Robust Dynamic Optimization

Moritz Diehl

Systems Control and Optimization Laboratory, University of Freiburg, Germany

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slides jointly developed with **Florian Messerer**, Katrin Baumgärtner, Titus Quah, Jim Rawlings
based on joint work with B. Houska

universität freiburg

Overview

1 Robust Dynamic Optimization Problem Statement
- Challenge 1: Robust constraint satisfaction
- Challenge 2: Feedback predictions
- Conceptual solution via Robust Dynamic Programming

2 Some exact NLP formulations for robust constraints (some with feedback)
- Scenario Tree for Finite Disturbances
- Scenario Tree for Polytopic Systems with Convex Costs and Constraints
- Dual norm formulation for systems that are affine in disturbances

3 Tube Based Formulations
- Ellipsoidal tubes - equivalent to robust ℓ_2-norm formulation
- Affine Disturbance Feedback Parameterization
- Overapproximating ellipsoidal tubes for stagewise bounded uncertainty
- Tube approximation for robust nonlinear MPC

Uncertain optimal control problem statement

Uncertain optimal control problem in discrete time

$$
\begin{aligned}
\min_{x,\,u} \quad & \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_\mathrm{f}(x_N) \\
\text{s.t.} \quad & x_0 = \bar{x}_0, \\
& x_{k+1} = f(x_k, u_k, w_k), \quad k = 0, \dots, N-1, \\
& 0 \geq h(x_k, u_k), \qquad\quad k = 0, \dots, N-1, \\
& 0 \geq r(x_N).
\end{aligned}
$$

All difficulty of robust model predictive control comes from the fact that the disturbance trajectory $(w_0, \dots, w_{N-1})$ is unknown. If it were known in advance, one could simply solve a nominal MPC problem with time varying dynamics.

Three challenges of robust dynamic optimization

When formulating and solving the robust dynamic optimization problems, one needs to address three major challenges:

- **Challenge 1: Robust constraint satisfaction.** How can the state uncertainty be approximated and propagated over the prediction horizon in order to guarantee robust constraint satisfaction?
- **Challenge 2: Feedback predictions.** How can feedback control policies be approximated and incorporated into the robust MPC optimization problem in order to reduce its conservatism?
- **Challenge 3: Dual control.** How can the optimality loss due to imperfect state estimation be computed and how can the uncertainty be reduced optimally, by proper choice of controls? (explore-exploit-tradeoff)

In this course, we will only address Challenges 1 and 2 (we assume perfect state estimates)

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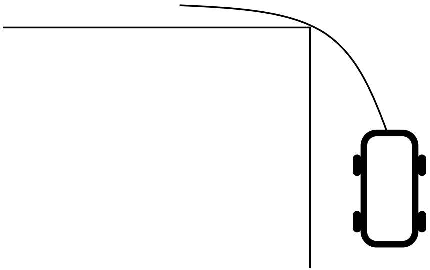
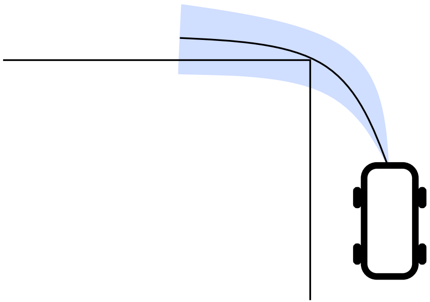
Challenge 1: Robust constraint satisfaction

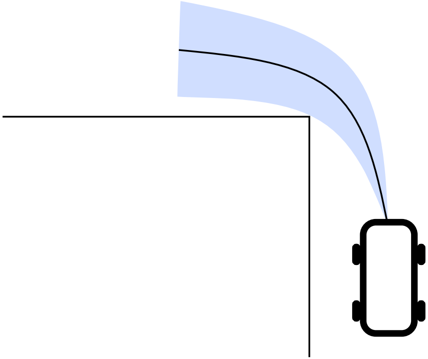


The nominal predicted trajectory cuts the corner tightly, in *Nominal MPC*.

Challenge 1: Robust constraint satisfaction



Predicting an uncertainty set ("tube"), we see that the car would often crash.



Due to uncertainty, the center of the tube needs to keep a distance ("backoff") from the corner.

Open-Loop Robust MPC addresses Challenge 1 (Robust Constraint Satisfaction).

Challenge 1: Robust constraint satisfaction

We will use two perspectives to derive a robust version of the uncertain OCP. These two perspectives are not necessarily in disagreement and can lead to identical formulations.

- **Perspective 1: Robust Optimization.** Bring into high-level standard form and use results from *Robust Optimization* lecture

$$\begin{aligned} \min_{u} \quad & \max_{w \in \mathbb{W}} F_0(u, w) \\ \text{s.t.} \quad & \max_{w \in \mathbb{W}} F_i(u, w) \leq 0, \quad i = 1, \dots, n_F. \end{aligned}$$

- **Perspective 2: OCP with set-valued trajectories.** Given initial state $\bar{x}_0$, a control trajectory $u = (u_0, \dots, u_{N-1})$, and the disturbance set W, what is the set $\mathbb{X}_k$ of possible values for x_k?

$$\begin{aligned} \mathbb{X}_0 &= \{x_0\}, \\ \mathbb{X}_{k+1} &= \mathcal{F}(\mathbb{X}_k, u_k) := \{f(x, u_k, w) \mid x \in \mathbb{X}_k, w \in \bar{\mathbb{W}}\} \end{aligned}$$

Perspective 1: Robust Optimization

Eliminate state trajectory - as in single shooting - via a recursion started at $\tilde{x}_0(u,w) := \bar{x}_0$ and looping through the state transitions $\boxed{\tilde{x}_{k+1}(u,w) := f(\tilde{x}_k(u,w), u_k, w_k)}$ for $k = 0, \dots, N-1$:

Min-max robust optimal control problem (as in single shooting)

$$
\begin{aligned}
\min_{u} \ & \max_{w \in \mathbb{W}} \ \sum_{k=0}^{N-1} \ell(\tilde{x}_k(u,w), u_k) + V_{\mathrm{f}}(\tilde{x}_N(u,w)) \\
\text{s.t.} \ & \max_{w \in \mathbb{W}} \ h(\tilde{x}_k(u,w), u_k) \leq 0, \quad k = 0, \dots, N-1 \\
& \max_{w \in \mathbb{W}} \quad r(\tilde{x}_N(u,w)) \leq 0
\end{aligned}
$$

Identify the cost with $F_0(u,w)$ and the constraints componentwise with $F_i(u,w)$:

$$
\boxed{\min_{u} \max_{w \in \mathbb{W}} F_0(u,w) \quad \text{s.t.} \quad \max_{w \in \mathbb{W}} F_i(u,w) \leq 0, \quad i = 1, \dots, n_F}
$$

Thus, all methods from the *Robust Optimization* lecture apply. We will look at their specific instantiation later.

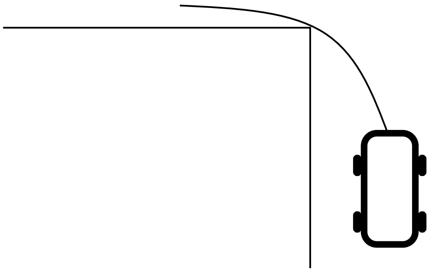
Perspective 2: OCP with set-valued trajectories

- Set dynamics: $\mathcal{F}(\mathbb{X}_k, u_k) = \{f(x_k, u_k, w_k) \mid x_k \in \mathbb{X}_k, w_k \in \bar{\mathbb{W}}\}$
- Assign costs $\mathcal{L}(\mathbb{X}_k, u)$ to set $\mathbb{X}_k$ based on $\ell(x_k, u_k)$, e.g., worst-case or average.

Set-based robust OCP

$$
\begin{aligned}
\min_{\mathbb{X},\,\kappa} \quad & \sum_{k=0}^{N-1} \mathcal{L}(\mathbb{X}_k, u_k) + \mathcal{L}_{\mathrm{f}}(\mathbb{X}_N) \\
\text{s.t.} \quad & \mathbb{X}_0 = \{\bar{x}_0\}, \\
& \mathbb{X}_{k+1} = \mathcal{F}(\mathbb{X}_k, u_k), \qquad\qquad k = 0, \dots, N-1, \\
& 0 \geq h(x_k, u_k), \quad \forall x_k \in \mathbb{X}_k,\; k = 0, \dots, N-1, \\
& 0 \geq r(x_N), \quad \forall x_N \in \mathbb{X}_N,
\end{aligned}
$$

Challenge 2: Feedback prediction



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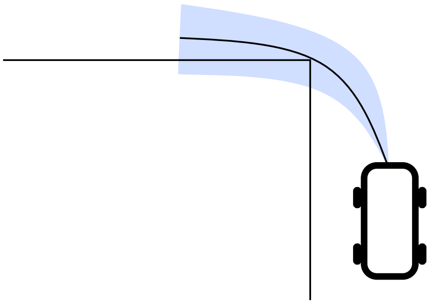

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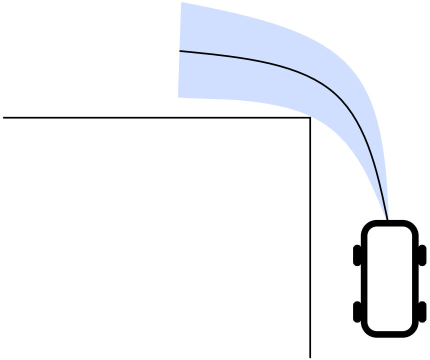

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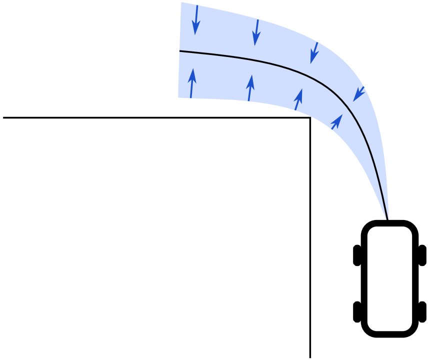
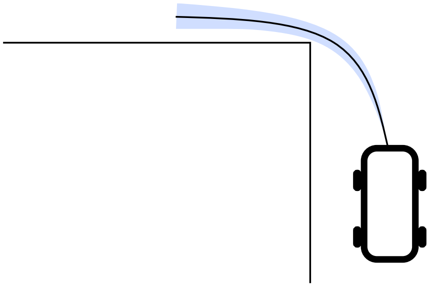
Challenge 2: Feedback prediction



But: we know that in the future we will apply feedback.

Challenge 2 - Feedback prediction - needs to be addressed.

Challenge 2: Feedback prediction



Considering future feedback allows for a more realistic, less conservative prediction.

Closed-Loop Robust MPC addresses Challenge 2.

Exact Feedback Prediction via Robust Dynamic Programming (RDP)

Exact feedback prediction requires us to optimize not in the space of control sequences $(u_0, \ldots, u_{N-1})$, but in the space of **feedback policies** $\kappa_0(\cdot), \ldots, \kappa_{N-1}(\cdot)$ with

$$\kappa_k : \mathbb{R}^{n_x} \to \mathbb{R}^{n_u}, \; x_k \mapsto u_k = \kappa_k(x)$$

This is an infinite dimensional function space, making it an infinite optimization problem that is impossible to solve numerically. In practice, one often parameterizes the feedback law, for example in the form of **linear state feedback policies**

$$u_k = K_k x_k + v_k$$

where the minimization variables are $K_k \in \mathbb{R}^{n_u \times n_x}$ and $v_k \in \mathbb{R}^{n_u}$.

Conceptually, the problem can exactly be solved via **Robust Dynamic Programming (RDP)**, which we present next. It delivers important insights w.r.t. to solution structure, e.g. convexity.

Robust Dynamic Programming Prelude: Extended Cost Values

Can assign infinite cost to infeasible points, using extended reals $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$

Constrained Optimal Control Problem

$$\begin{aligned}
\min_{x,u} \quad & \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_\mathrm{f}(s_N) \\
\text{s.t.} \quad & x_0 = \bar{x}_0 \\
& x_{k+1} = f(x_k, u_k, w_k) \\
& 0 \geq h(x_k, u_k), \; k = 0, \ldots, N-1 \\
& 0 \geq r(x_N)
\end{aligned}$$

Equivalent Unconstrained Formulation

$$\begin{aligned}
\min_{x,u} \quad & \sum_{k=0}^{N-1} \bar{\ell}(x_k, u_k) + \bar{V}_\mathrm{f}(x_N) \\
\text{s.t.} \quad & x_0 = \bar{x}_0 \\
& x_{k+1} = f(x_k, u_k, w_k), \; k = 0, \ldots, N-1
\end{aligned}$$

with

$$\bar{\ell}(x, u) = \left\{ \begin{array}{cc} \ell(x, u) & \text{if } h(x, u) \leq 0 \\ \infty & \text{else} \end{array} \right\}$$

and $\bar{V}_\mathrm{f}(x) = \left\{ \begin{array}{cc} V_\mathrm{f}(x) & \text{if } r(x) \leq 0 \\ \infty & \text{else} \end{array} \right\}$

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Equivalent Unconstrained Formulation

$$\min_{x,u} \sum_{k=0}^{N-1} \bar{\ell}(x_k, u_k) + \bar{V}_\mathrm{f}(x_N)$$

$$\text{s.t.} \quad x_0 = \bar{x}_0$$

$$x_{k+1} = f(x_k, u_k, w_k), \; k = 0, \ldots, N-1$$

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$$\min_{x,u} \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_\mathrm{f}(x_N)$$

$$\text{s.t.} \quad x_0 = \bar{x}_0$$

$$x_{k+1} = f(x_k, u_k, w_k), \; k = 0, \dots, N-1$$

with $\quad \ell : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \bar{\mathbb{R}}$

and $\quad V_\mathrm{f} : \mathbb{R}^{n_x} \to \bar{\mathbb{R}}$

Robust Dynamic Programming (RDP)

Assume uncertainty is restricted to set $w_k \in \bar{\mathbb{W}}$ in each time step.

Robust dynamic programming proceeds as follows.

Robust Dynamic Programming (RDP) Recursion

Iterate backwards, from $k = N-1$ down to $k = 0$, using the robust Bellman equation

$$J_k(x_k) = \min_{u_k} \left(\max_{w_k \in \bar{\mathbb{W}}} \ \ell(x_k, u_k) + J_{k+1}(f(x_k, u_k, w_k)) \right)$$

starting with terminal cost

$$J_N(x) = V_{\mathrm{f}}(x)$$

Monotonicity of Robust Dynamic Programming

The "cost-to-go" J_k is often also called the "value function".

The *robust dynamic programming operator* T mapping between value functions is defined by

$$\boxed{T[J](x) := \min_{u} \max_{w \in \bar{\mathbb{W}}} \ \ell(x,u) + J(f(x,u,w))}$$

Dynamic programming recursion now compactly written as $J_k = T[J_{k+1}]$.

We write $J \geq J'$ if $J(x) \geq J'(x)$ for all $x \in \mathbb{R}^{n_x}$.

One can prove that

$$J \geq J' \quad \Rightarrow \quad T[J] \geq T[J']$$

This is called "monotonicity" of dynamic programming. It holds also for deterministic or stochastic dynamic programming. It can e.g. be used in existence proofs for solutions of the stationary Bellman equation, or in stability proofs for MPC ($J_N \geq J_{N-1} \Rightarrow J_1 \geq J_0$).

Convex Robust Dynamic Programming

Certain RDP operators T preserve convexity of the value function $J : \mathbb{R}^{n_x} \rightarrow \bar{\mathbb{R}}$:

Theorem [D.: Formulation of Closed-Loop Min–Max MPC as a QCQP. IEEE TAC 2007]

If

- system is affine $\boxed{f(x, u, w) = A(w)x + B(w)u + c(w)}$ and
- stage cost $\ell(x, u)$ convex in (x, u)

then the **robust DP operator** T **preserves convexity** of J, i.e.

$$\boxed{J \text{ convex} \quad \Rightarrow \quad T[J] \text{ convex}}$$

Note: no assumptions on disturbance set $\bar{\mathbb{W}}$ or on how w enters cost and dynamics.

Proof of Convexity Preservation

The function

$$\ell(x,u) + J(\ A(w)x + B(w)u + c(w)\ )$$

is convex in (x,u) for any fixed w, as concatenation of an affine function inside a convex one.

Because the maximum over convex functions (indexed by w) preserves convexity, the function

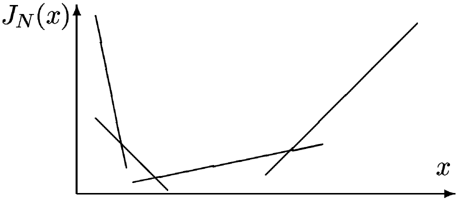
$$Q(x,u) := \max_{w \in \bar{\mathbb{W}}} \ \ell(x,u) + J(\ A(w)x + B(w)u + c(w)\ )$$

is also convex in (x,u).
Finally, the minimization of a convex function over one of its arguments preserves convexity, i.e. the resulting value function $T[J]$ defined by

$$T[J](x) = \min_u Q(x,u)$$

is convex. □

Why is convexity of the value function important?



- value function $J(x)$ can be represented (or approximated) as the maximum of affine functions with vectors $a_i \in \mathbb{R}^{1+n_x}$ with indices i in some (finite or infinite) set S

$$J(x) = \max_{i \in S} \; a_i^\top \begin{bmatrix} 1 \\ x \end{bmatrix}$$

- computation of feedback law $\arg\min_u Q(x,u)$ is convex and can be solved reliably
- convexity of value function allows us to conclude, in case of polytopic uncertainty, that worst case is assumed on boundary of the polytope, making scenario-tree formulation possible [D.: Formulation of Closed-Loop Min–Max MPC as a QCQP. IEEE TAC 2007]

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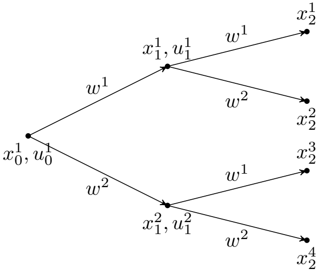
Scenario Tree for Finite Disturbances

Assumption: m possible disturbance values in each time step $\bar{\mathbb{W}} = \{w^1, \ldots, w^m\}$

- in each stage, m disturbance values $\{w^1, \ldots, w^m\}$
- m^k state values $\mathbb{X}_k = \{x_k^1, \ldots, x_k^{m^k}\}$ at stage k
- one control u_k^i for each state x_k^i encodes feedback, and "epigraph slack control" v_k^i collects worst-case objective



For each state and control pair, simulate dynamics for every possible disturbance value.

Exact Scenario Tree Formulation

$$
\begin{aligned}
\min_{x,u,v} \quad & \ell(x_0^1, u_0^1) + v_0^1 \\
\text{s.t.} \quad & x_0^1 = \bar{x}_0, \\
& x_{k+1}^i = f(x_k^{\lceil i/m^k \rceil}, u_k^{\lceil i/m^k \rceil}, w^{\,i]_1^m}), \\
& v_k^{\lceil i/m^k \rceil} \geq \ell(x_{k+1}^i, u_{k+1}^i) + v_{k+1}^i, \qquad k = 0, \ldots, N-1, \\
& 0 \geq h(x_k^{\lceil i/m^k \rceil}, u_k^{\lceil i/m^k \rceil}), \qquad i = 1, \ldots, m^{k+1}, \\
& 0 \geq r(x_N^j),\ v_N^j \geq V_{\mathrm{f}}(x_N^j), \qquad j = 1, \ldots, m^N
\end{aligned}
$$

$\lceil \cdot \rceil$: ceiling function, $i]_1^m$ wraps i to $\{1, \ldots, m\}$.

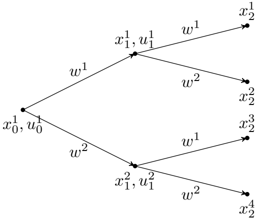
Scenario Tree for Polytopic Systems with Convex Costs and Constraints

Extension of scenario-tree formulation to infinite polytopic disturbance sets, using convexity of RDP cost-to-go

Assume

- polytopic uncertainty $\bar{\mathbb{W}} = \mathrm{conv}\{w^1, \ldots, w^m\} \subset \mathbb{R}^{n_w}$
- affine dynamics $x_{k+1} = A(w_k)x_k + B(w_k)u_k + c(w_k)$
- affine dependendence of $A(w), B(w), c(w)$ on $w \in \mathbb{R}^{n_w}$
- convexity of functions ℓ, h, V_f, r

then worst-case is taken in vertices of $\bar{\mathbb{W}}$ and scenario-tree suffices



Exact Convex Scenario Tree for Polytopic Systems [D., IEEE TAC 2007]

$$
\begin{aligned}
\min_{x,u,v} \quad & \ell(x_0^1, u_0^1) + v_0^1 \\
\text{s.t.} \quad & x_0^1 = \bar{x}_0, \\
& x_{k+1}^i = A\big(w^{i]_1^m}\big) x_k^{\lceil i/m^k \rceil} + B\big(w^{i]_1^m}\big) u_k^{\lceil i/m^k \rceil} + c\big(w^{i]_1^m}\big), \\
& v_k^{\lceil i/m^k \rceil} \geq \ell(x_{k+1}^i, u_{k+1}^i) + v_{k+1}^i, \qquad k = 0, \ldots, N-1, \\
& 0 \geq h(x_k^{\lceil i/m^k \rceil}, u_k^{\lceil i/m^k \rceil}), \qquad i = 1, \ldots, m^{k+1}, \\
& 0 \geq r(x_N^j), \; v_N^j \geq V_\mathrm{f}(x_N^j), \qquad j = 1, \ldots, m^N
\end{aligned}
$$

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Dual norm formulation for systems that are affine in disturbances

Regard disturbance trajectories $w = (\bar{w}_0, \ldots, \bar{w}_{N-1}) \in \mathbb{R}^{N n_{\bar{w}}}$ in norm ball $\mathbb{W} = \{w \in \mathbb{R}^{n_w} \mid \|w\| \leq 1\}$ for any norm $\|\cdot\|$, with $n_w = N n_{\bar{w}}$.[1]

Again define "single shooting" state trajectory $\tilde{x}_k(u, w)$ at time k as function of (u, w) trajectories, where $u = (\bar{u}_0, \ldots, \bar{u}_{N-1}) \in \mathbb{R}^{n_u}$, and $n_u = N n_{\bar{u}}$.

For simplicity, omit terminal constraint and uncertainty in objective.

Open loop robust optimal control problem

$$\begin{aligned} &\min_u \; F_0(u) \\ &\text{s.t.} \; \max_{w \in \mathbb{W}} \underbrace{h_j(\tilde{x}_k(u, w), \bar{u}_k)}_{=: F_{k,j}(u,w)} \leq 0, \quad k = 0, \ldots, N-1, \; j = 1, \ldots, n_h, \end{aligned}$$

If functions $F_{k,j}(u, w)$ are affine in uncertainty w, dual norm formulation is applicable.

[1] A mixed ℓ_∞-ℓ_p-norm covers the case of independent, stage-wise p-norm bounded uncertainties, $\mathbb{W} = \bar{\mathbb{W}} \times \ldots \times \bar{\mathbb{W}}$ with ℓ_p-norm balls $\bar{\mathbb{W}} = \{\bar{w} \in \mathbb{R}^{n_{\bar{w}}} \mid \|\bar{w}\|_p \leq 1\}$.

Dual norm formulation for systems that are affine in disturbances

For constraints affine in the uncertainty trajectory we obtain (using $\nabla_w f(w) = \frac{\partial f}{\partial w}(w)^\top$)

$$\max_{w \in \mathbb{W}} F_{k,j}(u,w) \quad = \quad h_j(\tilde{x}_k(u,0), \bar{u}_k) \quad + \quad \|\nabla_w \tilde{x}_k(u,w) \nabla_x h_j(\tilde{x}_k(u,0), \bar{u}_k)\|_*$$

For uncertainty affine systems $\boxed{x_{k+1} = a(u_k) + A(u_k)x_k + \Gamma(u_k)w_k}$

the derivative of state x_k w.r.t. disturbance w_m is given by

$$\boxed{G_{k,m}(u) := \frac{\partial \tilde{x}_k}{\partial w_m}(u,w) = A(u_{k-1}) \cdots A(u_{m+1})\Gamma(u_m)}$$

so that we obtain

$$\max_{w \in \mathbb{W}} F_{k,j}(u,w) \quad = \quad h_j(\tilde{x}_k(u,0), \bar{u}_k) \quad + \quad \left\| \begin{bmatrix} G_{k,0}(u)^\top \\ \vdots \\ G_{k,k-1}(u)^\top \\ 0 \\ \vdots \end{bmatrix} \underbrace{\nabla_x h_j(\tilde{x}_k(u,0), \bar{u}_k)}_{=: g_{k,j}(u)} \right\|_*$$

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$$\boxed{G_{k,m}(u) := \frac{\partial \tilde{x}_k}{\partial w_m}(u,w) = A(u_{k-1}) \cdots A(u_{m+1})\Gamma(u_m)}$$

so that we obtain

$$\max_{w \in \mathbb{W}} F_{k,j}(u,w) \quad = \quad h_j(\tilde{x}_k(u,0), \bar{u}_k) \quad + \quad \left\| \begin{bmatrix} G_{k,0}(u)^\top \\ \vdots \\ G_{k,k-1}(u)^\top \\ 0 \\ \vdots \end{bmatrix} \underbrace{\nabla_x h_j(\tilde{x}_k(u,0), \bar{u}_k)}_{=: g_{k,j}(u)} \right\|_*$$

In detail, this looks different for different norms...

Infinity Norm - Exact Dual Norm Formulation

Dual of infinity norm is ℓ_1-norm.

$$\left\| \begin{bmatrix} G_{k,0}(u)^\top \\ \vdots \\ G_{k,k-1}(u)^\top \\ 0 \\ \vdots \end{bmatrix} g_{k,j}(u) \right\|_1 \quad = \quad \sum_{m=0}^{k-1} \| G_{k,m}(u)^\top g_{k,j}(u) \|_1$$

Infinity Norm - Exact Dual Norm Formulation

Dual of infinity norm is ℓ_1-norm.

Exact robust problem for ℓ_∞-norm bounded disturbances

$$\begin{aligned} \min_{u} \quad & F_0(u) \\ \text{s.t.} \quad & h_j(\tilde{x}_k(u,w), \bar{u}_k) + \sum_{m=0}^{k-1} \| G_{k,m}(u)^\top g_{k,j}(u) \|_1 \leq 0, \\ & \qquad k = 0, \ldots, N-1, \ j = 1, \ldots, n_h, \end{aligned}$$

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This formulation is very expensive, because one needs to compute all matrices $G_{k,m}(u)$ for $k = 1, \dots, N-1$ and $m = 0, \dots, k-1$, resulting in $O(N^2 n_x n_{\bar{w}})$ extra variables.

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Let us now see how the robustified problem looks for the Euclidean norm.

Euclidean Norm - Exact Formulation

Euclidean ℓ_2-norm is self-dual, so its dual is also the ℓ_2-norm.

Euclidean Norm - Exact Formulation

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$$\left\| \begin{bmatrix} G_{k,0}(u)^\top \\ \vdots \\ G_{k,k-1}(u)^\top \\ 0 \\ \vdots \end{bmatrix} g_{k,j}(u) \right\|_2^2 = g_{k,j}(u)^\top \begin{bmatrix} G_{k,0}(u)^\top \\ \vdots \\ G_{k,k-1}(u)^\top \\ 0 \\ \vdots \end{bmatrix}^\top \begin{bmatrix} G_{k,0}(u)^\top \\ \vdots \\ G_{k,k-1}(u)^\top \\ 0 \\ \vdots \end{bmatrix} g_{k,j}(u)$$

$$= g_{k,j}(u)^\top \left(\sum_{m=0}^{k-1} G_{k,m}(u) G_{k,m}(u)^\top \right) g_{k,j}(u)$$

Euclidean Norm - Exact Formulation

Euclidean ℓ_2-norm is self-dual, so its dual is also the ℓ_2-norm.

Exact robust problem for ℓ_2-norm bounded disturbances

$$\begin{aligned} \min_{u} \quad & F_0(u) \\ \text{s.t.} \quad & h_j(\tilde{x}_k(u,w),\bar{u}_k) + \sqrt{g_{k,j}(u)^\top \left(\sum_{m=0}^{k-1} G_{k,m}(u) G_{k,m}(u)^\top \right) g_{k,j}(u)} \leq 0, \\ & \qquad k = 0,\dots,N-1,\ j = 1,\dots,n_h, \end{aligned}$$

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Exact robust problem for ℓ_2-norm bounded disturbances

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The computations can be much more efficient if one computes the matrix sums differently...

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Exact robust problem for ℓ_2-norm bounded disturbances

$$\begin{aligned}\min_u \quad & F_0(u) \\ \text{s.t.} \quad & h_j(\tilde{x}_k(u,w),\bar{u}_k) + \sqrt{g_{k,j}(u)^\top \left(\sum_{m=0}^{k-1} G_{k,m}(u)G_{k,m}(u)^\top\right) g_{k,j}(u)} \leq 0, \\ & k = 0,\dots,N-1,\ j = 1,\dots,n_h,\end{aligned}$$

The computations can be much more efficient if one computes the matrix sums differently...

$$\underbrace{\sum_{m=0}^{k} G_{k+1,m}(u)G_{k+1,m}(u)^\top}_{=P_{k+1}(u)} = A(u_k)\underbrace{\left(\sum_{m=0}^{k-1} G_{k,m}(u)G_{k,m}(u)^\top\right)}_{=P_k(u)} A(u_k)^\top + \underbrace{G_{k+1,k}(u)G_{k+1,k}(u)^\top}_{=\Gamma(u_k)\Gamma(u_k)^\top}$$

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Exact robust problem for ℓ_2-norm bounded disturbances

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The computations can be much more efficient if one computes the matrix sums differently...

$$\underbrace{\sum_{m=0}^{k} G_{k+1,m}(u) G_{k+1,m}(u)^\top}_{=P_{k+1}(u)} = A(u_k) \underbrace{\left(\sum_{m=0}^{k-1} G_{k,m}(u) G_{k,m}(u)^\top \right)}_{=P_k(u)} A(u_k)^\top + \underbrace{G_{k+1,k}(u) G_{k+1,k}(u)^\top}_{=\Gamma(u_k)\Gamma(u_k)^\top}$$

Start at $P_0(u) := 0 \in \mathbb{R}^{n_x \times n_x}$, compute $\boxed{P_{k+1}(u) := A(u_k) P_k(u) A(u_k)^\top + \Gamma(u_k)\Gamma(u_k)^\top}$

Euclidean Norm - Exact Formulation with Lyapunov Matrix Equations

One can make all dependencies explicit again, resulting in a sparse NLP in only $O(N)$ - exactly $Nn_{\bar{u}} + (N+1)({}_x + n_x^2)$ - variables $u = (u_0, \ldots, u_{N-1})$, $x = (x_0, \ldots, x_N)$, $P = (P_0, \ldots, P_N)$

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Exact robust problem for ℓ_2-norm bounded disturbances

$$
\begin{aligned}
\min_{u,x,P} \quad & F_0(u) \\
\text{s.t.} \quad & x_0 = \bar{x}_0, \quad P_0 = 0 \\
& x_{k+1} = f(x_k, u_k, 0) \\
& P_{k+1} = A(x_k, u_k)\ P_k\ A(x_k, u_k)^\top \ + \ \Gamma(x_k, u_k)\Gamma(x_k, u_k)^\top \\
& 0 \geq h_j(x_k, u_k) + \sqrt{\nabla_x h_j(x_k, u_k)^\top P_k \nabla_x h_j(x_k, u_k)}, \\
& \qquad k = 0, \ldots, N-1,\ j = 1, \ldots, n_h
\end{aligned}
$$

where we use $A(x_k, u_k) = \frac{\partial f}{\partial x}(x_k, u_k)$ and $\Gamma(x_k, u_k) = \frac{\partial f}{\partial w}(x_k, u_k)$.

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Recall that the formulation is exact for $f(x, u, w) = a(u_k) + A(u_k)x_k + \Gamma(u_k)w_k$.

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Exact robust problem for ℓ_2-norm bounded disturbances

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& x_{k+1} = f(x_k, u_k, 0) \\
& P_{k+1} = A(x_k, u_k)\; P_k\; A(x_k, u_k)^\top \;+\; \Gamma(x_k, u_k)\Gamma(x_k, u_k)^\top \\
& 0 \geq h_j(x_k, u_k) + \sqrt{\nabla_x h_j(x_k, u_k)^\top P_k \nabla_x h_j(x_k, u_k)}, \\
& \qquad k = 0, \ldots, N-1,\; j = 1, \ldots, n_h
\end{aligned}$$

where we use $A(x_k, u_k) = \frac{\partial f}{\partial x}(x_k, u_k)$ and $\Gamma(x_k, u_k) = \frac{\partial f}{\partial w}(x_k, u_k)$.
Recall that the formulation is exact for $f(x, u, w) = a(u_k) + A(u_k)x_k + \Gamma(u_k)w_k$.
Can also use as **linearization-based approximation** for any nonlinear system $x_+ = f(x, u, w)$.

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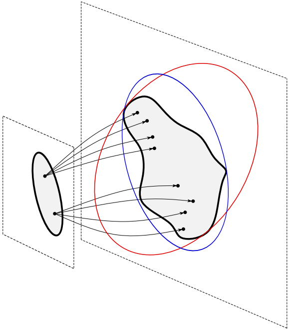
Tube-based Approximation Methods

- Regard again norm-bounded $\mathbb{W} = \{w \mid \|w\| \leq 1\}$
- formulate optimization problem in set-valued trajectory ("tube") $\mathbb{X} = (\mathbb{X}_0, \ldots, \mathbb{X}_N)$ and policy $\kappa = (\kappa_0, \ldots, \kappa_{N-1})$

Tube-based optimal control problem

$$
\begin{aligned}
\min_{\mathbb{X}, \kappa} \quad & \sum_{k=0}^{N-1} \mathcal{L}(\mathbb{X}_k, \kappa_k) + \mathcal{L}_N(\mathbb{X}_N) \\
\text{s.t.} \quad & \mathbb{X}_0 = \{\bar{x}_0\}, \\
& \mathbb{X}_{k+1} = \mathcal{F}(\mathbb{X}_k, \kappa_k), && k = 0, \ldots, N-1, \\
& 0 \geq h(x_k, \kappa_k(x_k)), \forall x_k \in \mathbb{X}_k, && k = 0, \ldots, N-1, \\
& 0 \geq r(x_N), \qquad \forall x_N \in \mathbb{X}_N,
\end{aligned}
$$

- Need to parametrize $\mathbb{X}$ and κ to obtain a tractable NLP
- Nonlinearity in general leads to non-parametrizable sets $\rightarrow$ overapproximate by parametrizable sets, e.g. ellipsoids.



The nonlinear transformation of an ellipsoid is in general not ellipsoidal.

Ellipsoidal tubes - dynamics

Consider the linear time-varying system, for $k = 0, \dots, N-1$,

$$x_0 = \bar{\bar{x}}_0, \quad x_{k+1} = A_k x_k + B_k u_k + \Gamma_k w_k, \quad \text{with} \quad w = (w_0, \dots, w_{N-1}) \in \mathbb{W} = \{w \mid \|w\|_2 \leq 1\}.$$

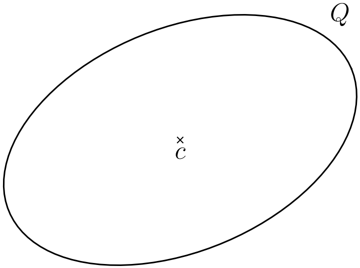
What is the sequence of sets $\mathbb{X}_k$, so that $x_k \in \mathbb{X}_k$ for all disturbance realizations ("tube")?

- **Variant 1, open-loop control trajectory:**
  $(\bar{u}_0, \dots, \bar{u}_{N-1})$
  This results in ellipsoidal state uncertainty sets
  $\mathbb{X}_k = \mathcal{E}(\bar{x}_k, P_k)$, with

$$\bar{x}_0 = \bar{\bar{x}}_0, \quad \bar{x}_{k+1} = A_k \bar{x}_k + B_k \bar{u}_k$$
$$P_0 = 0, \quad P_{k+1} = A_k P_k A_k^\top + \Gamma_k \Gamma_k^\top$$

- **Variant 2, with additional linear feedback:**
  $u_k = \bar{u}_k + K_k(x_k - \bar{x}_k)$
  Only the ellipsoid dynamics are modified

$$P_{k+1} = (A_k - B_k K_k) P_k (A_k - B_k K_k)^\top + \Gamma_k \Gamma_k^\top$$



Ellipsoids can be defined via center c and shape matrix ("variance") $Q \succ 0$.
$\mathcal{E}(c, Q) := \{x \mid (x-c)^\top Q^{-1}(x-c) \leq 1\}$

Ellipsoidal tubes - constraints

Given ellipsoidal uncertainty set $\mathbb{X}_k = \mathcal{E}(\bar{x}_k, P_k)$, how to treat constraints?

$$\boxed{b + a^\top x_k \le 0 \quad \forall x_k \in \mathcal{E}(\bar{x}_k, P_k)}$$

Reformulate as

$$b + \max_{x_k \in \mathcal{E}(\bar{x}_k, P_k)} a^\top x_k \le 0.$$

For affine constraints we can compute the maximum analytically as

$$\max_{x_k \in \mathcal{E}(\bar{x}_k, P_k)} a^\top x_k = a^\top \bar{x}_k + \sqrt{a^\top P_k a},$$

resulting in

$$\boxed{b + c^\top \bar{x}_k + \sqrt{a^\top P_k a} \le 0.}$$

Ellipsoidal tubes - Resulting optimal control problem

Robust optimal control for linear systems with linear state feedback

$$
\begin{aligned}
\min_{\bar{x}, \bar{u}, P, K} \quad & \sum_{k=0}^{N-1} \ell(\bar{x}_k, \bar{u}_k) + V_\mathrm{f}(\bar{x}_N) \\
\text{s.t.} \quad & \bar{x}_0 = \bar{\bar{x}}_0, \quad P_0 = 0, \\
& \bar{x}_{k+1} = A_k \bar{x}_k + B_k \bar{u}_k, & k = 0, \dots, N-1, \\
& P_{k+1} = (A_k - B_k K_k) P_k (A_k - B_k K_k)^\top + \Gamma_k \Gamma_k^\top, \ , \\
& 0 \geq b_i + a_i^\top \bar{x}_k + \sqrt{a_i^\top P_k a_i}, & i = 1, \dots, n_c, \\
& 0 \geq \tilde{b}_j + \tilde{a}_j^\top \bar{u}_k + \sqrt{\tilde{a}_j^\top K_k P_k K_k^\top \tilde{a}_j}, & j = 1, \dots, n_{\tilde{c}}
\end{aligned}
$$

- same OCP as from dual norm derivation
- exact constraint satisfaction (Challenge 1), but suboptimal feedback (Challenge 2)
- nonconvex due to optimization over state feedback gains K_k
- if K_k fix, then also P_k fix, resulting in standard OCP with backoff (convex)

Alternative Feedback Parameterizations

Optimization over state feedback matrices $(K_1, \ldots, K_{N-1}$ is nonconvex and can be challenging to solve (though not impossible)

- Alternative 1: No feedback in prediction, $K = 0$, or precomputed feedback gain $\bar{K}$.
  - For a *linear* system, the ellipsoids can be precomputed offline, resulting in constant constraint tightening (i.e., the structure of a nominal OCP).
  - No feedback, $K = 0$, leads to unrealistically conservative uncertainty sets.
  - Not necessarily obvious what would be a good choice of $\bar{K}$.
- Alternative 2: Disturbance feedback instead of state feedback

$$u_k = \bar{u}_k + \sum_{m=0}^{k-1} M_{k,m} w_m$$

  - For linear systems (some assumptions on the noise): equivalent to state feedback *and* leads to convex optimization problems [Goulart2006].
  - Many feedback gains $\rightarrow$ large-dimensional, expensive optimization problems

Affine disturbance feedback formulation for ℓ_2-norm

Exact robust optimal control with affine disturbance feedback (convex)

$$
\begin{aligned}
\min_{\bar{x}, \bar{u}, G, M} \quad & \sum_{k=0}^{N-1} \ell_k(\bar{x}_k, \bar{u}_k) + V_{\mathrm{f}}(\bar{x}_N) \\
\text{s.t.} \quad & \bar{x}_0 = \bar{\bar{x}}_0, \\
& \bar{x}_{k+1} = A_k \bar{x}_k + B_k \bar{u}_k, && k = 0, \dots, N-1, \\
& G_{k+1,k} = \Gamma_k, \\
& G_{k+1,n} = A_k G_{k,n} + B_k M_{k,n} && n = 0, \dots, k-1, \\
& 0 \geq b_i + a_i^\top \bar{x}_k + \sqrt{a_i^\top \left(\sum_{m=0}^{k-1} G_{k,m} G_{k,m}^\top \right) a_i}, && i = 1, \dots, n_c, \\
& 0 \geq \tilde{b}_j + \tilde{a}_j^\top \bar{u}_k + \sqrt{\tilde{a}_j^\top \left(\sum_{m=0}^{k-1} M_{k,m} M_{k,m}^\top \right) \tilde{a}_j}, && j = 1, \dots, n_{\tilde{c}}
\end{aligned}
$$

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Extending ellipsoidal tubes to independent stage noises ?

$$x_0 = \bar{\bar{x}}_0, \quad x_{k+1} = A_k x_k + B_k u_k + \Gamma_k w_k, \quad .$$

So far, we assumed $w = (w_0, \ldots, w_N) \in \mathbb{W} = \{w \in \mathbb{R}^{N n_w} \mid \|w\|_2 \leq 1\}$. This contains the assumption that the noise is dependent across time.
Alternative assumption: noise is norm-bounded independently at each time

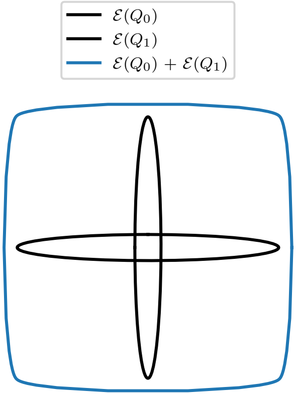
$$\mathbb{W} = \underbrace{\bar{\mathbb{W}} \times \cdots \times \bar{\mathbb{W}}}_{N-\text{times}} \quad \text{with} \quad \bar{\mathbb{W}} = \{w \in \mathbb{R}^{n_{\bar{w}}} \mid w^\top w \leq 1\}.$$

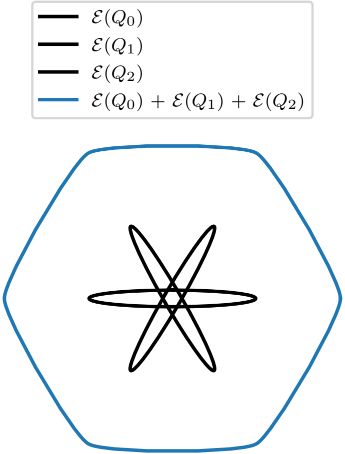
Can in principle be addressed using the affine case with mixed ℓ_∞-ℓ_2-norm, combined with any feedback parameterization - but this is expensive. Can we use ellipsoidal tubes instead?
Assume we have $\mathbb{X}_k = \mathcal{E}(\bar{x}_k, P_k)$. Then

$$\begin{aligned} \mathbb{X}_k &= A_k \mathbb{X}_k + B_k u_k + \Gamma_k \bar{\mathbb{W}} \\ &= \mathcal{E}(A_k \bar{x}_k + B_k u_k, A_k P_k A_k^\top) + \mathcal{E}(0, \Gamma_k \Gamma_k^\top) \end{aligned}$$

Problem: The sum of two ellipsoids is not an ellipsoid.

Sum of ellipsoids (Minkowski sum)





The sum of ellipsoids is not ellipsoidal.

Overapproximating sum of ellipsoids by ellipsoid

- Aim: find Q such that $\mathcal{E}(Q) \supseteq \mathcal{E}(Q_1) + \mathcal{E}(Q_2)$
- More general: Find Q such that $\mathcal{E}(Q) \supseteq \sum_{k=1}^{N} \mathcal{E}(Q_k)$
- Construct family of outer approximations parametrized by $\alpha \in \mathbb{R}^N_{++}$

$$Q(\alpha) = \sum_{k=1}^{N} \frac{1}{\alpha_k} Q_k \quad \Rightarrow \quad \mathcal{E}(Q(\alpha)) \supseteq \sum_{k=1}^{N} \mathcal{E}(Q_k) \quad \forall \alpha \in \mathbb{R}^N_{++} \quad \text{with} \quad \sum_{k=1}^{N} \alpha_k = 1$$

- Denote set of feasible α by $\mathcal{A}^N$ (basically a simplex)
- Parametrized outer approximation is tight

$$\bigcap_{\alpha \in \mathcal{A}^N} \mathcal{E}(Q(\alpha)) = \sum_{k=1}^{N} \mathcal{E}(Q_k)$$

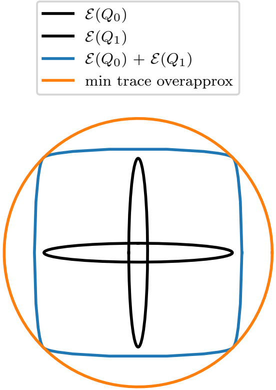
Overapproximating sum of ellipsoids by ellipsoid (cont.)

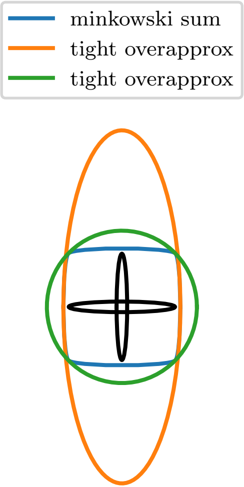
- In general: Choose α according to some criterion
  - e.g., such that $\mathcal{E}(Q(\alpha))$ has minimal size, e.g., $\min_{\alpha \in \mathcal{A}^N} \operatorname{tr}(Q(\alpha))$
  - or $\mathcal{E}(Q(\alpha))$ tight in a given direction $g \in \mathbb{R}^n$ (approximation touches true sum)

$$\min_{\alpha \in \mathcal{A}^N} \left(\max_{x \in \mathbb{R}^n} \; g^\top x \quad \text{s.t.} \quad x \in \mathcal{E}(Q(\alpha)) \right) = \min_{\alpha \in \mathcal{A}^N} \sqrt{g^\top Q(\alpha) g} \;\; \hat{=} \;\; \min_{\alpha \in \mathcal{A}^N} \operatorname{tr}(g g^\top Q(\alpha))$$

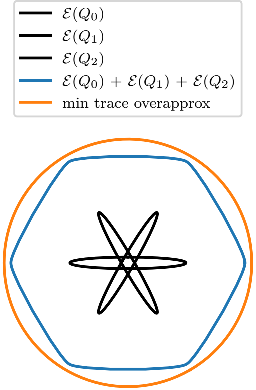
- Special case $N = 2$
  - $Q(\alpha) = \frac{1}{\alpha_1} Q_1 + \frac{1}{\alpha_2} Q_2$ with $\alpha_1 + \alpha_2 = 1$
  - Reparametrize: $\alpha_2 = 1 - \alpha_1$, $\beta = \frac{1}{1 - \alpha_1} > 0$
  - $\tilde{Q}(\beta) = (1 + \frac{1}{\beta}) Q_1 + (1 + \beta) Q_2$

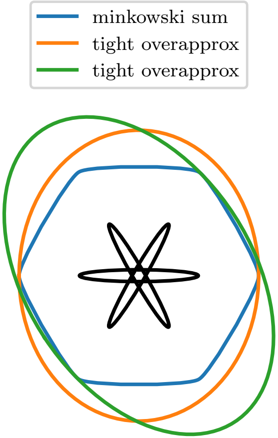
Overapproximations of sum of two ellipsoids





Overapproximations of sum of three ellipsoids





Uncertain linear dynamical system

$$x_{k+1} = A_k x_k + B_k u_k + \Gamma_k w_k,$$

- Reachable set

$$\begin{aligned} & x_k \in \mathcal{E}(\bar{x}_k, P_k),\ w_k \in \bar{\mathbb{W}} \\ \Rightarrow\ & x_{k+1} \in \tilde{\mathbb{X}}_{k+1} = \mathcal{E}(A_k \bar{x}_k + B_k u_k, A_k P_k A_k^\top) + \mathcal{E}(\Gamma_k \Gamma_k^\top) \end{aligned}$$

  - $\tilde{\mathbb{X}}_{k+1}$ not ellipsoidal
  - Overapproximate by ellipsoid

- Overapproximation of reachable set

$$\begin{aligned} & P_{k+1} = (1+\beta_k) A_k P_k A_k^\top + (1 + \tfrac{1}{\beta_k}) \Gamma_k \Gamma_k^\top \\ \Rightarrow\ & \tilde{\mathbb{X}}_{k+1} \subseteq \mathcal{E}(P_{k+1}) \\ \Rightarrow\ & x_{k+1} \in \mathcal{E}(P_{k+1}) \end{aligned}$$

Overapproximating tubes for stagewise ellipsoidal uncertainty

Robust optimal control for linear systems with linear state feedback

$$
\begin{aligned}
\min_{\bar{x},\bar{u},\beta,P,K} \quad & \sum_{k=0}^{N-1} \ell(\bar{x}_k,\bar{u}_k) + V_{\mathrm{f}}(\bar{x}_N) \\
\text{s.t.} \quad & \bar{x}_0 = \bar{\bar{x}}_0, \quad P_0 = 0, \\
& \bar{x}_{k+1} = A_k\bar{x}_k + B_k\bar{u}_k, \qquad k = 0,\dots,N-1, \\
& P_{k+1} = (1+\beta_k)(A_k - B_kK_k)P_k(A_k - B_kK_k)^\top + (1+(1/\beta_k))\Gamma_k\Gamma_k^\top, \\
& 0 \geq b_i + a_i^\top \bar{x}_k + \sqrt{a_i^\top P_k a_i}, \qquad i = 1,\dots,n_c, \\
& 0 \geq \tilde{b}_j + \tilde{a}_j^\top \bar{u}_k + \sqrt{\tilde{a}_j^\top K_k P_k K_k^\top \tilde{a}_j}, \quad j = 1,\dots,n_{\tilde{c}}
\end{aligned}
$$

- Conservative constraint satisfaction (Challenge 1) but suboptimal feedback. Non-convex.
- Not the same as - and cheaper than - dual norm formulation for ℓ_∞-ℓ_2-norm.
- Three types of "controls" with two different tasks
  - nominal $\bar{u} = (\bar{u}_0,\dots,\bar{u}_{N-1})$ influence $\bar{x}_k$
  - gains $K = (K_0,\dots,K_{N-1})$ and "Minkowski-multipliers" $\beta = (\beta_0,\dots,\beta_{N-1})$ influence P_k

Overview

1 Robust Dynamic Optimization Problem Statement
- Challenge 1: Robust constraint satisfaction
- Challenge 2: Feedback predictions
- Conceptual solution via Robust Dynamic Programming

2 Some exact NLP formulations for robust constraints (some with feedback)
- Scenario Tree for Finite Disturbances
- Scenario Tree for Polytopic Systems with Convex Costs and Constraints
- Dual norm formulation for systems that are affine in disturbances

3 Tube Based Formulations
- Ellipsoidal tubes - equivalent to robust ℓ_2-norm formulation
- Affine Disturbance Feedback Parameterization
- Overapproximating ellipsoidal tubes for stagewise bounded uncertainty
- Tube approximation for robust nonlinear MPC

Tube approximation for robust nonlinear MPC

- We switch to a nonlinear system

$$x_0 = \bar{\bar{x}}_0, \qquad x_{k+1} = f_k(x_k, u_k, w_k), \qquad k = 0, \ldots, N-1.$$

- $w = (w_0, \ldots, w_{N-1})$ is drawn from ℓ_2-ball with radius σ, i.e., $w \in \mathcal{E}(0, \sigma^2 I)$
- Similar approach with ellipsoids as before, but we will only have "approximate robustness" based on linearization at nominal trajectory

$$\bar{x}_0 = \bar{\bar{x}}_0, \qquad \bar{x}_{k+1} = f_k(\bar{x}_k, \bar{u}_k, 0)$$

$$A_k = \frac{\partial f_k}{\partial x_k}(\bar{x}_k, \bar{u}_k, 0), \quad B_k = \frac{\partial f_k}{\partial u_k}(\bar{x}_k, \bar{u}_k, 0), \quad \Gamma_k = \frac{\partial f_k}{\partial w_k}(\bar{x}_k, \bar{u}_k, 0), \qquad k = 0, \ldots, N-1.$$

Feedback to reduce the uncertainty

- Plan with linear feedback law to reduce uncertainty

$$u_k = \kappa_k(x_k) = \bar{u}_k + K_k(x_k - \bar{x}_k), \quad k = 0, \ldots, N-1, \quad K_0 = 0.$$

- Propagate ellipsoids according to linearized dynamics

$$P_0 = 0, \quad P_{k+1} = \underbrace{(A_k + B_k K_k) P_k (A_k + B_k K_k)^\top + \sigma^2 \Gamma_k \Gamma_k^\top}_{=: \psi(\bar{x}_k, \bar{u}_k, P_k, K_k)}$$

- Left out here, but could also generalize to ℓ_∞-ℓ_2-norms by including Minkowski-multipliers β_k, or to affine disturbance feedback

Closed-loop Robustified NMPC problem

$$
\begin{aligned}
\min_{\bar{x},\bar{u},P,K} \quad & \sum_{k=0}^{N-1} \ell_k(\bar{x}_k,\bar{u}_k) + V_{\mathrm{f}}(\bar{x}_N) \\
\text{s.t.} \quad & \bar{x}_0 = \bar{\bar{x}}_0, \quad P_0 = 0, \\
& \bar{x}_{k+1} = f_k(\bar{x}_k,\bar{u}_k,0), \qquad\qquad k = 0,\dots,N-1, \\
& P_{k+1} = \psi_k(\bar{x}_k,\bar{u}_k,P_k,K_k), \\
& 0 \geq h_k(\bar{x}_k,\bar{u}_k) + b_k(\bar{x}_k,\bar{u}_k,P_k,K_k), \\
& 0 \geq h_N(\bar{x}_N) + b_N(\bar{x}_N,P_N).
\end{aligned}
$$

$$
\begin{aligned}
b_k^i(\bar{x}_k,\bar{u}_k,P_k,K_k) &= \sqrt{\nabla h_k^i(\bar{x}_k,\bar{u}_k)^\top \begin{bmatrix} I & K_k^\top \end{bmatrix}^\top P_k \begin{bmatrix} I & K_k^\top \end{bmatrix} \nabla h_k^i(\bar{x}_k,\bar{u}_k)}, \\
b_N^i(\bar{x}_N,P_N) &= \sqrt{\nabla h_N^i(\bar{x}_N)^\top P_N \nabla h_N^i(\bar{x}_N)},
\end{aligned}
$$

Zero-Order Robust Optimization (ZORO) algorithm

[Zanelli et al.: Zero-order robust nonlinear model predictive control with ellipsoidal uncertainty sets, IFAC, 2021], [Frey et al.: Efficient Zero-Order Robust Optimization with acados, ECC, 2024]

- fix gains K_k (e.g. set to zero)
- iterate between (A) nominal problem with fixed backoffs, and (B) matrix propagation
- converges to feasible but suboptimal solution of combined problem on previous slide

(A) Nominal problem with backoffs - standard NMPC problem

$$\begin{aligned}
\min_{\bar{x},\bar{u}} \quad & \sum_{k=0}^{N-1} \ell_k(\bar{x}_k, \bar{u}_k) + V_{\mathrm{f}}(\bar{x}_N) \\
\text{s.t.} \quad & \bar{x}_0 = \bar{\bar{x}}_0, \quad x_{k+1} = f_k(\bar{x}_k, \bar{u}_k, 0), \qquad k = 0, \dots, N-1, \\
& 0 \geq h_k(\bar{x}_k, \bar{u}_k) + b_k, \quad 0 \geq h_N(\bar{x}_N) + b_N
\end{aligned}$$

(B) Matrix propagation to compute backoffs

$$\begin{aligned}
& P_0 := 0, \quad P_{k+1} := \psi_k(\bar{x}_k, \bar{u}_k, P_k, K_k), \\
& b_k^i := \sqrt{\nabla h_k^i(\bar{x}_k, \bar{u}_k)^\top \begin{bmatrix} I & K_k^\top \end{bmatrix}^\top P_k \begin{bmatrix} I & K_k^\top \end{bmatrix} \nabla h_k^i(\bar{x}_k, \bar{u}_k)}, \qquad k = 0, \dots, N-1 \\
& b_N^i := \sqrt{\nabla h_N^i(\bar{x}_N)^\top P_N \nabla h_N^i(\bar{x}_N)}
\end{aligned}$$

Conclusions

- Robust optimal control needs to address two challenges: robust constraint satisfaction, and feedback predictions
- Robust Dynamic Programming (RDP) conceptually solves the robust OCP exactly
- Scenario-trees allow one to exactly solve the problem for finite uncertainties and polytopic systems, but suffer from exponential growth
- dual-norm based approaches can guarantee robust constraint satisfaction for systems affine in the uncertainty
- affine disturbance feedback is an elegant but expensive way to incorporate feedback
- ellipsoidal tube based uncertainty propagations can lead to conservative approximations
- robust nonlinear MPC problems can be addressed by linearization
- zero-order robust optimization (ZORO) quickly computes feasible but suboptimal solutions

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