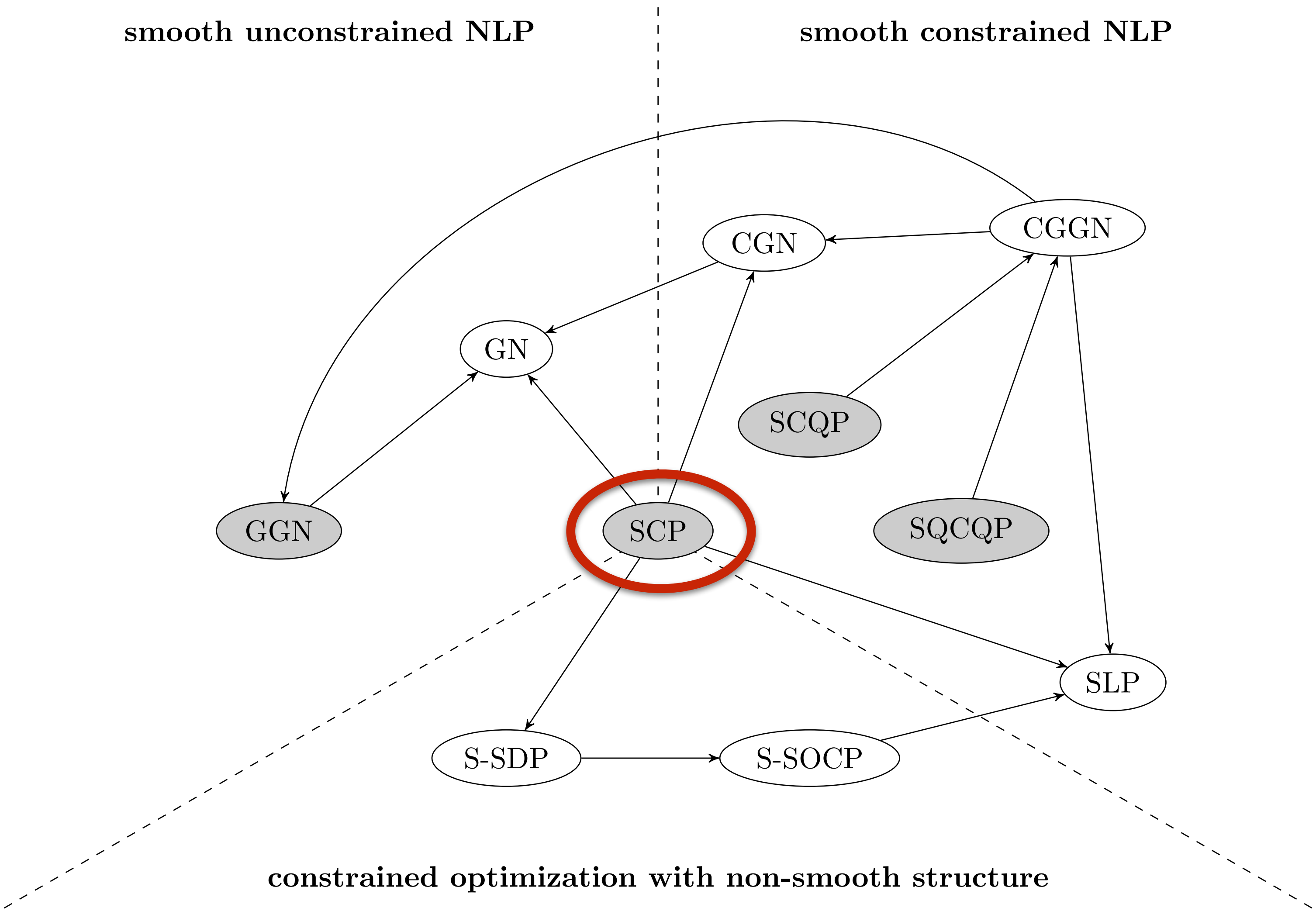

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Sequential Convex Programming (SCP)

- linearize $\boxed{F_i^{\mathrm{lin}}(w;\bar{w}) := F_i(\bar{w}) + J_i(\bar{w})\ (w - \bar{w})}$ with $J_i(\bar{w}) := \frac{\partial F_i}{\partial w}(\bar{w})$
- formulate convex subproblems:

$$
\begin{aligned}
\underset{w \in \mathbb{R}^{n_w}}{\text{minimize}} \quad & \phi_0(F_0^{\mathrm{lin}}(w;\bar{w})) \\
\text{subject to} \quad & F_i^{\mathrm{lin}}(w;\bar{w}) \in \Omega_i, \quad i = 1,\ldots,m, \\
& G^{\mathrm{lin}}(w;\bar{w}) = 0
\end{aligned}
$$

- start at w_0 with $k = 0$
- solve convex subproblem at $\bar{w} = w_k$ to obtain next iterate w_{k+1}

Simplest case: smooth unconstrained problems

Unconstrained minimization of "convex over nonlinear" function

$$\underset{w \in \mathbb{R}^n}{\text{minimize}} \quad \underbrace{\phi(F(w))}_{=:f(w)}$$

Assumptions:
- Inner function $F : \mathbb{R}^n \to \mathbb{R}^N$ of class C^2
- Outer function $\phi : \mathbb{R}^N \to \mathbb{R}$ of class C^2 and convex

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- Inner function $F : \mathbb{R}^n \to \mathbb{R}^N$ of class C^2
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SCP subproblem becomes

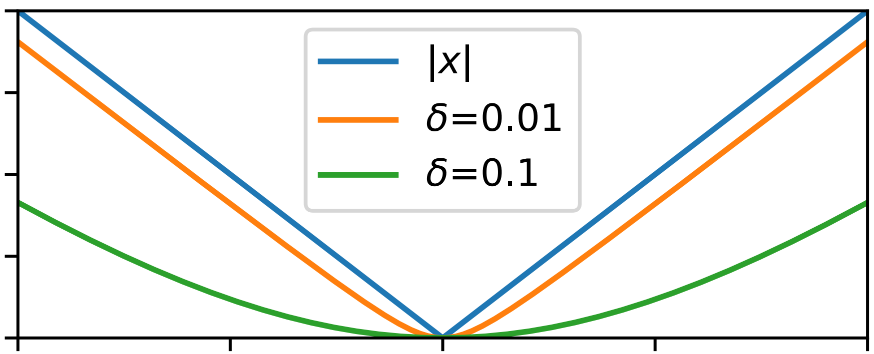
$$\underset{w \in \mathbb{R}^n}{\text{minimize}} \quad \underbrace{\phi\left(F^{\text{lin}}(w; \bar{w})\right)}_{=: f_{\text{SCP}}(w; \bar{w})} \tag{1}$$

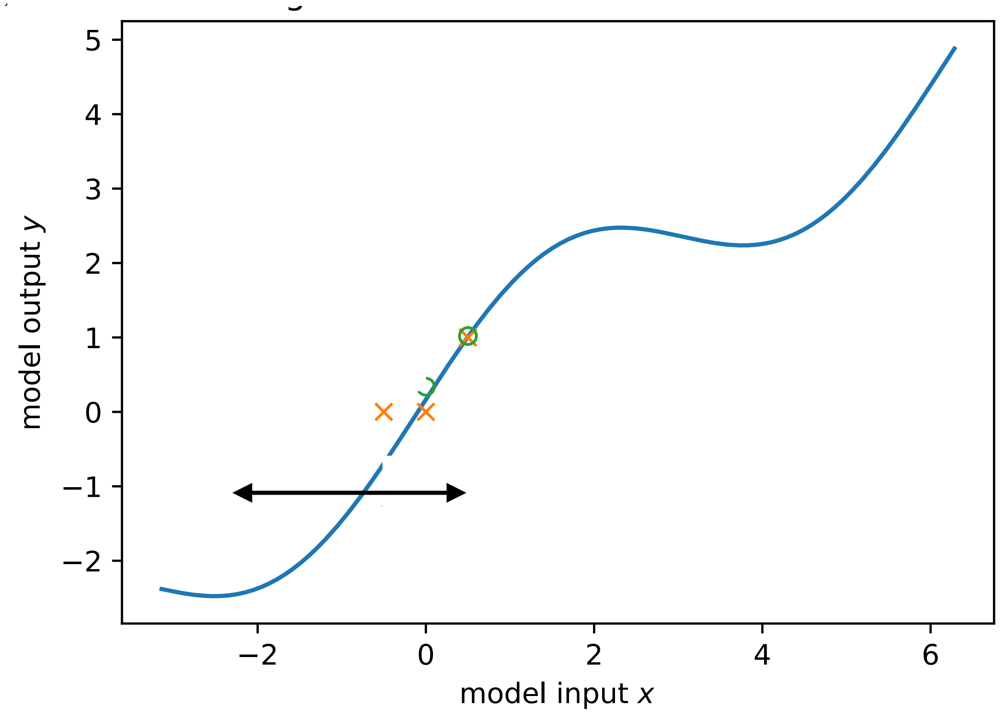
Tutorial Example: Pseudo Huber Loss Minimization

Experiments conducted by Florian Messerer

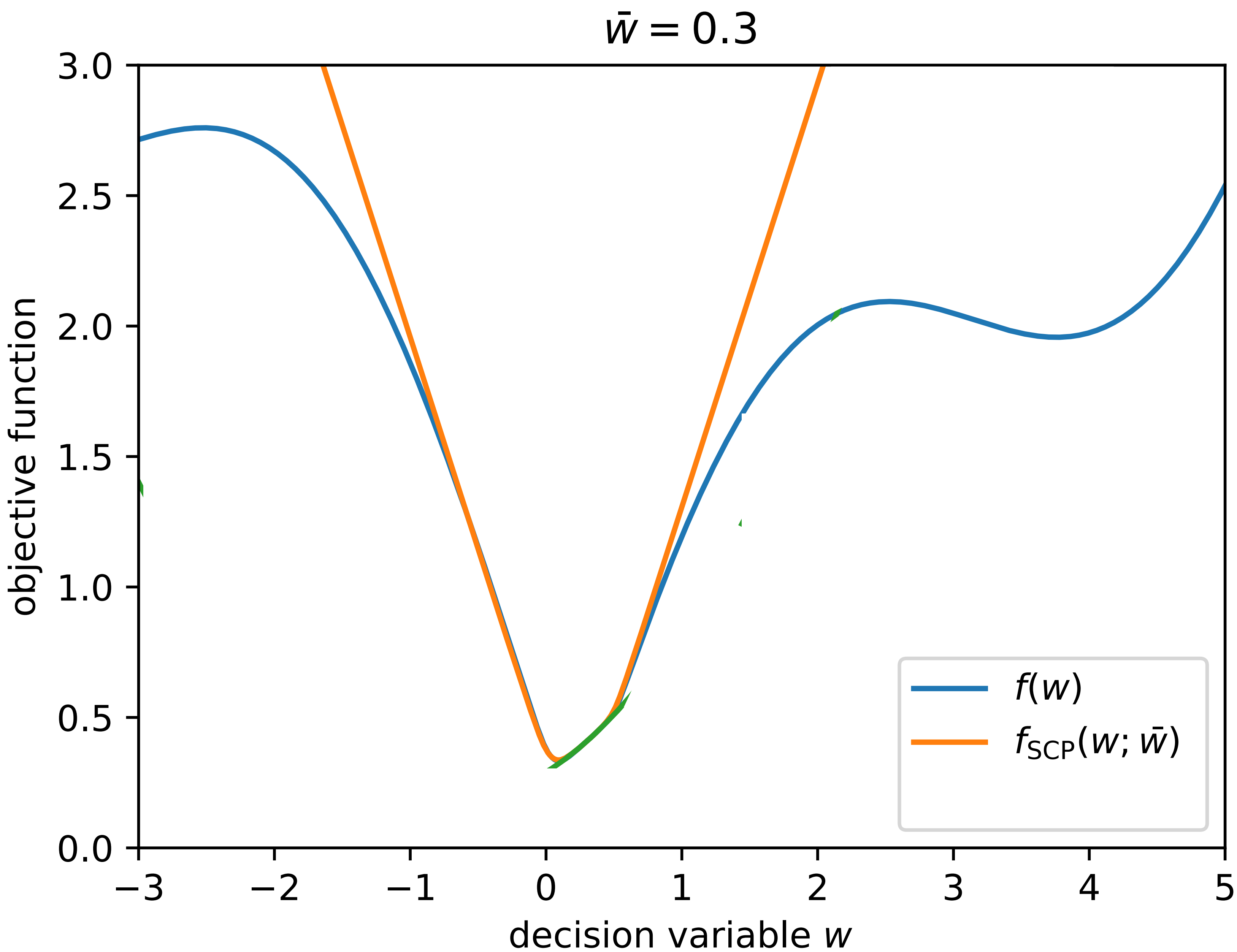
$$\underset{w \in \mathbb{R}}{\text{minimize}} \quad \underbrace{\sum_{i=1}^{n} \phi_\delta(y_i - m(x_i + w))}_{=:f(w)}$$

Aim: fit n=3 measurements y_i to a model $m(w + x_i)$ with $m(x) = \frac{3}{4}x + \sin(x)$ using the pseudo Huber loss $\phi_\delta(x) := \sqrt{\delta^2 + x^2}$





Cost function and SCP approximation



SCP for Least Squares = Gauss-Newton

With quadratic $\phi(z) = \frac{1}{2}\|z\|_2^2 = \frac{1}{2}z^\top z$, SCP subproblems become

$$\underset{w \in \mathbb{R}^n}{\text{minimize}} \quad \frac{1}{2}\|F(w_k) + J(w_k)(w - w_k)\|_2^2 \tag{2}$$

If $\operatorname{rank}(J) = n$ this is uniquely solvable, giving

$$w_{k+1} = w_k - \Big(\underbrace{J(w_k)^\top J(w_k)}_{=:B_{\mathrm{GN}}(w_k)}\Big)^{-1} \underbrace{J(w_k)^\top F(w_k)}_{=\nabla f(w_k)}$$

SCP applied to LS = Newton method with "Gauss-Newton Hessian"

$$B_{\mathrm{GN}}(w) \approx \nabla^2 f(w)$$

Generalized Gauss-Newton (GGN) [Schraudolph 2002]

For general convex $\phi(\cdot)$ we have for $f(w) = \phi(F(w))$

$$\nabla^2 f(w) = \underbrace{J(w)^\top \, \nabla^2 \phi(F(w)) \, J(w)}_{=: B_{\mathrm{GGN}}(w)} + \underbrace{\textstyle\sum_{j=1}^N \nabla^2 F_j(w) \, \nabla_{z_j} \phi(F(w))}_{=: E_{\mathrm{GGN}}(w)}$$

"GGN Hessian" "Error matrix"

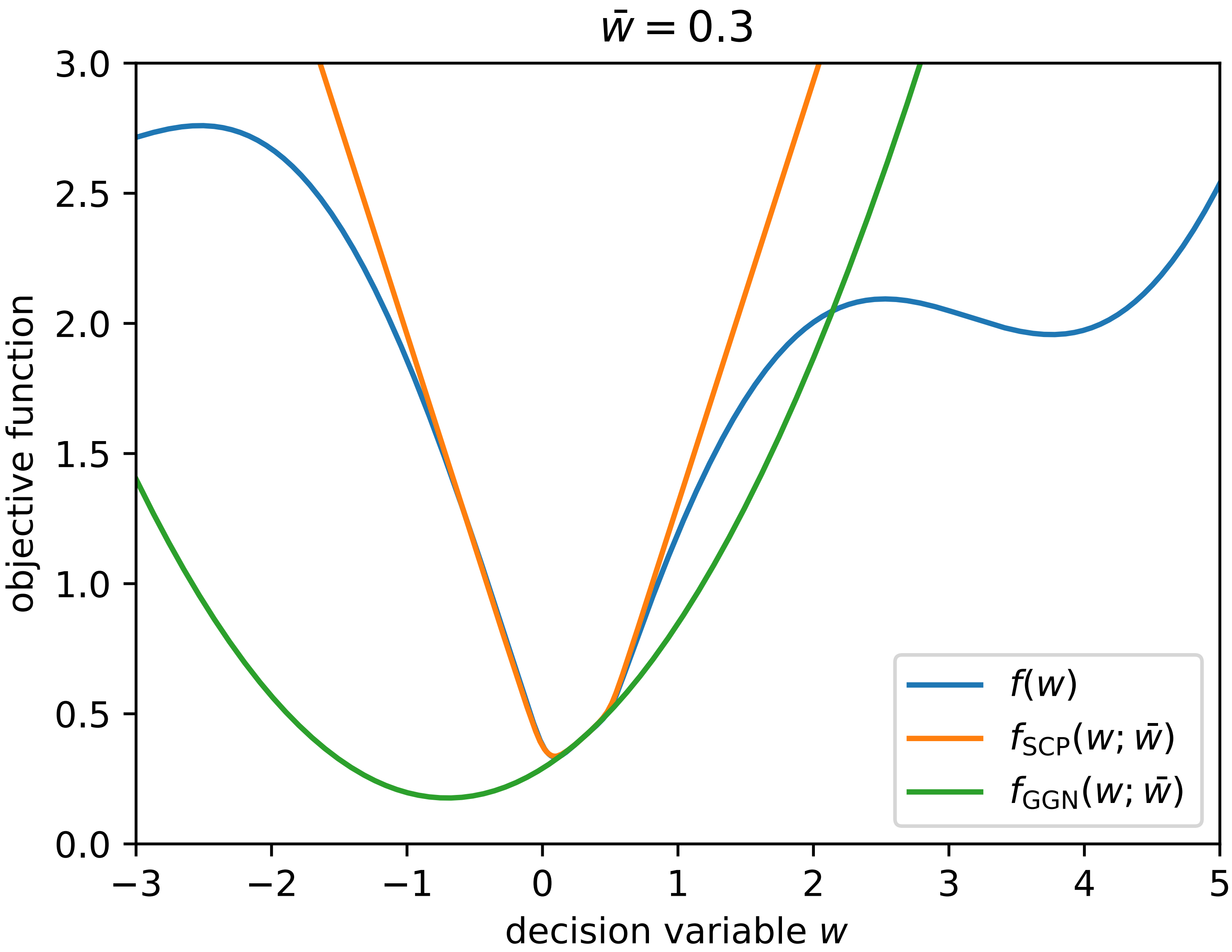
Generalized Gauss-Newton (GGN) method iterates according to

$$w_{k+1} = w_k - B_{\mathrm{GGN}}(w)^{-1} \nabla f(w_k)$$

Note: GGN solves convex quadratic subproblems

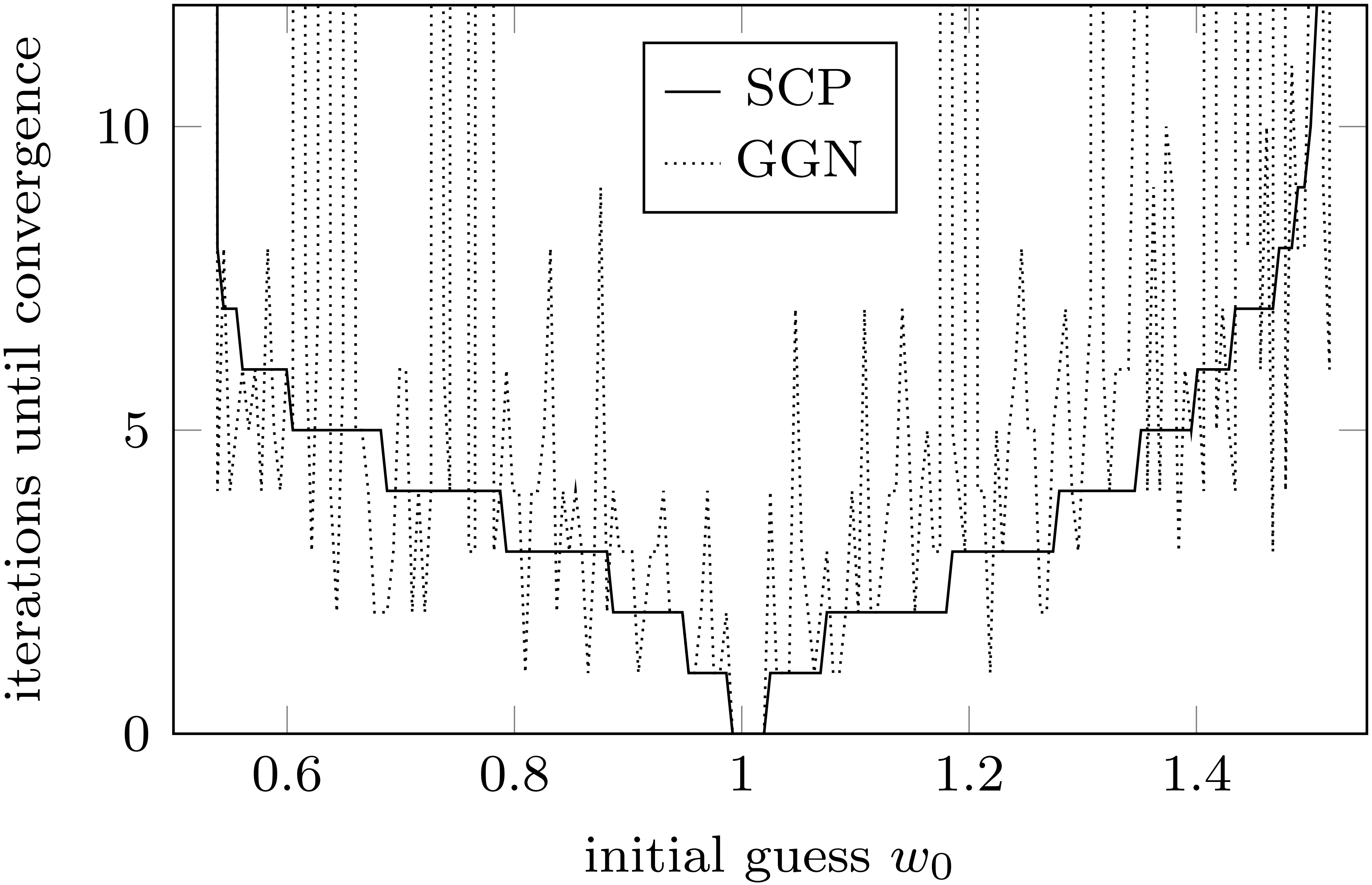
$$\min_{w \in \mathbb{R}^n} \; \underbrace{f(w_k) + \nabla f(w_k)^\top (w - w_k) + \frac{1}{2}(w - w_k)^\top B_{\mathrm{GGN}}(w_k)(w - w_k)}_{=: f_{\mathrm{GGN}}(w; w_k)}$$

Tutorial Example: SCP and GGN Approximation



Iteration count: SCP more predictable than GGN

(on a similar example)



General smooth NLP formulation with constraints

Now regard an NLP with smooth convex $\phi_0, \phi_1, \ldots, \phi_m$

$$\begin{aligned}
\underset{w \in \mathbb{R}^{n_w}}{\text{minimize}} \quad & \underbrace{\phi_0(F_0(w))}_{=: f_0(w)} \\
\text{subject to} \quad & \underbrace{\phi_i(F_i(w))}_{=: f_i(w)} \leq 0, \quad i = 1, \ldots, m, \\
& G(w) = 0
\end{aligned}$$

SCP subproblem becomes

$$\begin{aligned}
\underset{w \in \mathbb{R}^{n_w}}{\text{minimize}} \quad & \phi_0(F_0^{\text{lin}}(w; \bar{w})) \\
\text{subject to} \quad & \phi_i(F_i^{\text{lin}}(w; \bar{w})) \leq 0, \quad i = 1, \ldots, m, \\
& G^{\text{lin}}(w; \bar{w}) = 0
\end{aligned}$$

(SCP algorithm is expensive, but multiplier-free and affine-invariant)

Constrained Gauss-Newton [Bock 1983]

Use $B_{\mathrm{CGN}}(w) := J_0(w)^\top \nabla^2 \phi_0(F_0(w)) J_0(w)$ and solve convex quadratic program (QP)

$$\begin{aligned} \underset{w \in \mathbb{R}^{n_w}}{\text{minimize}} \quad & f_0^{\mathrm{lin}}(w;\bar{w}) + \frac{1}{2}(w-\bar{w})^\top B_{\mathrm{CGN}}(\bar{w})(w-\bar{w}) \\ \text{subject to} \quad & f_i^{\mathrm{lin}}(w;\bar{w}) \leq 0, \quad i = 1,\ldots,m, \\ & G^{\mathrm{lin}}(w;\bar{w}) = 0 \end{aligned}$$

- like SCP, the method is multiplier free and affine invariant
- QPs are potentially cheaper to solve
- but CGN diverges on some problems where SCP converges

Remark: for least-squares objectives, this method is due to [Bock 1983]. In many papers, Bock's method is called "the Generalized Gauss-Newton (GGN) method". To avoid a notation clash with Schraudolph and the computer science literature, we prefer to call Bock's method "the Constrained Gauss-Newton (CGN) method".

Sequential Convex Quadratic Programming (SCQP)
[Verschueren et al 2016]

$$B_{\mathrm{SCQP}}(w,\mu) := J_0(w)^\top \nabla^2 \phi_0(F_0(w)) J_0(w) + \sum_{i=1}^{m} \mu_i J_i(w)^\top \nabla^2 \phi_i(F_i(w)) J_i(w)$$

$$\begin{aligned} \underset{w \in \mathbb{R}^{n_w}}{\text{minimize}} \quad & f_0^{\text{lin}}(w;\bar{w}) + \frac{1}{2}(w-\bar{w})^\top B_{\mathrm{SCQP}}(\bar{w},\bar{\mu})(w-\bar{w}) \\ \text{subject to} \quad & f_i^{\text{lin}}(w;\bar{w}) \leq 0, \quad i = 1,\dots,m, \quad \mid \quad \mu^+, \\ & G^{\text{lin}}(w;\bar{w}) = 0 \end{aligned}$$

- obtain pair (w_{k+1}, μ_{k+1}) from solution at $(\bar{w},\bar{\mu}) = (w_k, \mu_k)$
- "optimizer state" contains both, $\bar{w}$ and inequality multipliers $\bar{\mu}$
- again, only a QP needs to be solved in each iteration
- again, affine invariant
- $B_{\mathrm{SCQP}}(w,\mu) \succeq B_{\mathrm{CGN}}(w)$ (more likely to converge than CGN)
- for unconstrained problems, SCQP becomes GGN
- in fact, SCQP has same contraction rate as SCP [Messerer &D., ECC 2020]

Identical local convergence of SCP and SCQP/GGN

Theorem 1 [Messerer and Diehl, ECC 2020]

Regard KKT point $z^* := (w^*, \mu^*, \lambda^*)$ with LICQ and strict complementarity. Denote the reduced Hessian by $\tilde{\Lambda}_*$, the reduced SCQP Hessian by $\tilde{B}_*$ (*) and assume that $\tilde{B}_* \succ 0$. Then

- z^* is a fixed point for both the SCP and SCQP iterations
- both methods are well-defined in a neighborhood of z^*
- their linear contraction rates are equal and given by the smallest $\alpha \in \mathbb{R}$ that satisfies the linear matrix inequality

$$-\alpha \tilde{B}_* \preceq \tilde{\Lambda}_* - \tilde{B}_* \preceq \alpha \tilde{B}_* \tag{3}$$

(*) $\tilde{\Lambda}_* := Z^\top \nabla^2 \mathcal{L}(w^*, \mu^*, \lambda^*) Z$ and $\tilde{B}_* := Z^\top B_{\text{SCQP}}(w^*, \mu^*) Z$ with Z a fixed nullspace basis of the Jacobian of active constraints

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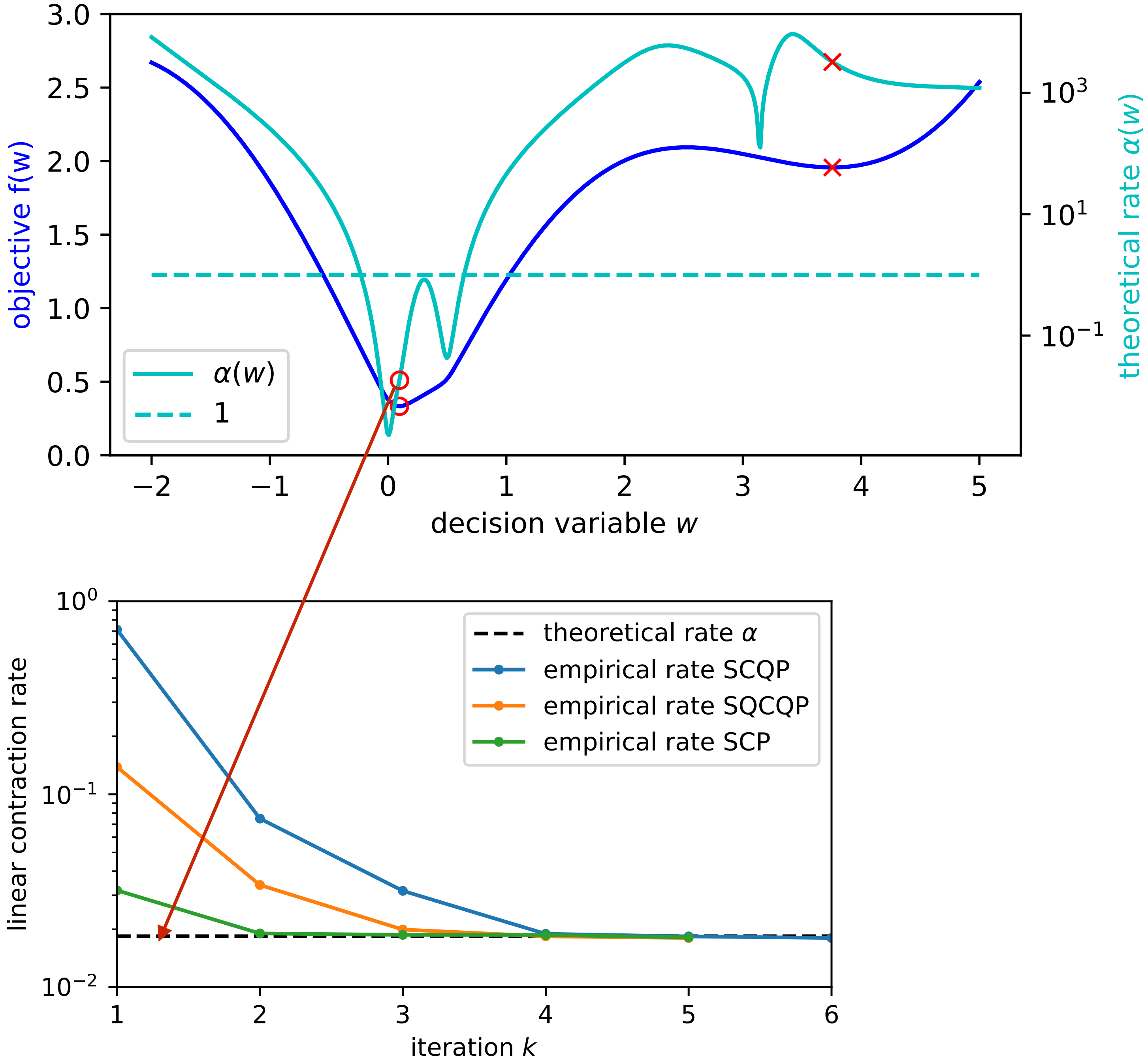
Corollary

Necessary condition for local convergence of both methods is $\tilde{B}_* \succeq \frac{1}{2}\tilde{\Lambda}_* \succeq 0$

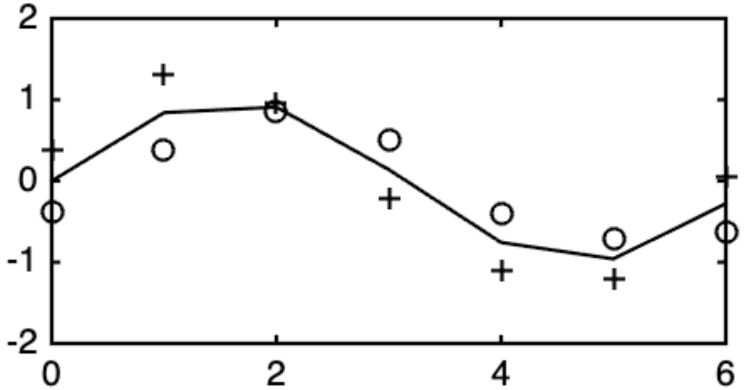
Proof of corollary: Set $\alpha = 1$ in (3).

Tutorial Example: Objective and Local Contraction Rate



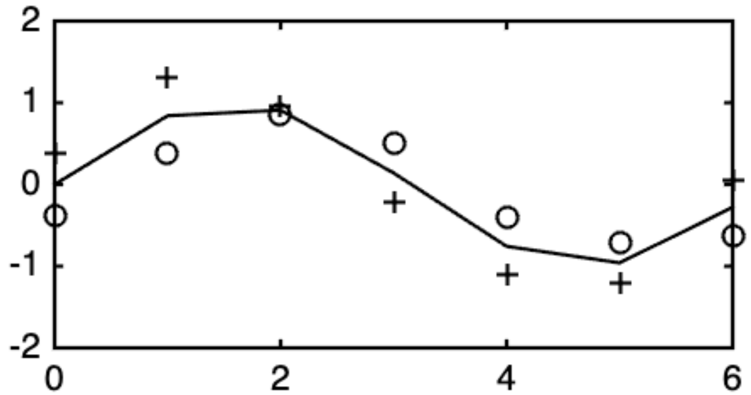
Desirable Divergence and Mirror Problem [cf. Bock 1987]

SCP and GGN do not converge to every local minimum. This can help to avoid "bad" local minima, as discussed next.



Regard maximum likelihood estimation problem $\boxed{\min_w \phi(M(w) - y)}$ with nonlinear model $M : \mathbb{R}^n \to \mathbb{R}^N$ and measurements $y \in \mathbb{R}^N$. Assume penalty ϕ is symmetric with $\phi(-z) = \phi(z)$ as is the case for symmetric error distributions. At a solution w^*, we can generate "mirror measurements" $y_{\mathrm{mr}} := 2M(w^*) - y$ obtained by reflecting the residuals.
From a statistical point of view, y_{mr} should be as likely as y.

SCP divergence ⇔ minimum unstable under mirroring



Theorem [Messerer and D., 2019/2020] generalizing [Bock 1987]

Regard a local minimizer w^* of $\phi(M(w) - y)$ that satisfies SOSC. If the necessary SCP convergence condition $\tilde{B}_* \succeq \frac{1}{2}\tilde{\Lambda}_*$ does not hold, then w^* is a stationary point of the mirror problem but **not** a local minimizer.

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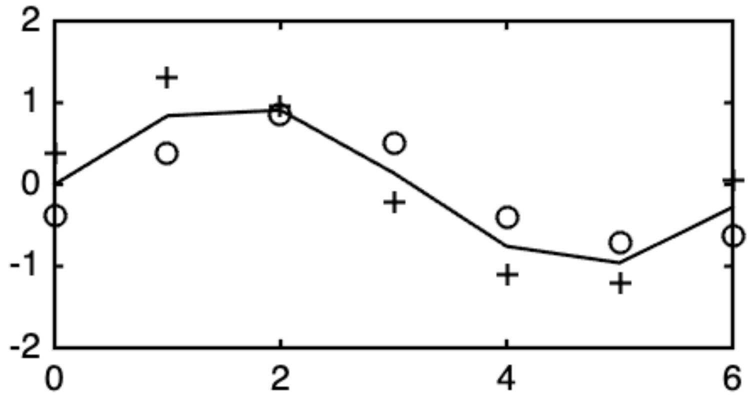

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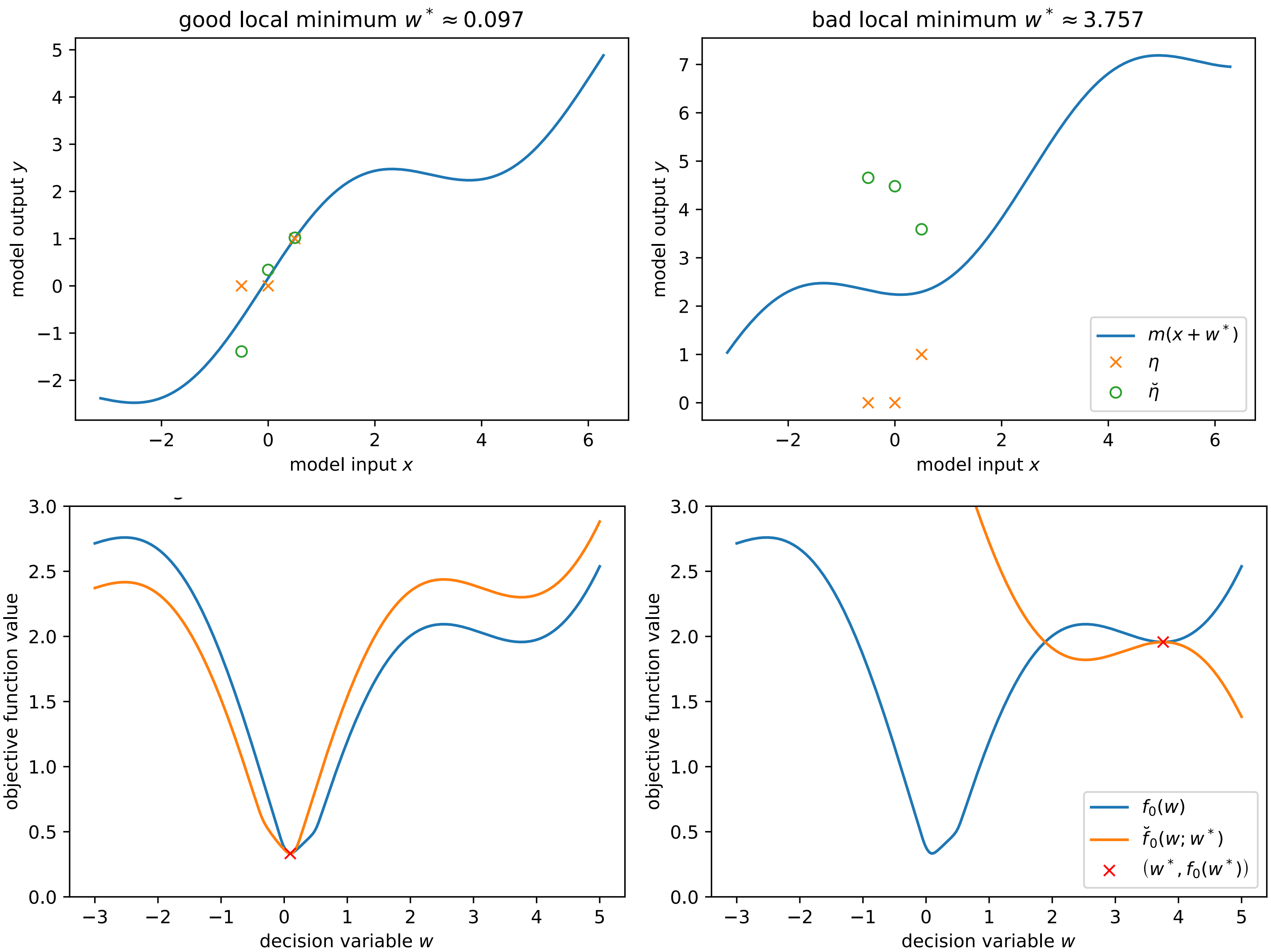
*Sketch of proof (unconstrained): use $M(w^*) - y_{\mathrm{mr}} = y - M(w^*)$ to show that $\nabla f_{\mathrm{mr}}(w^*) = J(w^*)^\top (y - M(w^*)) = 0$ and
$\nabla^2 f_{\mathrm{mr}}(w^*) = B_{\mathrm{GGN}}(w^*) - E_{\mathrm{GGN}}(w^*) = 2B_{\mathrm{GGN}}(w^*) - \nabla^2 f(w^*) \not\succeq 0$

Tutorial Example and Mirror Problems at Different Local Minima



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Two Ingredients of Newton-Type Optimization

The convexity exploiting algorithms presented so far need two ingredients:

1. a good nonlinear model and its linearisation, and
2. convex substructure in objective and constraints

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Which of the two is more important for success in data-driven optimization?

Two Ingredients of Newton-Type Optimization

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1. a good nonlinear model and its linearisation, and
2. **convex substructure** in objective and constraints

Which of the two is more important for success in data-driven optimization?

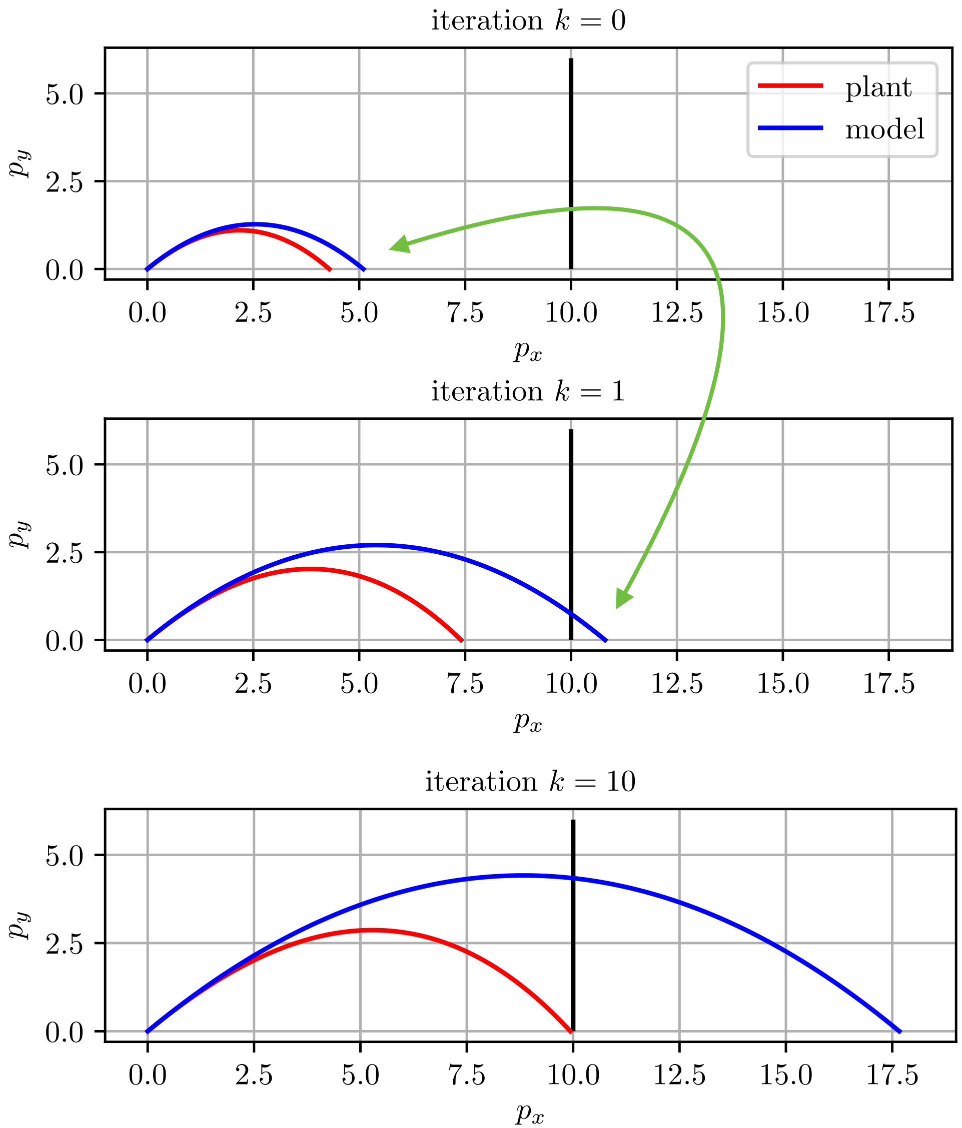
Iterative Learning Control for Lemon-Ball Throwing

Iterative Learning of Ball Throwing with Minimal Energy

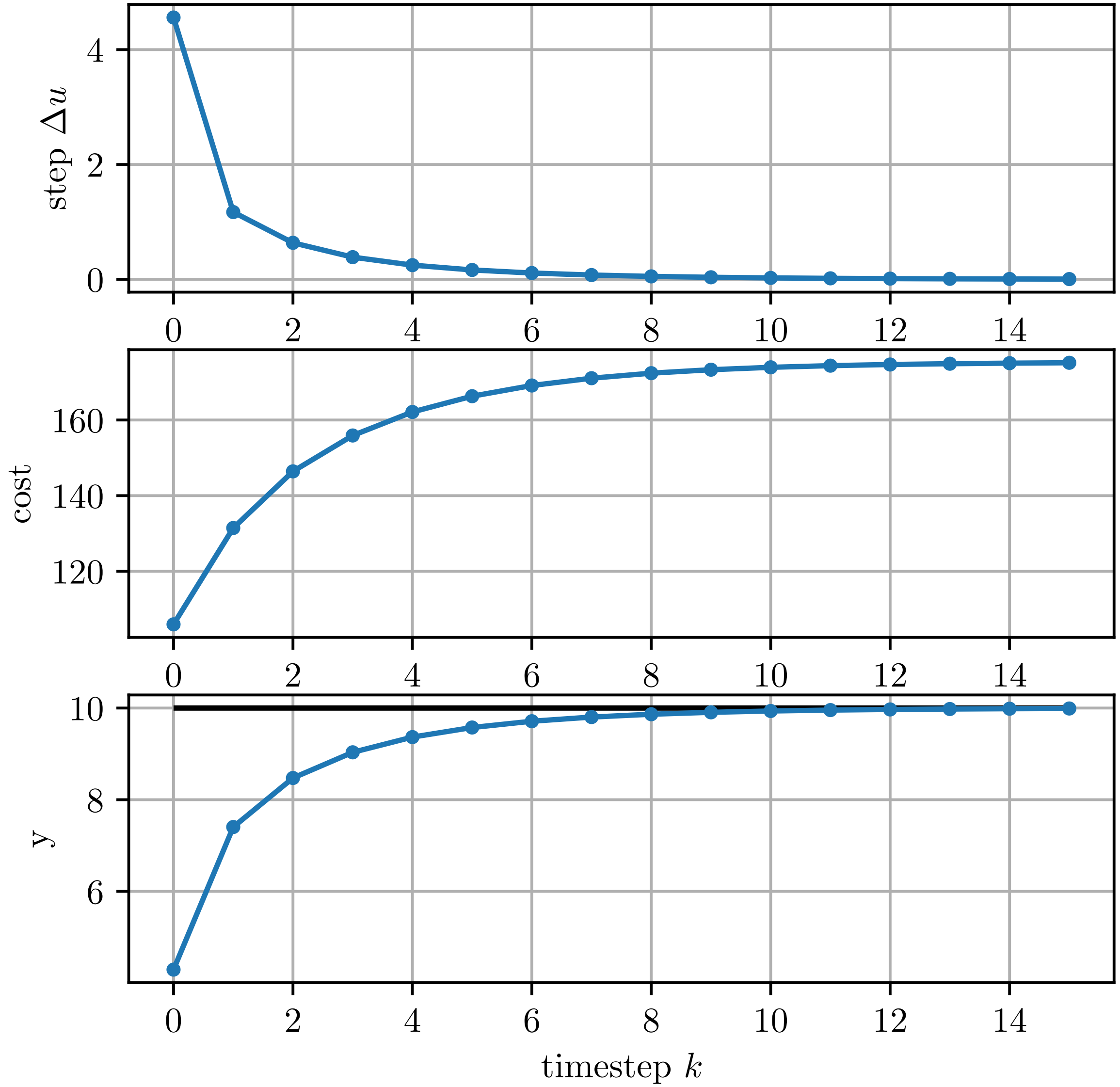
Experiments conducted by Katrin Baumgärtner

- Model $F_{\mathrm{M}}(u)$ maps initial velocity $u \in \mathbb{R}^2$ to landing position $y \in \mathbb{R}$
- Aim: throw ball further than $y \geq 10$ with minimal initial velocity
- Experiments with "real plant" give pairs (u_k, y_k) [shorter distance than predicted]
- We can use (u_k, y_k) to correct the model, and iteratively obtain u_{k+1} by solving the following optimization problem:

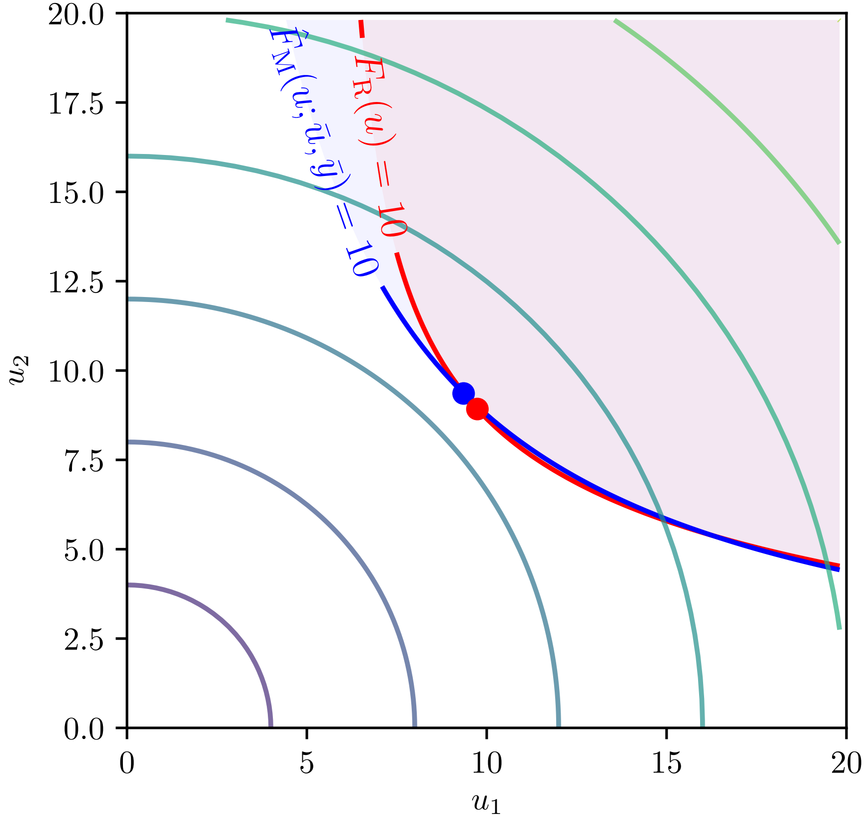
$$\begin{aligned} \underset{u \in \mathbb{R}^2, y \in R}{\text{minimize}} \quad & \|u\|_2^2 \\ \text{subject to} \quad & y = \underbrace{F_{\mathrm{M}}(u) - F_{\mathrm{M}}(u_k) + y_k}_{=:\tilde{F}_M(u; u_k, y_k)}, \\ & y \geq 10 \end{aligned}$$



Iterations of Algorithm and Reduced Problem Visualization



$$\begin{array}{ll} \underset{u \in \mathbb{R}^2}{\text{minimize}} & \|u\|_2^2 \\ \text{subject to} & \tilde{F}_M(u; u_k, y_k) \geq 10 \end{array}$$



Zero Order Optimization-based Iterative Learning Control (ZOO-ILC)

Aim: optimization with unknown input-output system $y = F_{\mathrm{R}}(u)$ ("reality"):

$$\begin{aligned} \underset{u,\,y}{\text{minimize}} \quad & \phi(u,y) \\ \text{subject to} \quad & F_{\mathrm{R}}(u) - y = 0, \\ & H(u,y) \leq 0 \end{aligned} \tag{4}$$

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ZOO-ILC idea [cf. Schöllig, Volkaert, Zeilinger]: use trial input u_k with output y_k and a model F_{M} to obtain new trial input u_{k+1} from solution of

$$\begin{aligned} \underset{u,\,y}{\text{minimize}} \quad & \phi(u,y) \\ \text{subject to} \quad & F_{\mathrm{M}}(u) - y = F_{\mathrm{M}}(u_k) - y_k, \\ & H(u,y) \le 0 \end{aligned} \tag{5}$$

Questions: Does this method converge? What is its loss of optimality?

Feasibility and Loss of Optimality of ZOO-ILC

Theorem 2 [Baumgärtner et al., in preparation]

For any fixed point $(\bar{u}, \bar{y})$ of the ZOO-ILC algorithm with multipliers $(\bar{\lambda}, \bar{\mu})$ holds under mild conditions:

- $(\bar{u}, \bar{y})$ is feasible for the real problem
- the loss of optimality compared to a real solution $(u_\mathrm{R}, y_\mathrm{R})$ is bounded by:

$$\phi(\bar{u}, \bar{y}) - \phi(u_\mathrm{R}, y_\mathrm{R}) \leq \bar{\lambda}^\top \left(J_\mathrm{M}(\bar{u}) - J_\mathrm{R}(\bar{u})\right)(u_\mathrm{R} - \bar{u})$$

Here, the Lagrangian of the model problem is given by

$$\mathcal{L}(u, y, \lambda, \mu) = \phi(u, y) + \lambda^\top (F_\mathrm{M}(u) - y - b_k) + \mu^\top H(u, y)$$

and $J_\mathrm{M}(u)$ and $J_\mathrm{R}(u)$ are the Jacobians of $F_\mathrm{M}(u)$ and $F_\mathrm{R}(u)$.

Special cases where ZOO-ILC delivers a lossless solution

$$\boxed{\phi(\bar{u},\bar{y}) - \phi(u_\mathrm{R}, y_\mathrm{R}) \le \bar{\lambda}^\top \left(J_\mathrm{M}(\bar{u}) - J_\mathrm{R}(\bar{u})\right)(u_\mathrm{R} - \bar{u})}$$

ZOO-ILC delivers lossless solution in the following three cases:

1. Tracking ILC with zero residual (standard ILC):
   $$\bar{\lambda} = 0$$

2. Model and real Jacobian coincide at solution (rarely the case):
   $$J_\mathrm{M}(\bar{u}) - J_\mathrm{R}(\bar{u}) = 0$$

3. Constrained problems where solution u_R is in vertex of the reduced feasible set:
   $$u_\mathrm{R} - \bar{u} = 0$$
   (if the Jacobian error is small enough, LICQ and strict complementarity hold)

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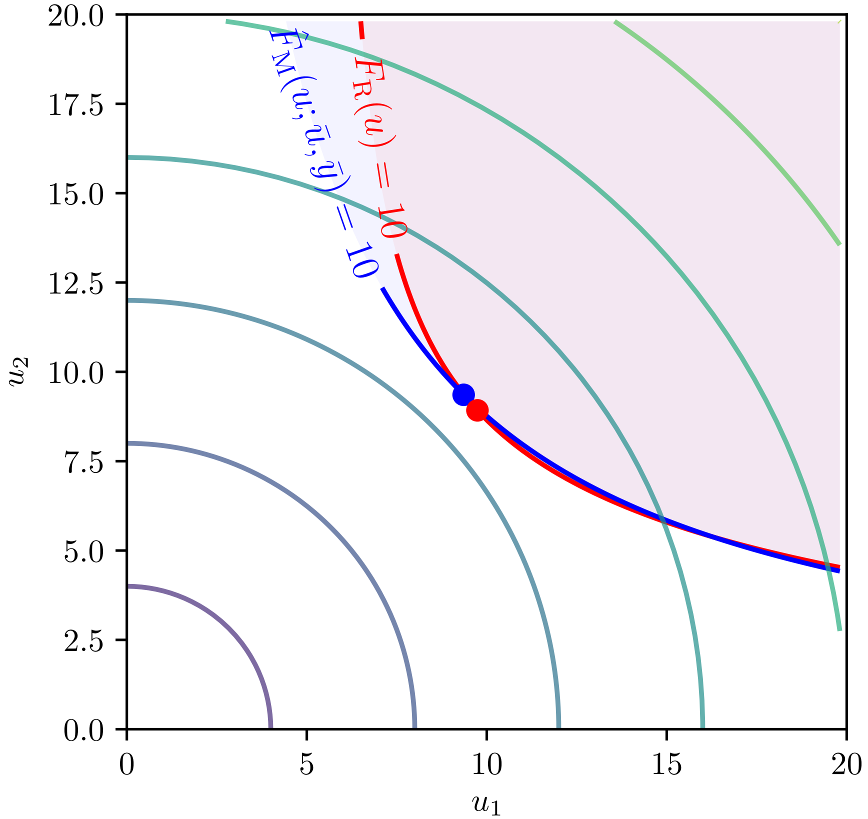
Solutions for L_2- and L_∞-norm minimisation

$$\begin{aligned} \underset{u \in \mathbb{R}^2}{\text{minimize}} \quad & \|u\|_2^2 \\ \text{subject to} \quad & \tilde{F}_M(u; u_k, y_k) \geq 10 \end{aligned}$$



suboptimality: $0.874 \leq 1.377$ (bound)

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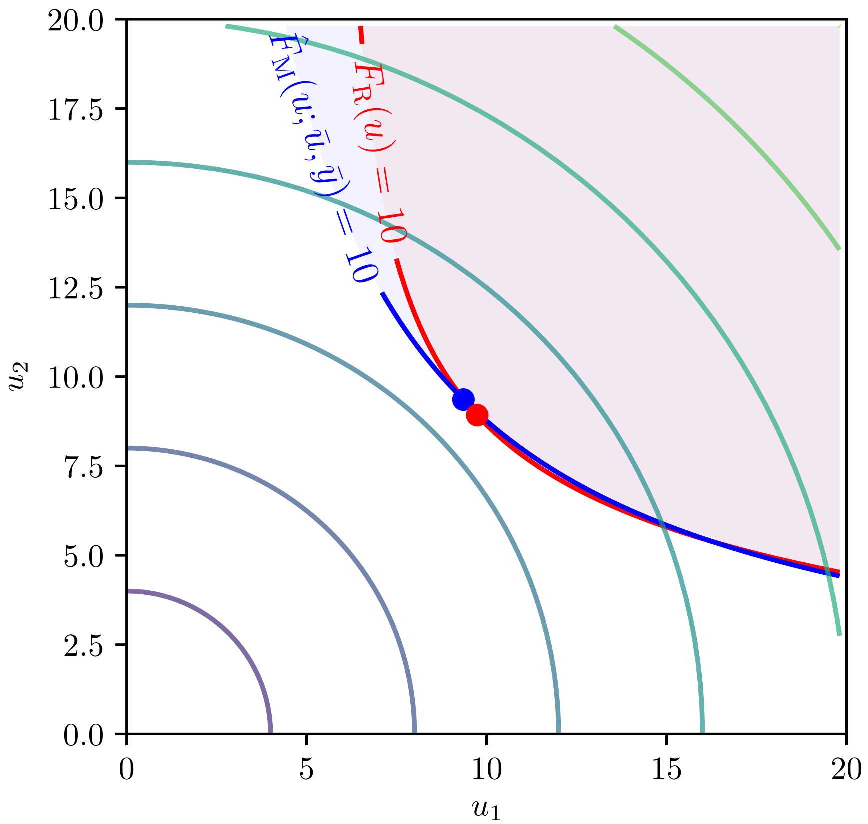

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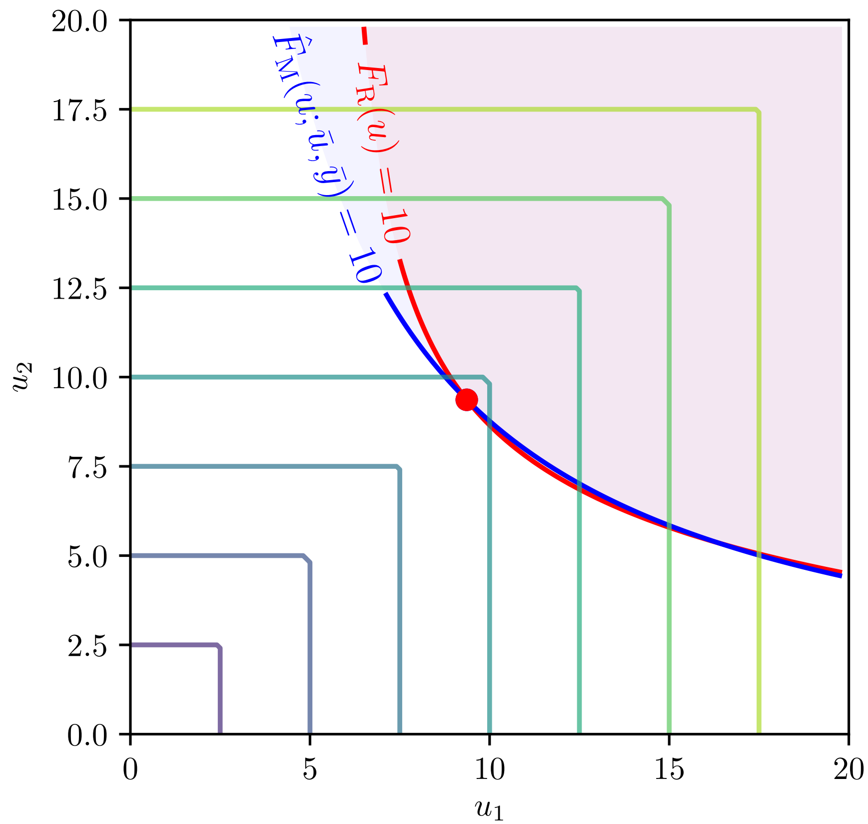


solution in vertex, no loss of optimaliy

Time-Optimal Motion of an oscillator (L1-tracking)

Real plant: $T^2\ddot{y} + 2Td\dot{y} + y + \beta y^3 = K_{\mathrm{R}}u$

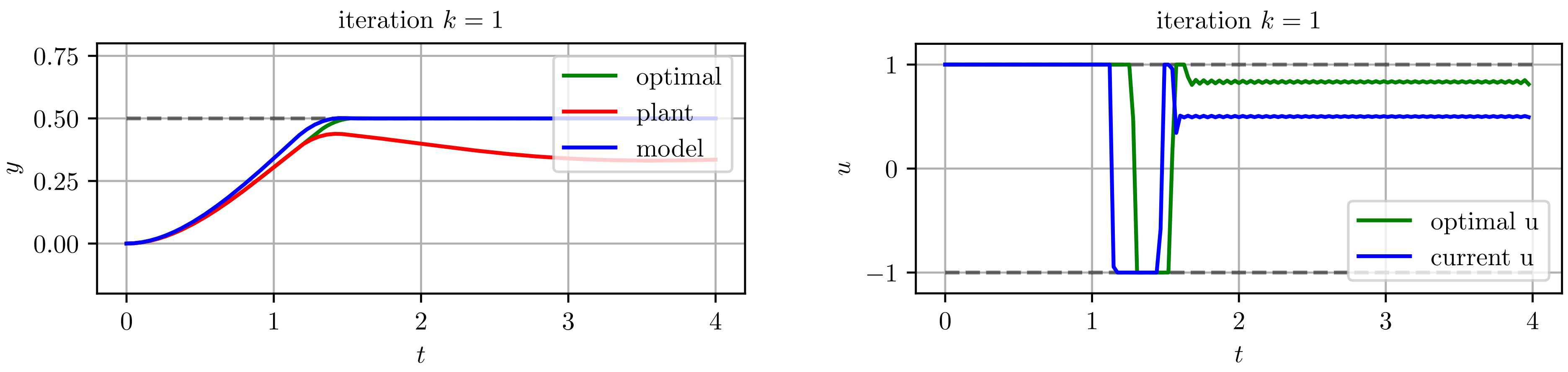
with $T = 1$, $d = 0.5$, $\beta = 2$, $K_{\mathrm{R}} = 0.9$

Model: $T^2\ddot{y} + 2Td\dot{y} + y = K_{\mathrm{M}}u$

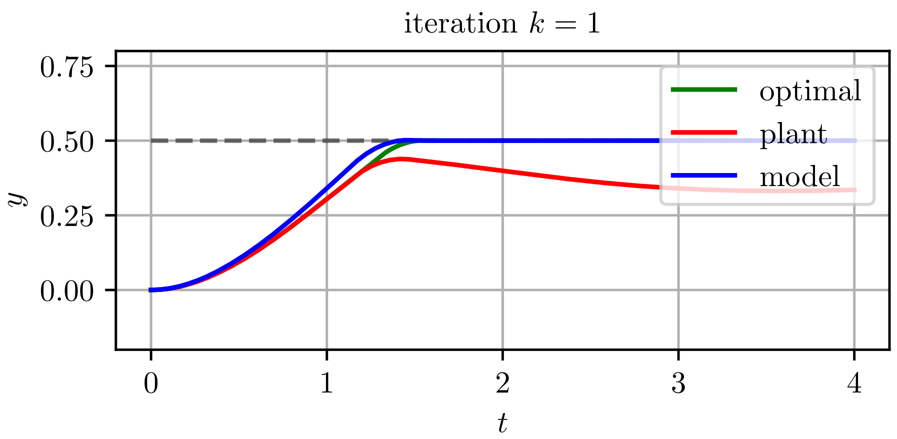
with $K_{\mathrm{M}} = 1$

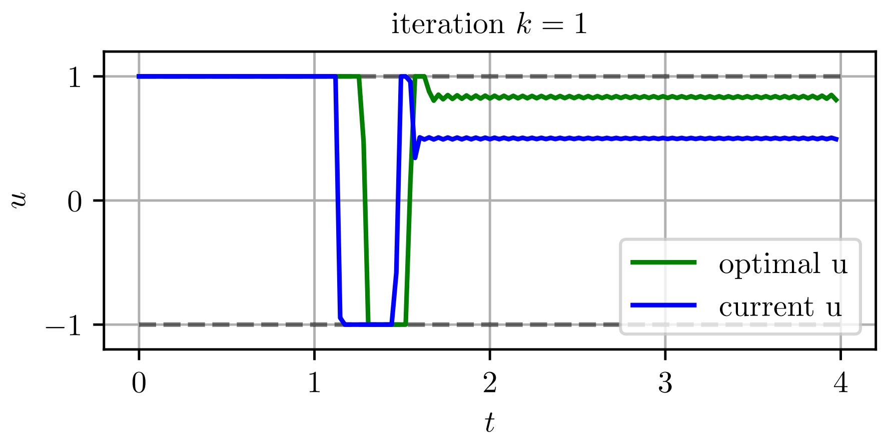
$$
\begin{aligned}
\underset{y(\cdot),\,u(\cdot)}{\text{minimize}} \quad & \int_0^{T_{\mathrm{H}}} |y(t) - y_{\mathrm{ref}}| + \alpha u(t)^2 \;\mathrm{d}t \\
\text{subject to} \quad & y(t) = F_{\mathrm{M}}(t;u) + y_k(t) - F_{\mathrm{M}}(t;u_k), \\
& |u(t)| \le 1, \quad t \in [0, T_{\mathrm{H}}]
\end{aligned}
$$

$\alpha = 10^{-4}$



Time-Optimal Motion of an oscillator (L1-tracking)





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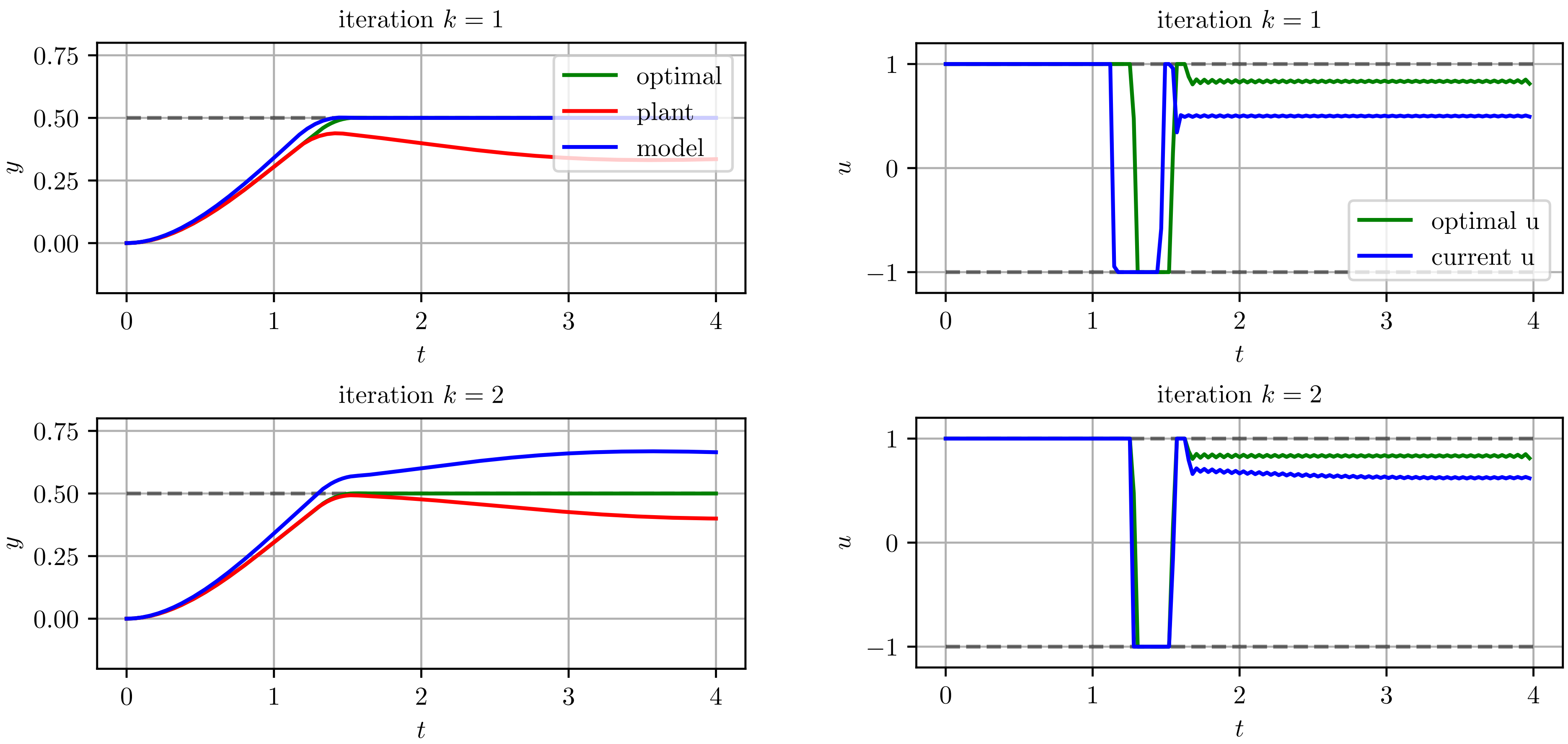

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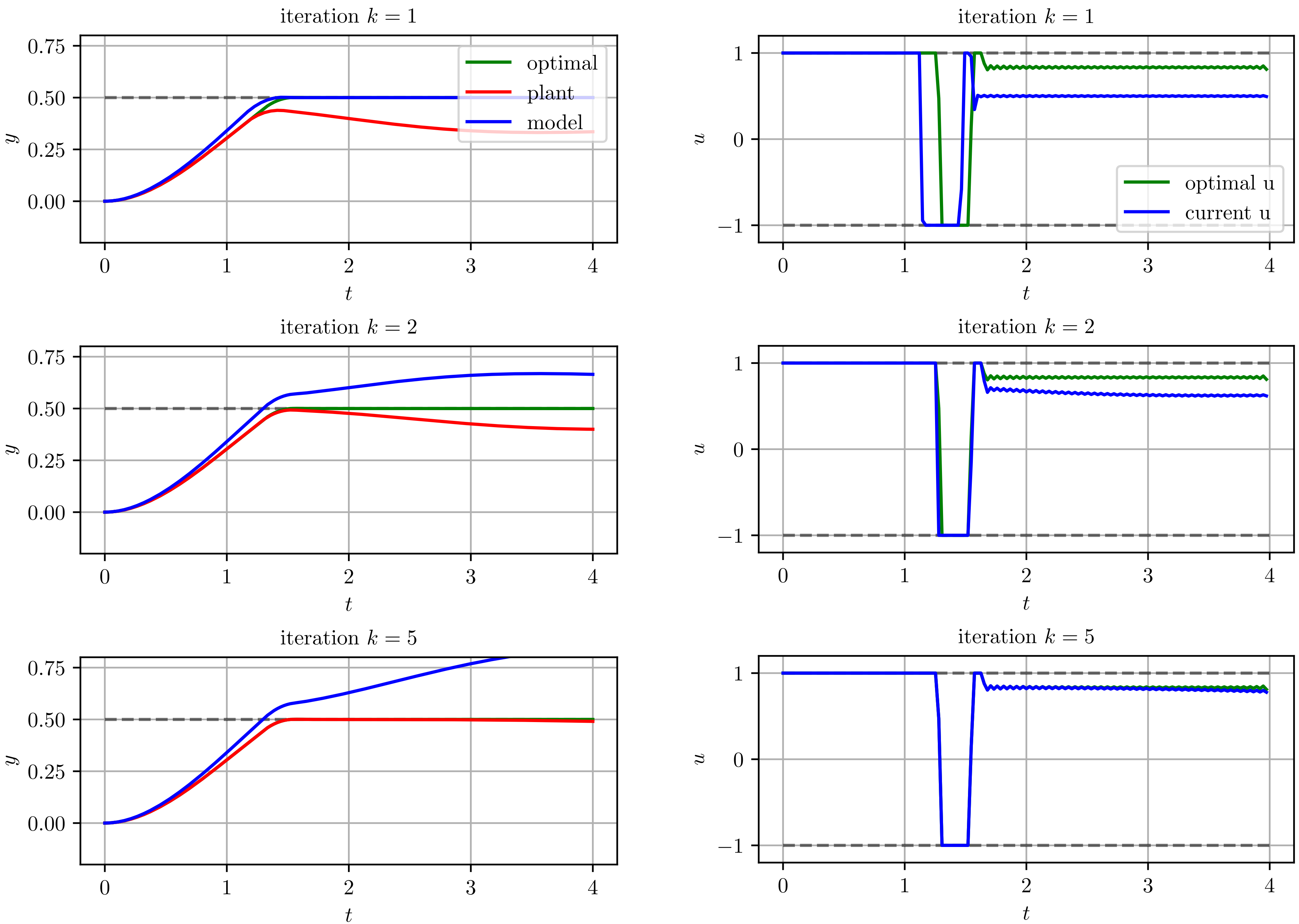

When does the ZOO-ILC method converge?

Theorem 3 (Convergence of ZOO-ILC) [Baumgärtner et al., in preparation]

Regard a fixed point $\bar{z} = (\bar{u}, \bar{y}, \bar{\lambda}, \bar{\mu}_{\mathcal{A}})$ of ZOO-ILC and assume it satisfies LICQ, SOSC and strict complementarity in the model problem. Then the local contraction rate is given by the spectral radius $\rho(A)$ of the matrix

$$A \quad := \quad [\mathbb{I}_{n_u} \quad 0 \quad 0 \quad 0] \left(\frac{\partial R}{\partial z}(\bar{z}; \bar{u}, \bar{y}) \right)^{-1} \begin{bmatrix} 0 \\ 0 \\ J_{\mathrm{M}}(\bar{u}) - J_{\mathrm{R}}(\bar{u}) \\ 0 \end{bmatrix}$$

The ZOO-ILC method converges if $\rho(A) < 1$ and diverges if $\rho(A) > 1$.

Here, $\mu_{\mathcal{A}}$ are the active constraint multipliers and $R(z; u', y')$ is defined by

$$R(z; u', y') := \begin{bmatrix} \nabla_u \mathcal{L}_{\mathrm{M}}(u, y, \lambda, \mu_{\mathcal{A}}; u', y') \\ \nabla_y \mathcal{L}_{\mathrm{M}}(u, y, \lambda, \mu_{\mathcal{A}}; u', y') \\ F_{\mathrm{M}}(u) - y + y' - F_{\mathrm{M}}(u') \\ H_{\mathcal{A}}(u, y) \end{bmatrix}$$

where the Lagrangian of the model problem is given by

$$\mathcal{L}_{\mathrm{M}}(u, y, \lambda, \mu_{\mathcal{A}}; u', y') = \phi(u, y) + \lambda^\top (F_{\mathrm{M}}(u) - y + y' - F_{\mathrm{M}}(u')) + \mu_{\mathcal{A}}^\top H_{\mathcal{A}}(u, y)$$

and $J_{\mathrm{M}}(u)$ and $J_{\mathrm{R}}(u)$ are the Jacobians of $F_{\mathrm{M}}(u)$ and $F_{\mathrm{R}}(u)$.

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$$\begin{bmatrix} 0 \\ 0 \\ J_{\mathrm{M}}(\bar{u}) - J_{\mathrm{R}}(\bar{u}) \\ 0 \end{bmatrix}$$

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Contraction rate grows with distance between model and real Jacobian.

Summary and Recent Software Developments

- Exploiting convex structures in nonlinear problems is key for reliable and fast nonlinear MPC algorithms.

- Sequential Convex Programming (SCP) and its variants converge linearly. They avoid “bad” minimizers (where the nonlinearity dominates the convex substructure).

- Zero-Order Optimization allows us to design theoretically solid Iterative Learning Control algorithms. They can recover an optimal solution in special cases.

- Latest open-source (BSD 3) software developments from the team are:
  - BLASFEO: Basic Linear Algebra Subroutines For Embedded Optimization (Frison et al.), targeting dense matrices from 10x10 to 400x400
  - HPIPM: interior point QP/QCQP solver for block-sparse problems with optimal control and tree structure, based on BLASFEO (Frison et al., IFAC 2020)
  - acados: Nonlinear MPC and MHE library implementing SCP type algorithms, using HPIPM and CasADi, with user interfaces from MATLAB and Python (Verschueren, Kouzoupis, Frison, Frey et al., successor of ACADO)

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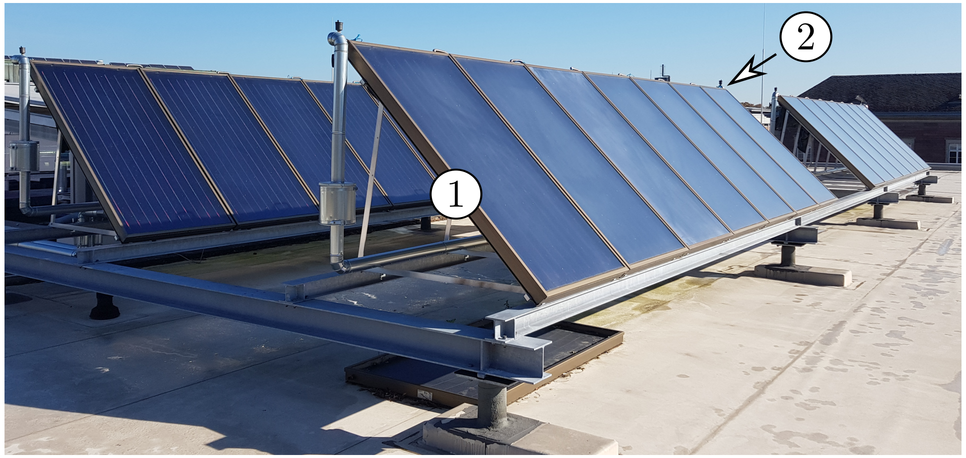
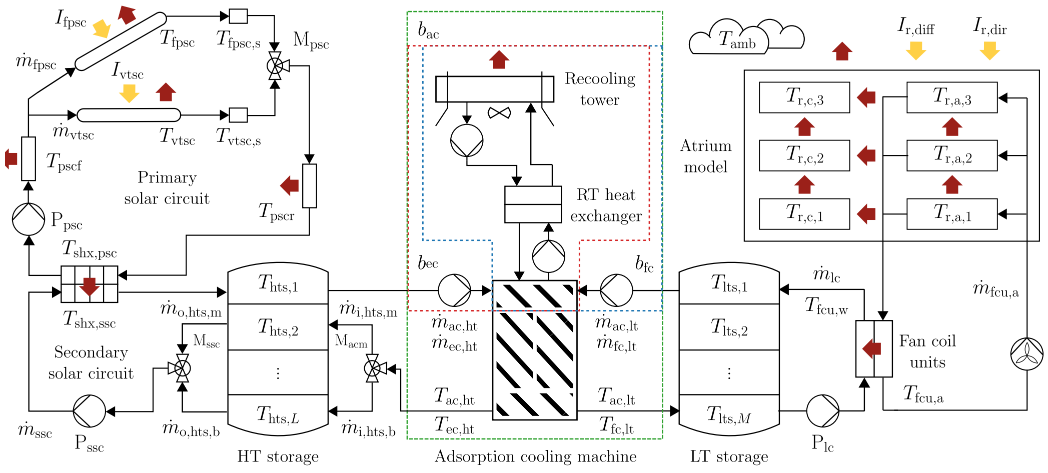
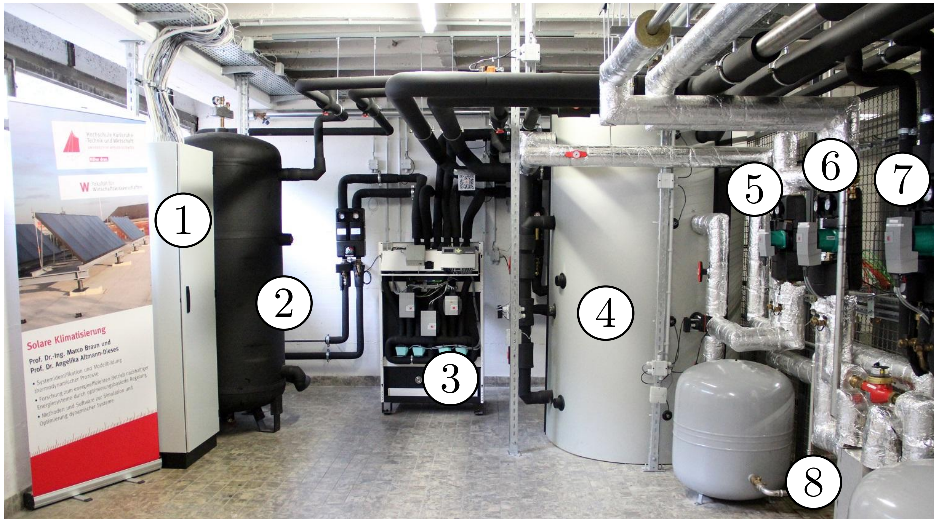
  - pycombina: fast solution of a special class of mixed integer linear programs arising in the combinatorial integral approximation (CIA) method for nonlinear mixed integer optimal control (Bürger et al., IFAC 2020)

Thank you

Nonlinear Mixed-Integer MPC of a Solar Adsorptive Cooling Machine
[Bürger et al., 2019]







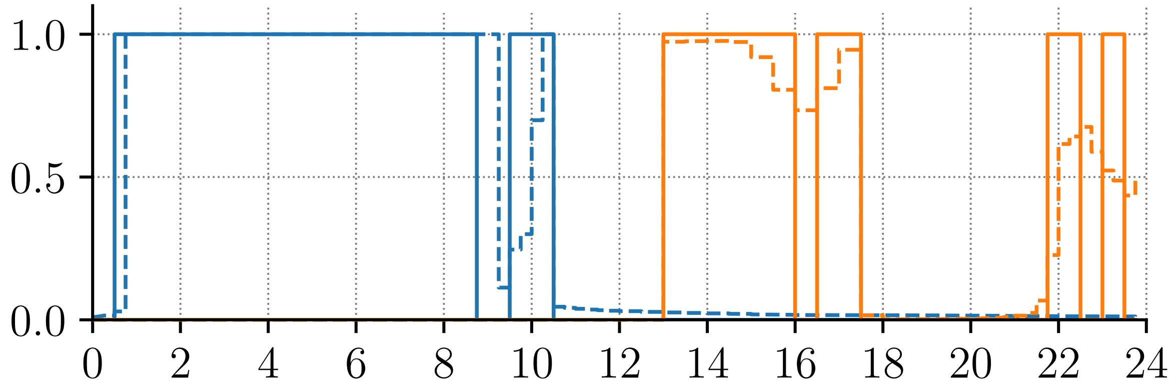
Nonlinear ODE with 39 states, 6 continuous and 2 binary inputs. Contains combinatorial constraints such as minimum uptime, minimum downtime, …

Predict 24 hours. Aim: minimise electricity consumption.

Three Stage Algorithm [Sager et al., Bürger et al.]

1. Solve Nonlinear Optimal Control Problem with Relaxed Integer Controls, using direct collocation or multiple shooting and a nonlinear programming (NLP) solver.

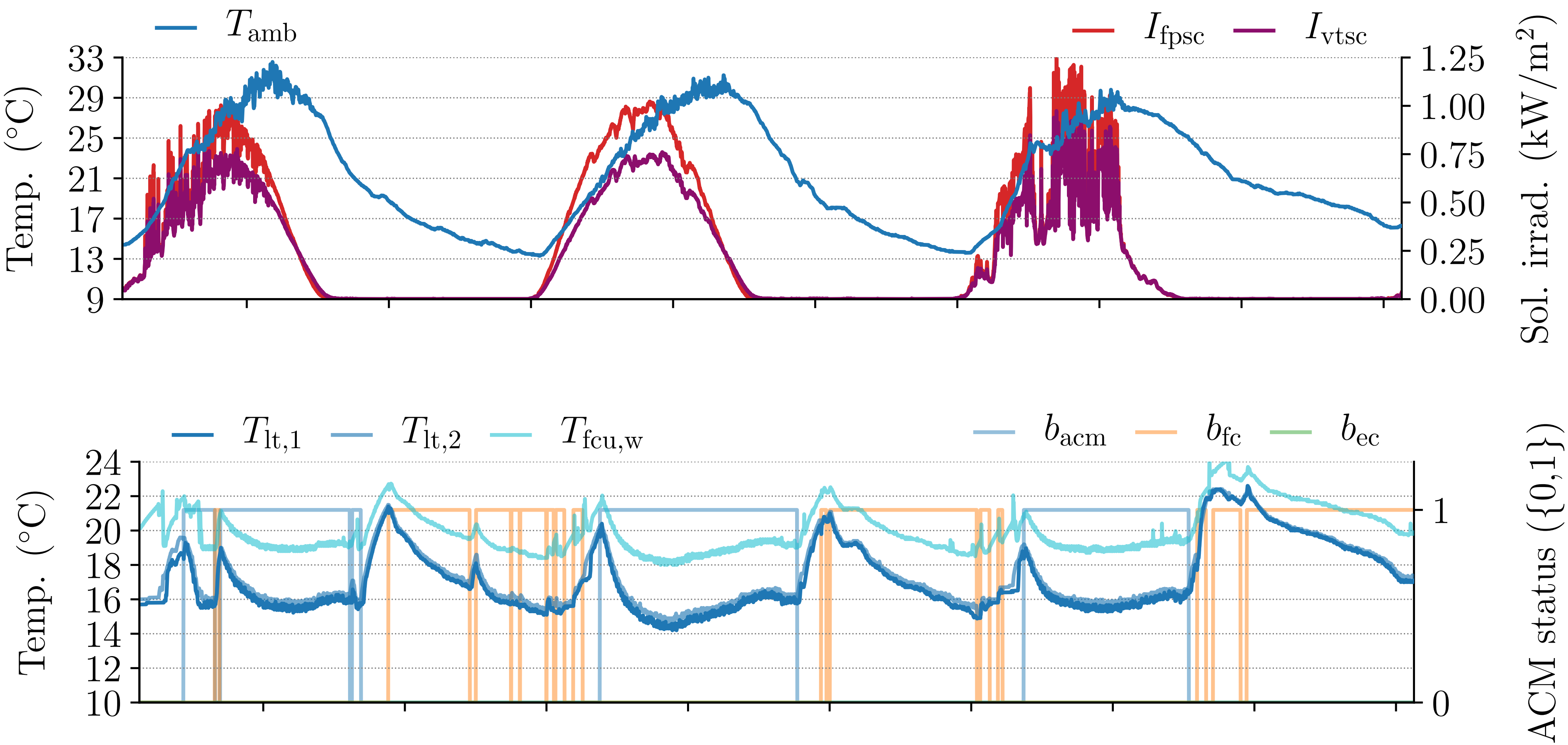
2. Find the integer input trajectory that
   (a) satisfies all combinatorial constraints and
   (b) minimises the distance to the relaxed input trajectory (L_∞- norm of the integrals)



(**pycombina** algorithm is 10-100x times faster than standard MILP solver)

3. Fix the integer inputs and reoptimize over all remaining variables by solving another NLP.

Experimental Results from Sept 14-17, 2019



Every 2 minutes, a new nonlinear mixed integer optimal control problem is solved, using a real-time algorithm based on CasADi, IPOPT [Wächter and Biegler 2006], and Pycombina [Bürger et al, 2019], an implementation of the combinatorial integral approximation (CIA) method [Sager 2009].

Details on recent algorithmic developments

- Inexact Newton with Iterated Sensitivities (INIS) [Quirynen et al, 2017]: partial answer to Potschka's problem,achieves same contraction rate for optimisation as for simulation problem with wrong Jacobian
    - Mutligrid INIS for Elliptic PDE Optimization (Pearse-Danker)
- Zero-Order SCP Methods for MPC [Zanelli et al. 2019]
- Zero-Order Moving Horizon Estimation [Baumgärtner et al. 2019]: surprisingly, consistently wrong derivatives can result in same estimation error as exact optimisation
    - iterative learning control via zero-order optimization
- BLASFEO [Frison et al, 2018, 2020]: Basic Linear Algebra Subroutines for Embedded Optimization: up to 5x speedup against BLAS on matrix dimensions below 300 x 300
- HPIPM [Frison et al., submitted]: BLASFEO based QP solver for optimal control problems
- General Nonlinear Static Feedback (GNSF) structure detection and exploitation in DAE solvers [Frey et al. 2019]
- acados [Verschueren et al., submitted]: stand-alone nonlinear optimal control package for embedded optimization, building on BLASFEO, HPIPM, qpOASES, GNSF Integrators, SCQP, CasADi,