# 3. Introduction to nonsmooth differential equations and hybrid systems

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# Outline of the lecture

# Introduction

**Definition**

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2.) **External**, time depended, on/off decisions - modeled with integer variables

Internal switches:

- Arise whenever first principles are coupled with *if-else* statements.
- From macroscopic empirical laws (Coulomb friction, contacts, flow reversal, ...).

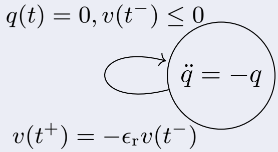
# Difference between hybrid automata and nonsmooth systems - Example 1

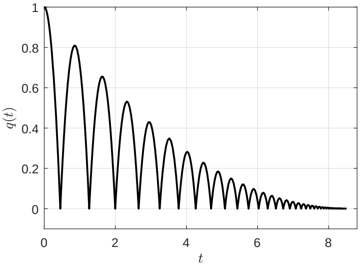
There are many different ways to model the same

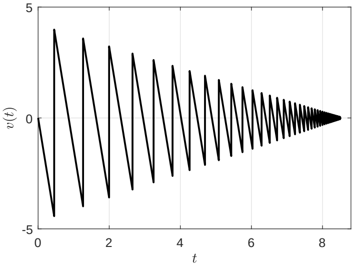
## Nonsmooth dynamical system

$$\ddot{q} = -g + \lambda$$
$$0 \le q \perp \lambda \ge 0$$
$$\text{if } q(t) = 0,\ v(t^-) \text{ and } \le 0,$$
$$\text{then } v(t^+) = -\epsilon_{\mathrm{r}} v(t^-)$$

## Hybrid dynamical system



# Difference between hybrid automata and nonsmooth systems - Example 2

### Nonsmooth dynamical system

$$\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{bmatrix} = \begin{bmatrix} u_1 + \lambda_1 \\ -g + u_2 + \lambda_2\end{bmatrix}$$

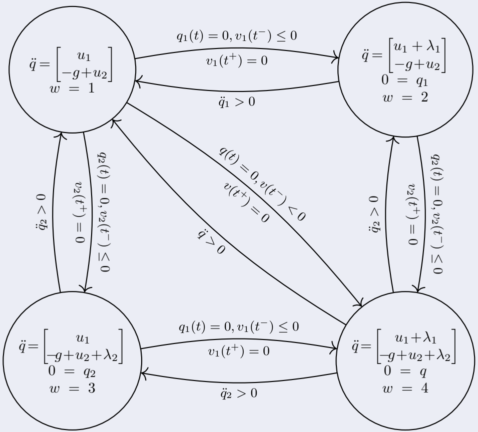
$$0 \leq q_1 \perp \lambda_1 \geq 0$$

$$0 \leq q_2 \perp \lambda_2 \geq 0$$

if $q_i(t) = 0$ and $v_i(t^-) \leq 0$,

then $v_i(t^+) = 0 \; i = 1, 2$

### Hybrid dynamical system




- Nonsmooth: easy to write down and do computations with, hard to analyze.
- Automaton: harder to write down and do computations, easier to analyze.

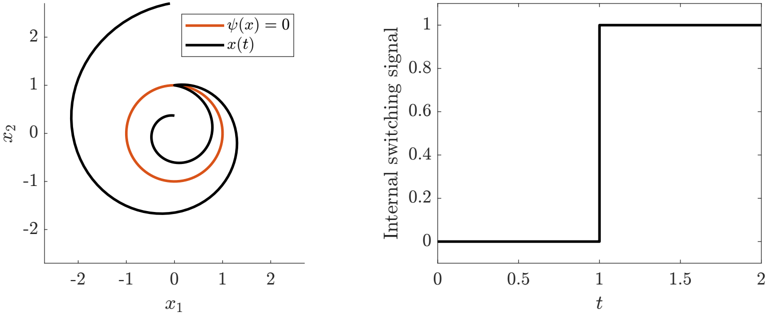
# Classification of hybrid systems w.r.t. what triggers a switch

Nonsmooth/hybrid systems experience switches and jumps

## Type of switches

Depending on how the discrete events or switches are triggered, we distinguish between:

1.) **internal switches**: triggered implicitly, depending on the systems' differential state




Switch can happen only when $x(t)$ reaches some $S = \{x \in \mathbb{R}^n \mid \psi(x) = 0\}$

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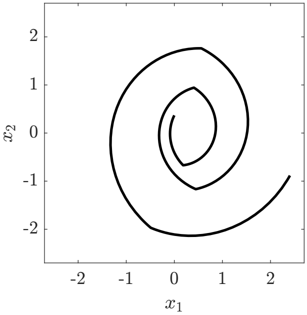
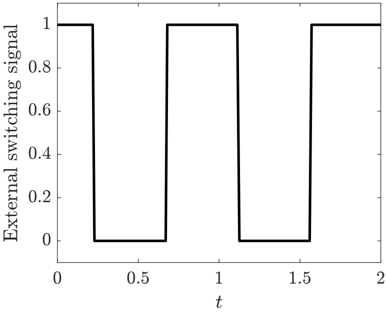
1.) **internal switches**: triggered implicitly, depending on the systems' differential state

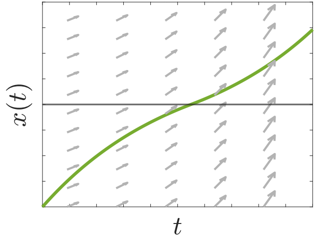
2.) external switches: triggered explicitly, independent of the differential state



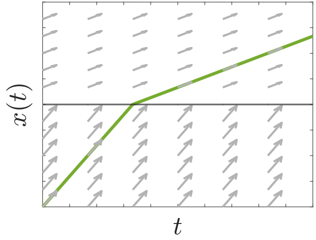




Switch can happen anytime - no matter where $x(t)$ is in the state space

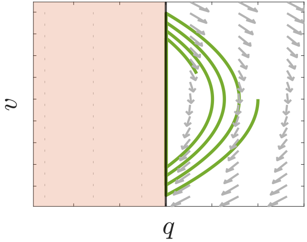
Ordinary differential equation (ODE) with a nonsmooth right-hand side (RHS).




**NSD1**
non-differentiable RHS




**NSD2**
discontinuous RHS




**NSD3**
state dependent jump

Ordinary differential equation (ODE) with a nonsmooth right-hand side (RHS).

Continuous activation functions in the RHS

$$\dot{x} = 1 + \max(0, x)$$

Continuous non-diff. ODEs

$$\dot{x} = 1 + |x|$$

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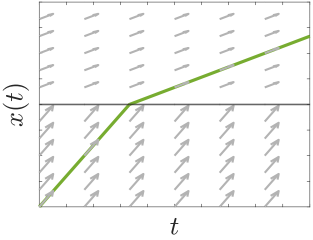


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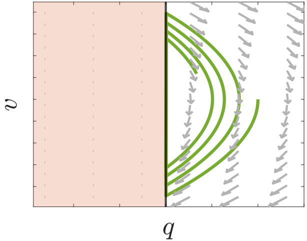


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Piecewise smooth systems

$$\dot{x} = f_i(x), \text{ if } x \in R_i$$
$$i = 1, \ldots, m$$

Projected dynamical systems

$$\dot{x} = \mathrm{P}_{\mathcal{T}_C(x)}(f(x))$$

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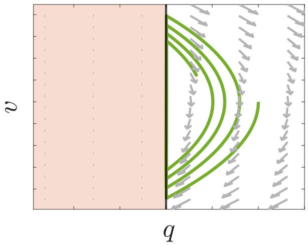


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Rigid bodies with impacts and friction

$$\begin{aligned} \dot{q} &= v \\ M(q)\dot{v} &= f_{\mathrm{v}}(q,v) + J_{\mathrm{n}}(q)\lambda_{\mathrm{n}} \\ 0 \leq \lambda_{\mathrm{n}} &\perp f_{\mathrm{c}}(q) \geq 0 \end{aligned}$$
$$(\text{state jump law for } v)$$

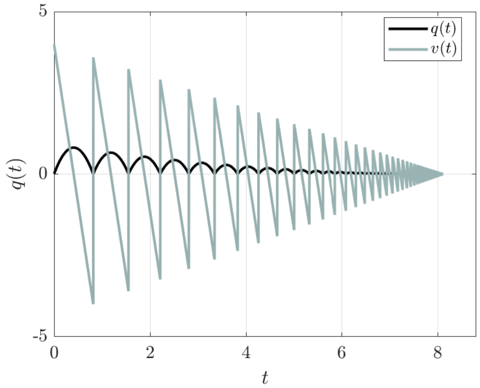
**NSD3**
state dependent jump

# Infinitely many switches in finite time - Zeno's phenomenon

## The bouncing ball example - NSD3

$$\begin{aligned} \dot{q}(t) &= v(t) \\ m\,\dot{v}(t) &= -g \\ v(t^+) &= -\epsilon_{\mathrm{r}} v(t^-),\ \text{if } v(t^-) \leq 0 \text{ and } q(t) = 0 \\ q(0) &= 0,\ \ v(0) > 0 \end{aligned}$$

- Coefficient of restitution $\epsilon_{\mathrm{r}} \in [0,1]$, e.g., $\epsilon_{\mathrm{r}} = 0.9$.
- $t_1 = \frac{2v(0)}{g},\ t_2 = t_1 + \frac{2\epsilon_{\mathrm{r}} v(0)}{g}, \ldots$.
- $\Delta_{k+1} = t_{k+1} - t_k = \frac{2\,\epsilon_{\mathrm{r}}^k\, v(0)}{g}$.
- Since $\epsilon_{\mathrm{r}} < 1$ it follows that $\lim_{k\to\infty} \Delta_k = 0$.

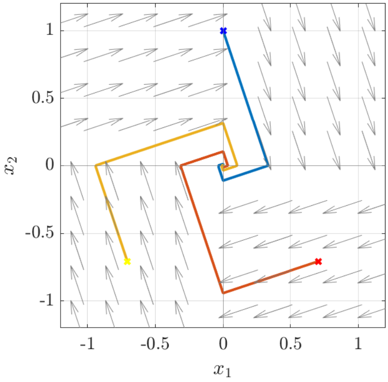


# Infinitely many switches in finite time - Zeno's phenomenon

**A Filippov system - NSD2**

$$\dot{x}_1 \in -\text{sign}(x_1) + 2\text{sign}(x_2)$$
$$\dot{x}_2 \in -2\text{sign}(x_1) - \text{sign}(x_2)$$

- Real world system do not experience Zeno.
- By modeling and design one wants to avoid this behavior.
- Might complicate the numerical computations sometimes.



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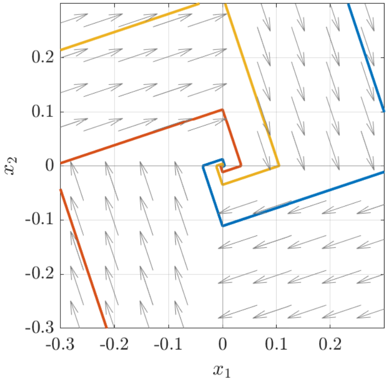


Zoom in: trajectories spiral down.

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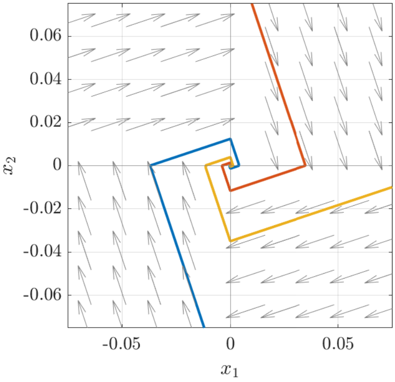


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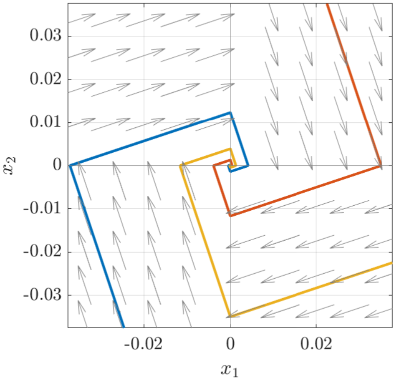


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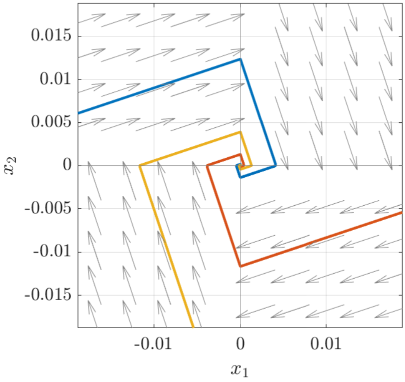


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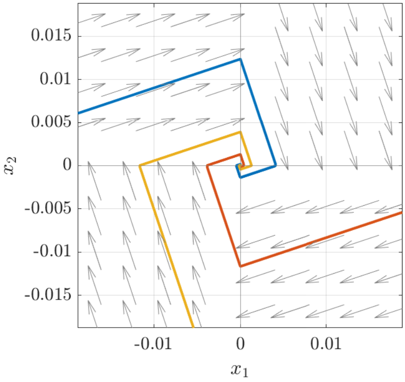


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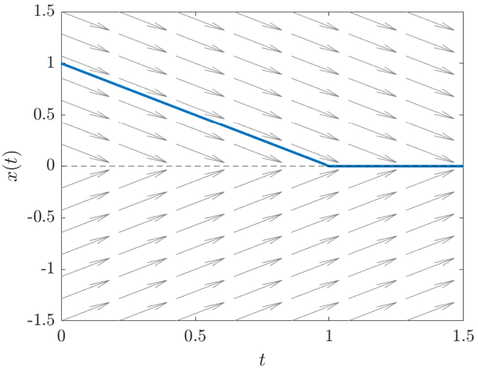
# Reduced systems dimensions and sliding modes

## A sliding mode example

$$\dot{x} \in -\text{sign}(x)$$

- System evolves on surface of discontinuity.
- Need to define meaningful dynamics (treated later in detail).



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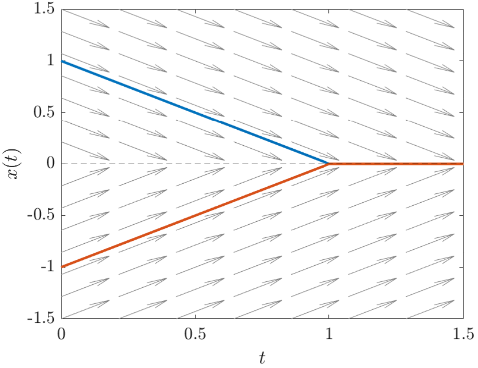

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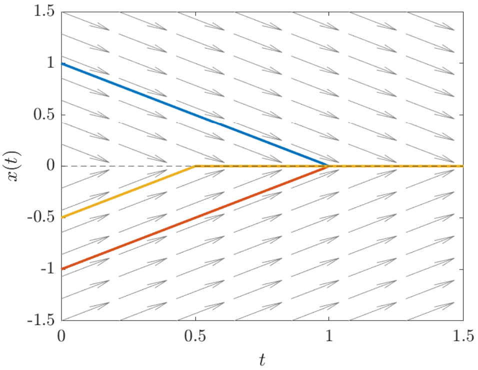

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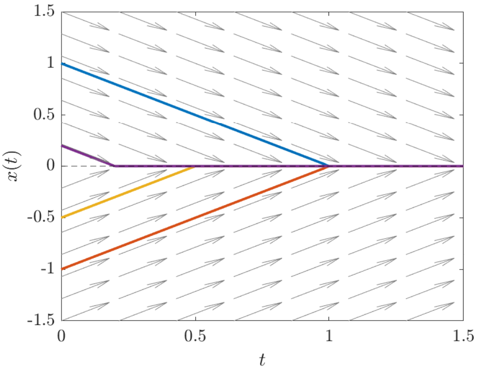
- Dynamics switch from ODE to DAE of higher index.

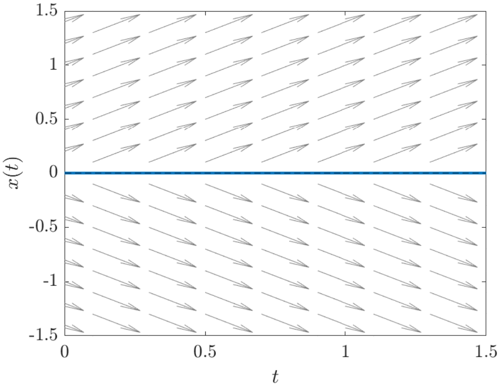


# Nonunique solutions

**Nonunique solutions example**

$$\dot{x} \in \operatorname{sign}(x),\ x(0) = 0$$

- In nonsmooth systems examples with nonunique solutions easily constructed.
- It may not be clear what numerical algorithms do.



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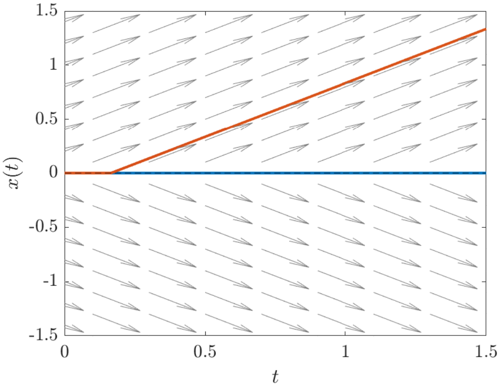

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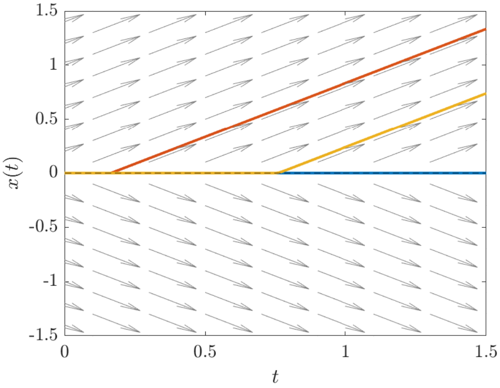

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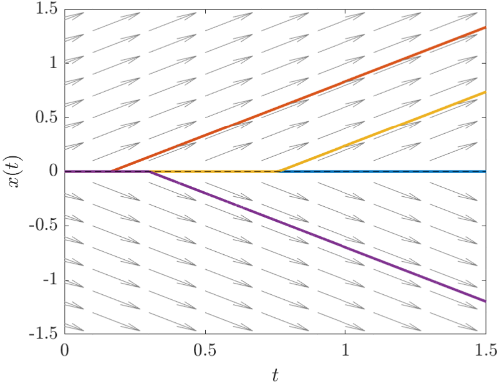


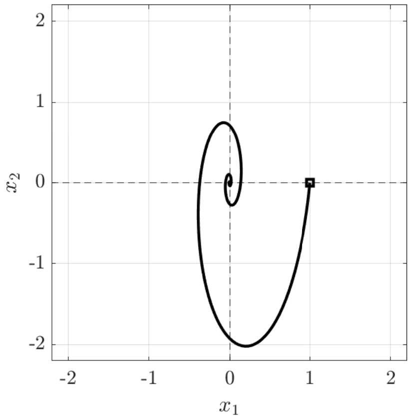
## Unstable piecewise affine systems

$$\dot{x} = f(x) := \begin{cases} A_1 x, & \text{if } x_1 x_2 \leq 0 \\ A_2 x, & \text{if } x_1 x_2 > 0 \end{cases}$$

with

$$A_1 = \begin{bmatrix} -1 & 1 \\ -10 & -1 \end{bmatrix}, \; A_2 = \begin{bmatrix} -1 & 10 \\ -1 & -1 \end{bmatrix}$$

- First and third quadrant: $\dot{x} = A_1 x$.




$\dot{x} = A_1 x$ - stable

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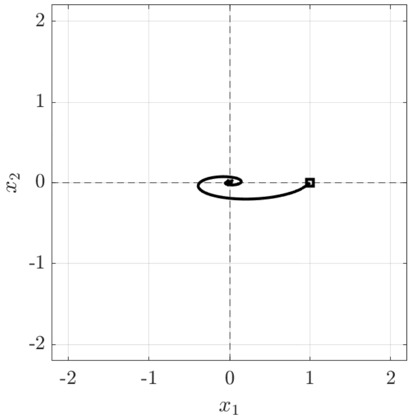


$\dot{x} = A_2 x$ - stable

# Stability and instability due to switches and jumps

Example from [Branicky, 1998]

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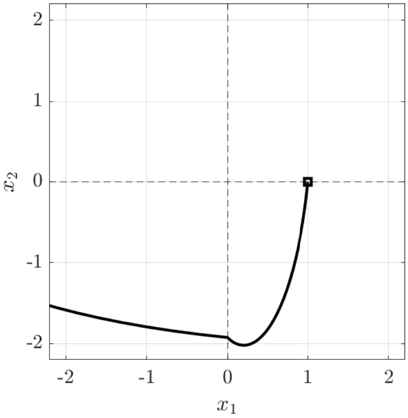
- Nonsmooth systems can have stable *modes* but still be overall unstable.

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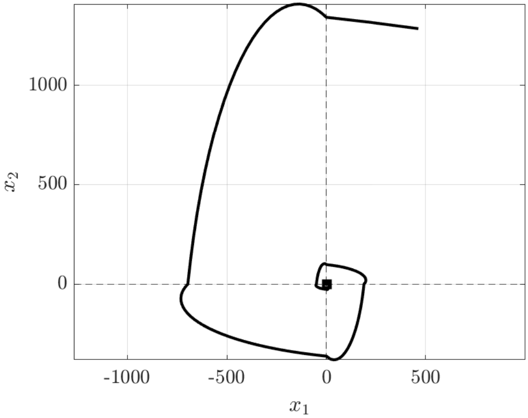


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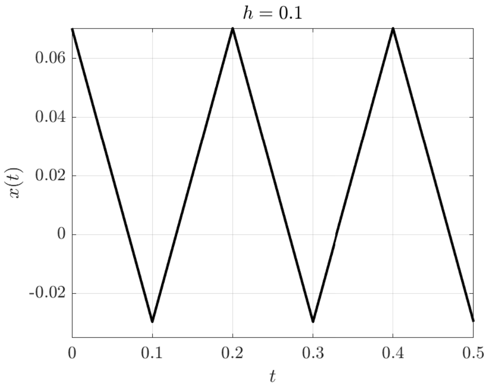
# Numerical chattering in sliding mode examples

## Explicit Euler for nonsmooth systems

$$x_{k+1} = x_k - h\,\mathrm{sign}(x_k)$$

- In presence of discontinuities numerical solutions can *chatter* around a discontinuity.



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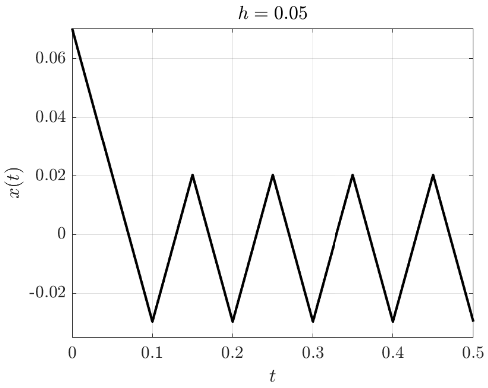

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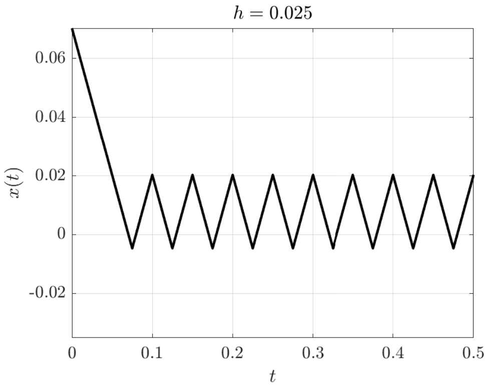

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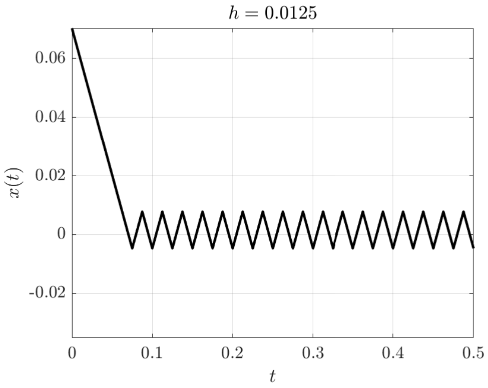

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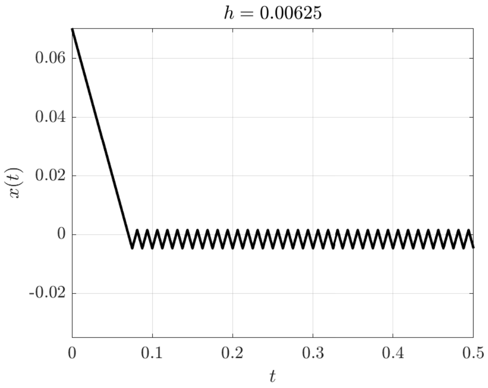

# Numerical chattering in sliding mode examples

## Explicit Euler for nonsmooth systems

$$x_{k+1} = x_k - h\,\mathrm{sign}(x_k)$$

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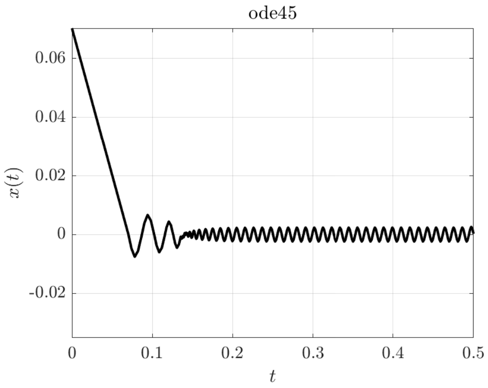
- Even sophisticated codes may struggle.



# Numerical chattering in sliding mode examples

**Explicit Euler for nonsmooth systems**

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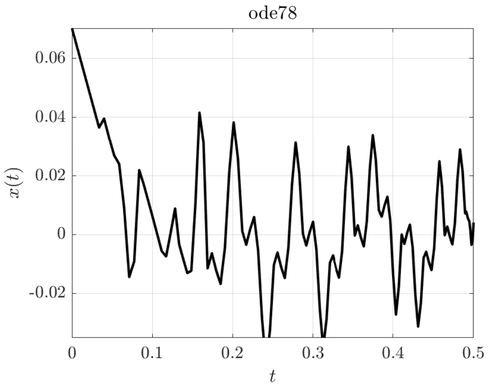

# Numerical chattering in sliding mode examples

**Explicit Euler for nonsmooth systems**

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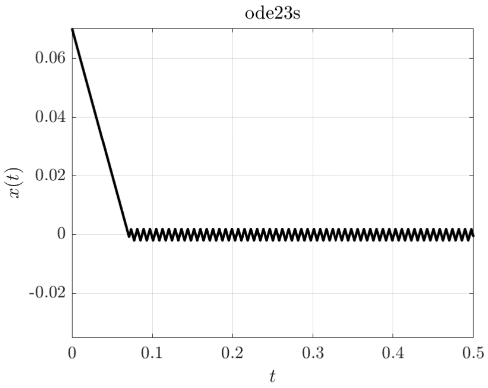

# Numerical chattering in sliding mode examples

## Explicit Euler for nonsmooth systems

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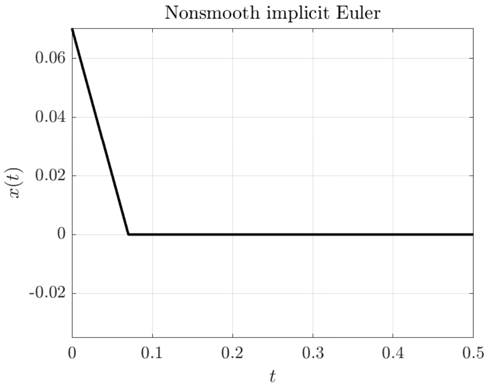
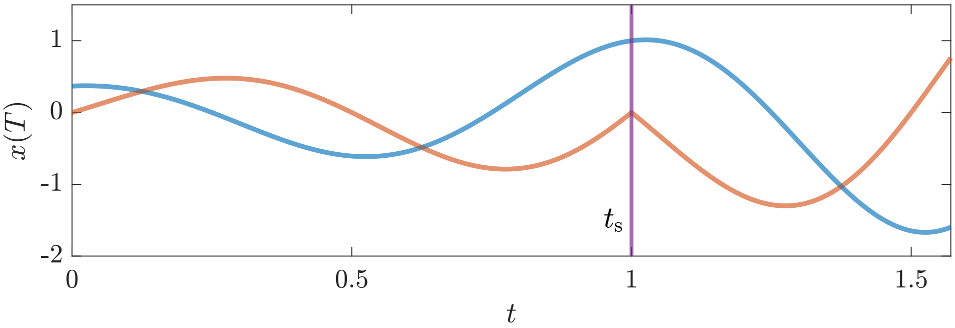
- In presence of discontinuities numerical solutions can *chatter* around a discontinuity.
- Decreasing the step size might worsen things.
- Even sophisticated codes may struggle.
- Method converges - but qualitative behavior is not good.
- Nonsmooth implicit methods resolve the issue (Lecture 4).



# Outline of this lecture

## Approaches to discretize and simulate a nonsmooth ODE

1) event-capturing, time-stepping methods (can handle Zeno, low accuracy)



# Time discretization methods for nonsmooth ODEs

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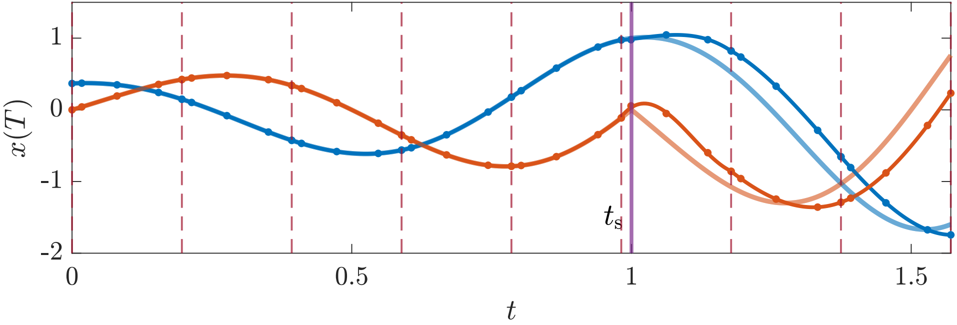

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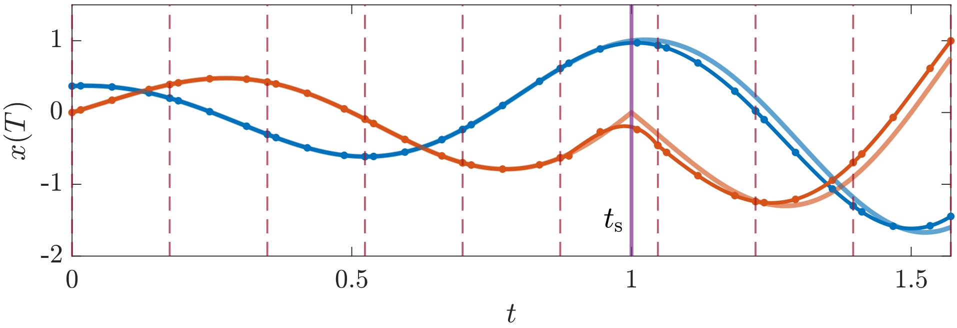
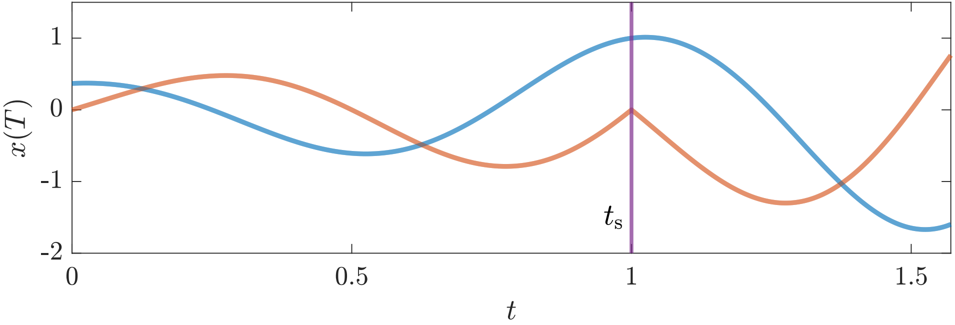


# Time discretization methods for nonsmooth ODEs

## Approaches to discretize and simulate a nonsmooth ODE

1) event-capturing, time-stepping methods (can handle Zeno, low accuracy)
2) smoothing and penalty methods (low accuracy, easy to implement)



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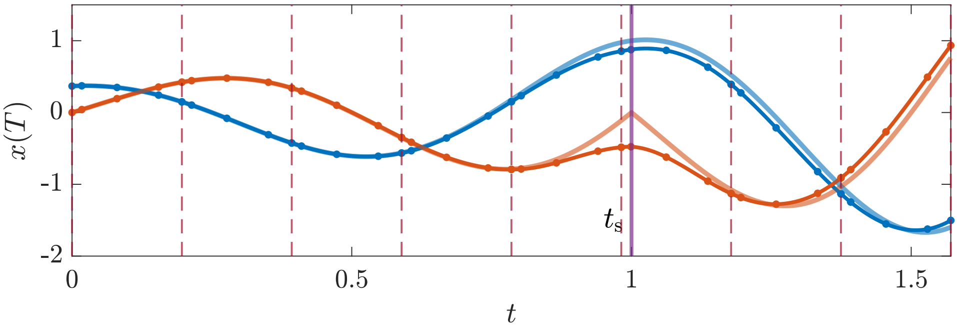

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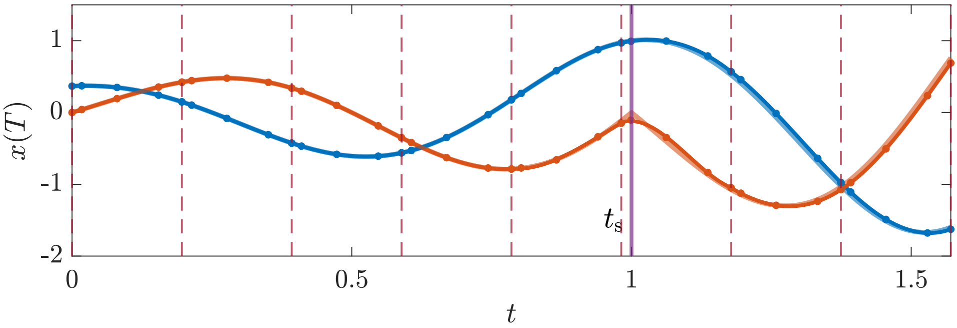

# Time discretization methods for nonsmooth ODEs

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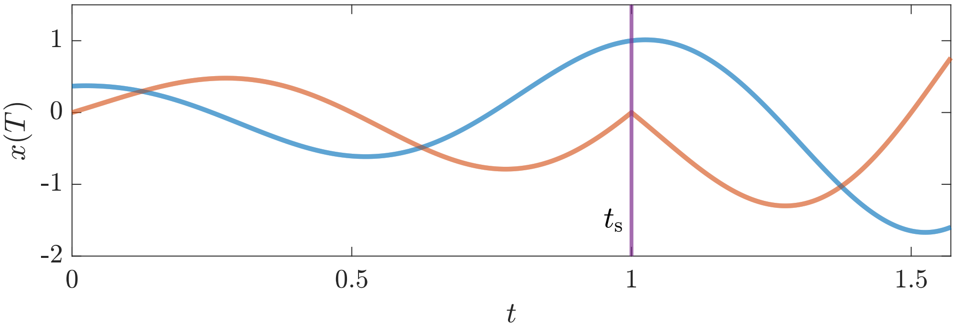
3) event-driven, switch-detecting, active-set methods (cannot handle Zeno, high accuracy)



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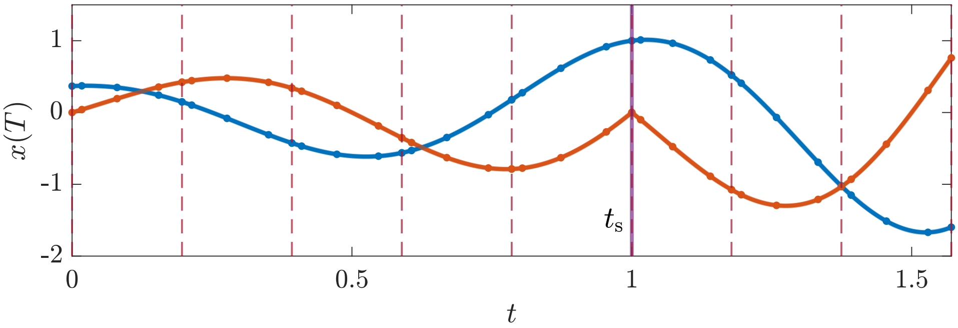


# Integration order plots for different simulation methods

Compute global integration error $E(T)$ using different strategies

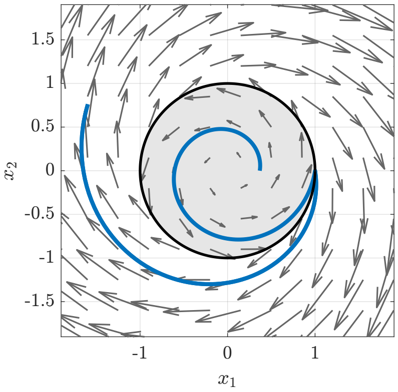
## Tutorial example

$$\dot{x} = \begin{cases} A_1 x, & \psi(x) < 0, \\ A_2 x, & \psi(x) > 0 \end{cases}$$

with $A_1 = \begin{bmatrix} 1 & 2\pi \\ -2\pi & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & -2\pi \\ 2\pi & 1 \end{bmatrix}$,
$\psi(x) = \|x\|_2^2 - 1$, $x(0) = (e^{-1}, 0)$,

Compute solution approximation:

1. with fixed step size IRK methods (time-stepping),



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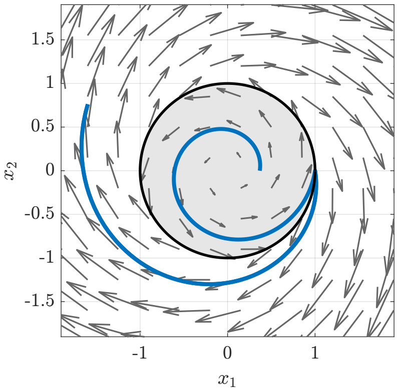
Compute solution approximation:

1. with fixed step size IRK methods (time-stepping),
2. with sophisticated adaptive step size methods (time-stepping),



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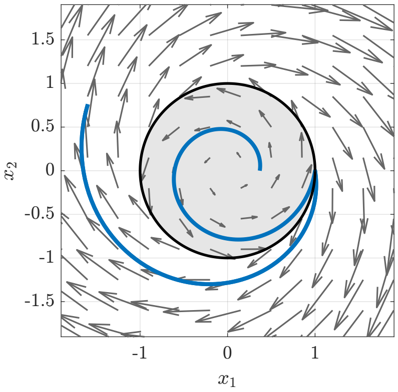


Compute solution approximation:

1. with fixed step size IRK methods (time-stepping),
2. with sophisticated adaptive step size methods (time-stepping),
3. with switch detecting integrators,

# Integration order plots for different simulation methods

Compute global integration error $E(T)$ using different strategies

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$$\dot{x} = \begin{cases} A_1 x, & \psi(x) < 0, \\ A_2 x, & \psi(x) > 0 \end{cases}$$

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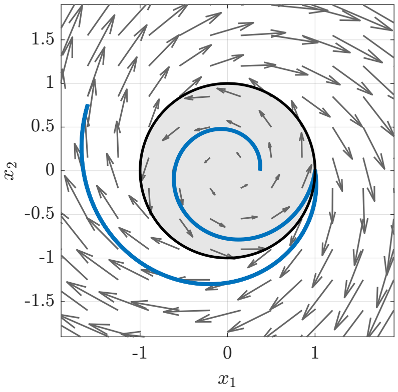
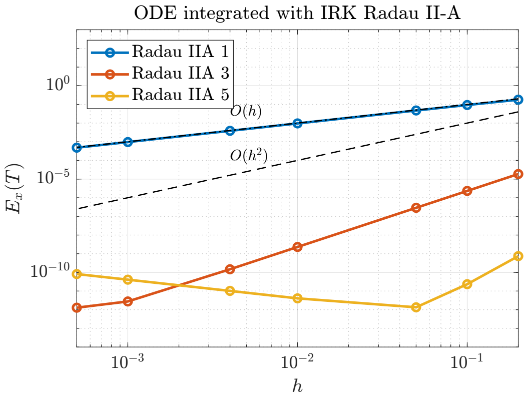
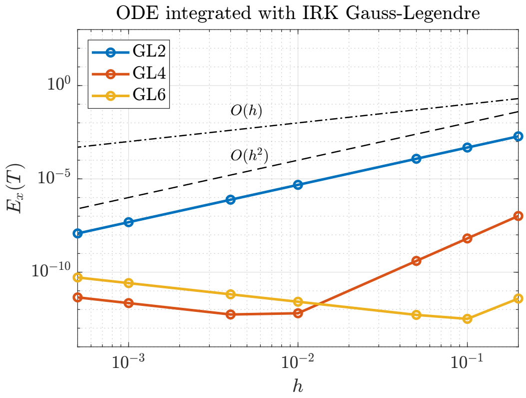



Compute solution approximation:

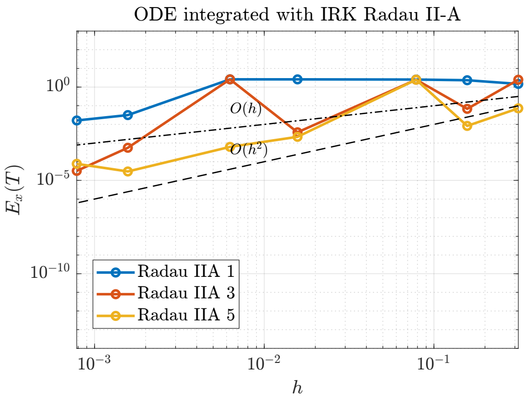
1. with fixed step size IRK methods (time-stepping),
2. with sophisticated adaptive step size methods (time-stepping),
3. with switch detecting integrators,
4. via smoothed approximations.

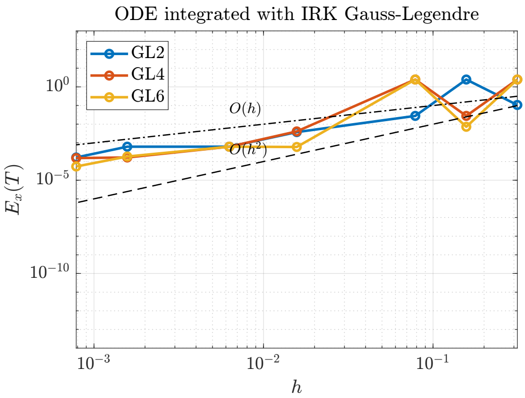


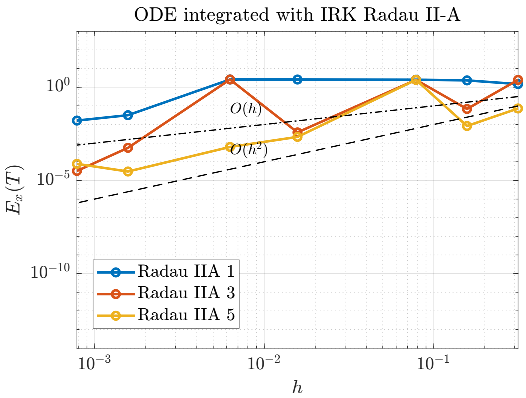

Simulation time $T = 1$, no switch yet, high accuracy

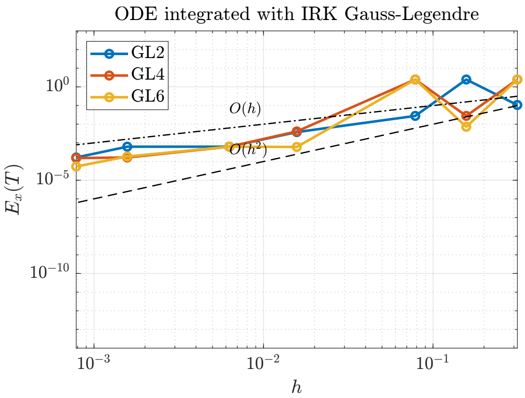
# 1. Integration order plots fixed step size Implicit Runge-Kutta methods



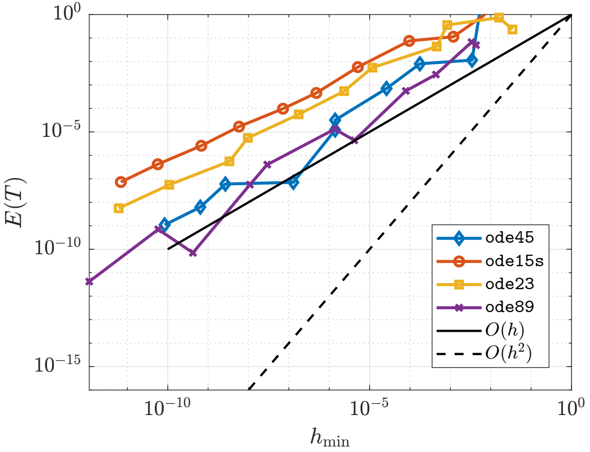

Simulation time $T = \pi/2$, switch happened, low accuracy



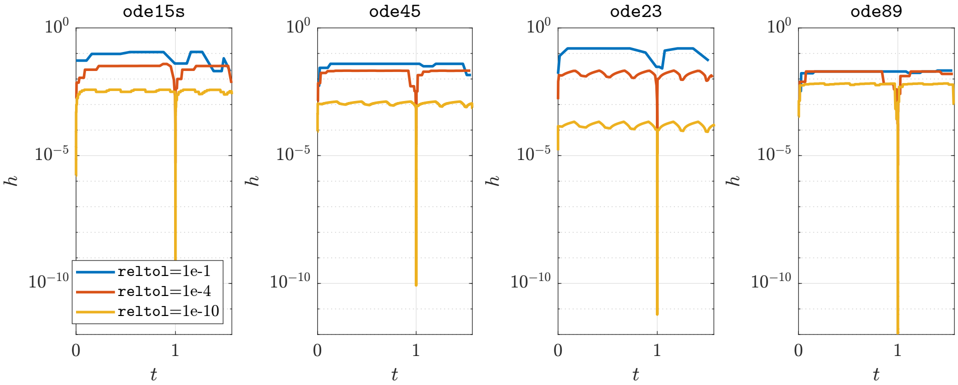

**The nonsmoothness leads to severe order reduction, all methods have $O(h)$ accuracy.**

# 2. Integration order plots adaptive step size methods

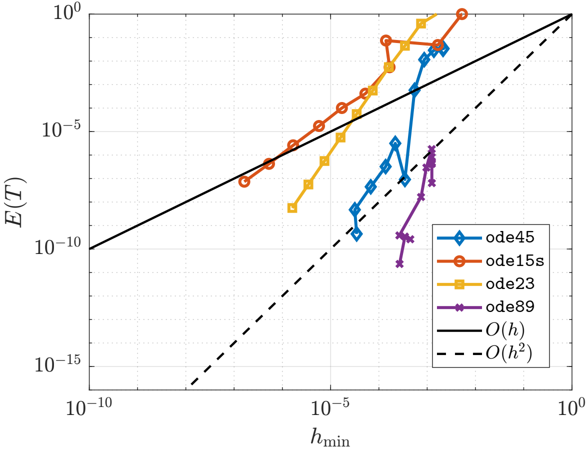



Very small step size necessary to achieve high accuracy even with very sophisticated methods.

# 2. Integration order plots adaptive step size methods

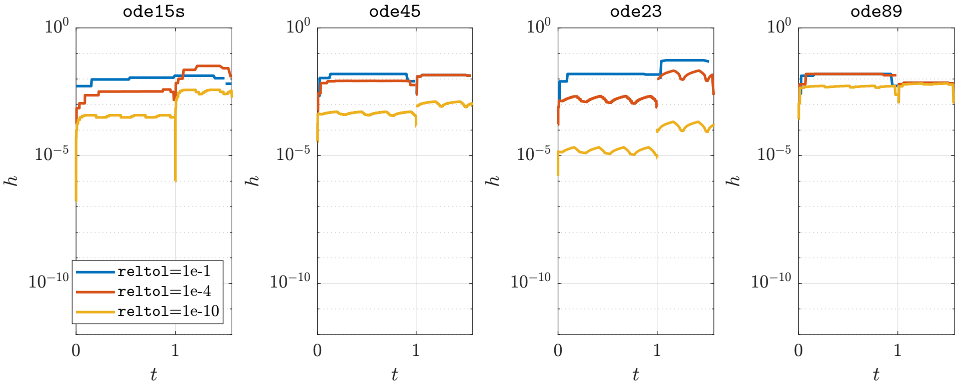



Step size small around switch - many switches = very slow integration.




Switch detected explicitly - high accuracy properties recovered.

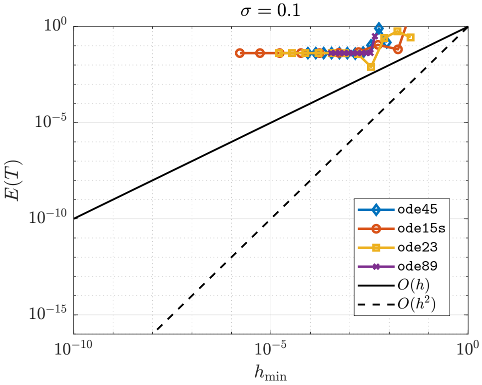
# 3. Adaptive step size methods **with switch detection**




No extremely small step sizes around the switch.

## Smooth approximation parameterized by $\sigma$

$$\dot{x} = (1-\alpha_\sigma(x))A_1 x + \alpha_\sigma(x) A_2 x, \;\; \alpha_\sigma(x) = \frac{1}{2}\left(1 - \tanh\left(\frac{\psi(x)}{\sigma}\right)\right)$$



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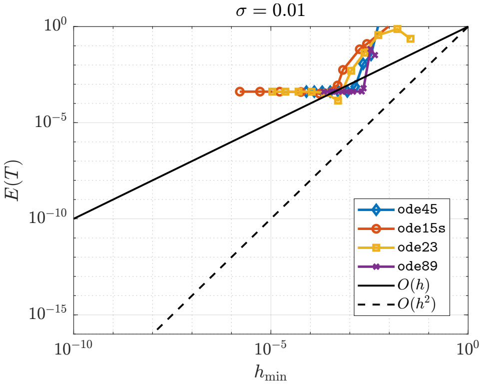

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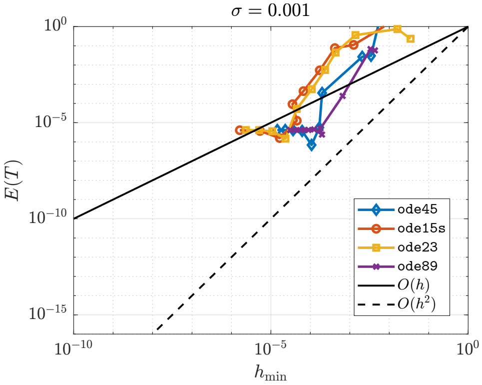

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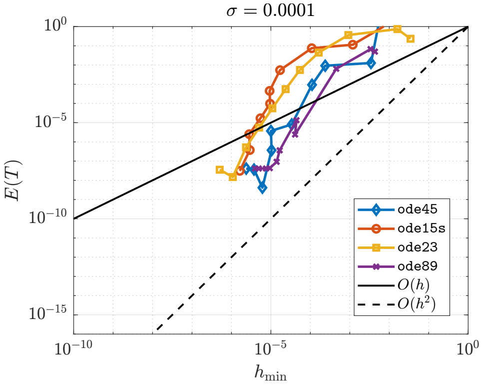
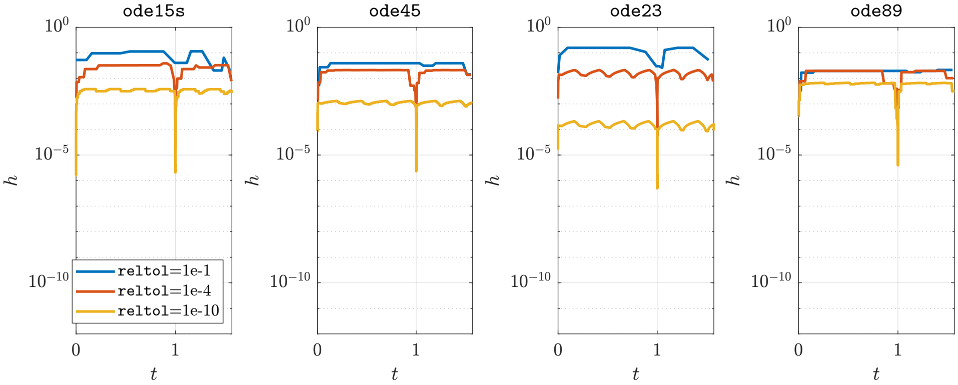


# 4. Accuracy order plots for smoothing

## Error dominated by $\sigma$

### Smooth approximation parameterized by $\sigma$

$$\dot{x} = (1-\alpha_\sigma(x))A_1x + \alpha_\sigma(x)A_2x,\ \ \alpha_\sigma(x) = \frac{1}{2}\left(1-\tanh\left(\frac{\psi(x)}{\sigma}\right)\right)$$

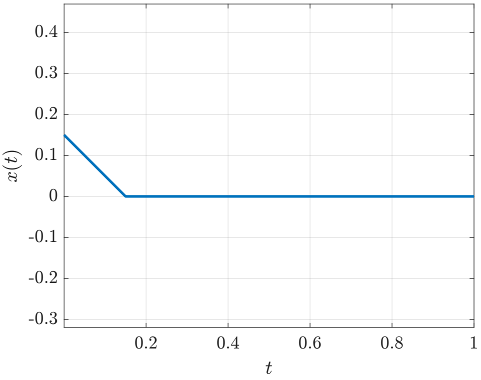


# Smoothed sliding mode example

Error dominated by $\sigma$

## Smooth approximation parameterized by $\sigma = 10^{-5}$

$$\dot{x} = -\text{sign}(x)$$

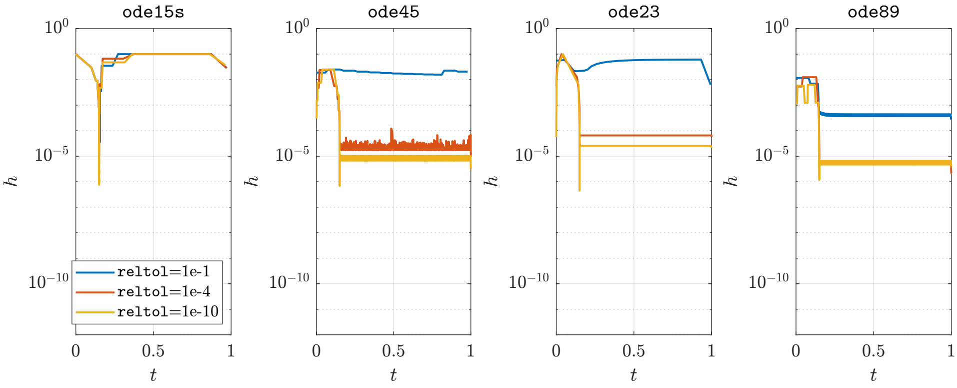


# Smoothed sliding mode example

Error dominated by $\sigma$

## Smooth approximation parameterized by $\sigma = 10^{-5}$

$$\dot{x} = -\tanh\left(\frac{x}{\sigma}\right)$$



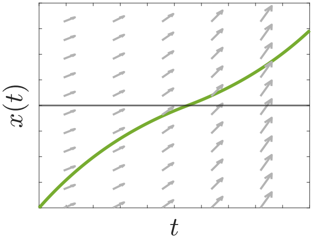

Small $\sigma$ makes system very stiff - small step sizes.

# Outline of the lecture

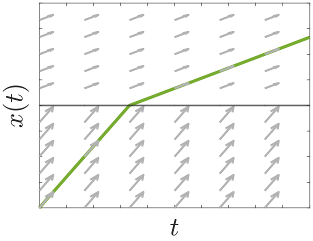
# Classification nonsmooth dynamical systems

## Classification of NonSmooth Dynamics (NSD)

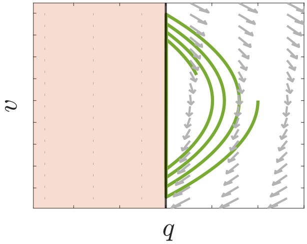
Ordinary differential equation (ODE) with a nonsmooth right-hand side (RHS).




**NSD1**
non-differentiable RHS




**NSD2**
discontinuous RHS




**NSD3**
state dependent jump

Ordinary differential equation (ODE) with a nonsmooth right-hand side (RHS).

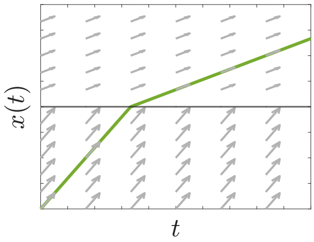
Continuous activation functions in the RHS

$$\dot{x} = 1 + \max(0, x)$$

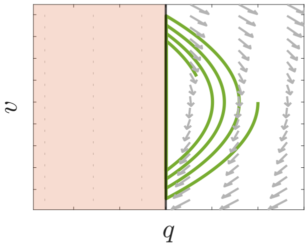
Continuous non-diff. ODEs

$$\dot{x} = 1 + |x|$$

**NSD1**
non-differentiable RHS




**NSD2**
discontinuous RHS

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Continuous non-diff. ODEs

$$\dot{x} = 1 + |x|$$

**NSD1**
non-differentiable RHS

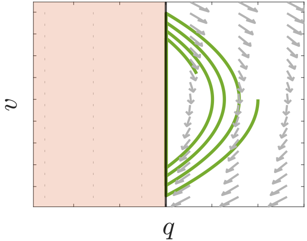
Piecewise smooth systems

$$\dot{x} = f_i(x), \text{ if } x \in R_i$$
$$i = 1, \dots, m$$

Projected dynamical systems

$$\dot{x} = \mathrm{P}_{\mathcal{T}_C(x)}(f(x))$$

**NSD2**
discontinuous RHS

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Projected dynamical systems

$$\dot{x} = \mathrm{P}_{\mathcal{T}_C(x)}(f(x))$$

**NSD2**

discontinuous RHS

Rigid bodies with impacts and friction

$$\begin{aligned} \dot{q} &= v \\ M(q)\dot{v} &= f_{\mathrm{v}}(q, v) + J_{\mathrm{n}}(q)\lambda_{\mathrm{n}} \\ 0 &\leq \lambda_{\mathrm{n}} \perp f_{\mathrm{c}}(q) \geq 0 \\ &\text{(state jump law for } v) \end{aligned}$$

**NSD3**

state dependent jump

# Differential-Algebraic Equations (DAE) - semi-explicit form

Extend the ODE by algebraic equations $g$ and algebraic states $z$:

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t), z(t))\\ 0 &= g(x(t), z(t), u(t))\end{aligned}$$

- differential states: $x(t) \in \mathbb{R}^{n_x}$
- algebraic states: $z(t) \in \mathbb{R}^{n_z}$
- control input: $u(t) \in \mathbb{R}^{n_u}$
- no $z(0)$ needed, implicitly determined via $0 = g(x_0, z(0), u(0))$

**Simplified view**: introduce $n_z$ new variables $z(t)$, and $n_z$ new algebraic equations $g(x, z, u) = 0$ to compute them.

# Modeling of nonsmoothness with convex optimization

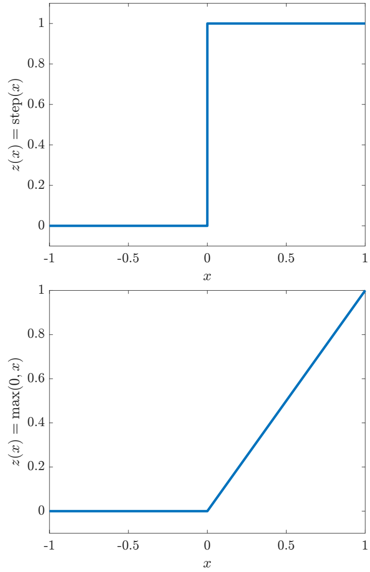
**Ordinary differential inclusion/equation**

$$\dot{x}(t) \in f(x(t), u(t), z(x(t)))$$

nonsmoothness modeled via $z(x)$:

**Convex optimization problem**

$$z(x) \in \underset{z}{\operatorname{argmin}}\ F(z, x) \quad \text{s.t.} \quad H(z, x) \geq 0$$



# Modeling of nonsmoothness with convex optimization

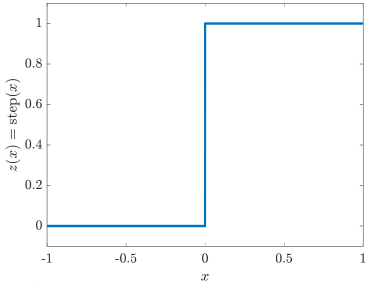
## Ordinary differential inclusion/equation

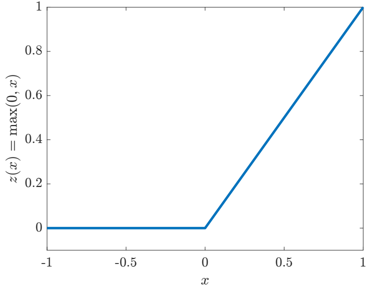
$$\dot{x}(t) \in f(x(t), u(t), z(x(t)))$$

nonsmoothness modeled via $z(x)$:

## Dynamic complementary system

$$\begin{aligned}\dot{x}(t) &= f(x, u, z) \\ 0 &= \nabla_z F(z, x) - \nabla_z H(z, x)\mu \\ 0 &\leq \mu \perp H(z, x) \geq 0\end{aligned}$$



# Modeling of nonsmoothness with convex optimization

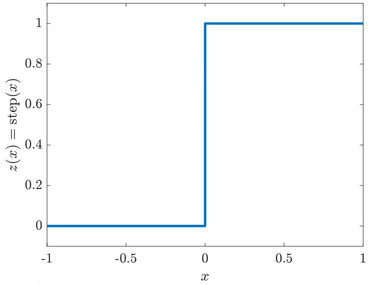
**Ordinary differential inclusion/equation**

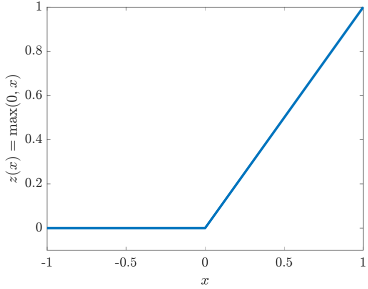
$$\dot{x}(t) \in f(x(t), u(t), z(x(t)))$$

nonsmoothness modeled via $z(x)$:

**Dynamic complementary system = nonsmooth DAE**

$$\dot{x}(t) = f(x, u, z)$$
$$0 = \nabla_z F(z, x) - \nabla_z H(z, x)\mu$$
$$0 = \min(\mu, H(z, x))$$



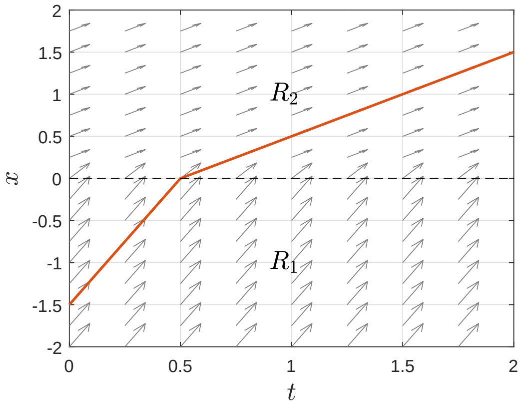



Consider the ODE

$$\dot{x} = 2 - \operatorname{sign}(x)$$

More explicitly:

$$\dot{x} = \begin{cases} 3, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$$



Consider the ODE

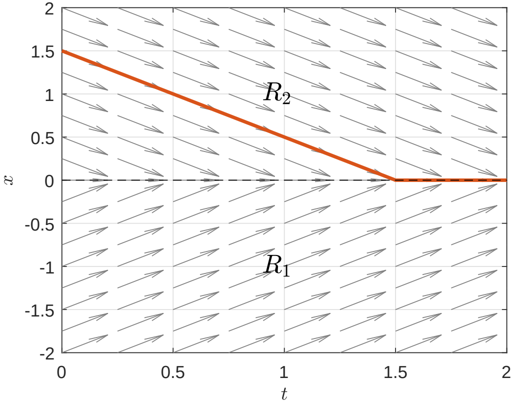
$$\dot{x} = -\text{sign}(x)$$

And let

$$\text{sign}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$$

Then...

$$\dot{x} = \begin{cases} 1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x > 0 \end{cases}$$



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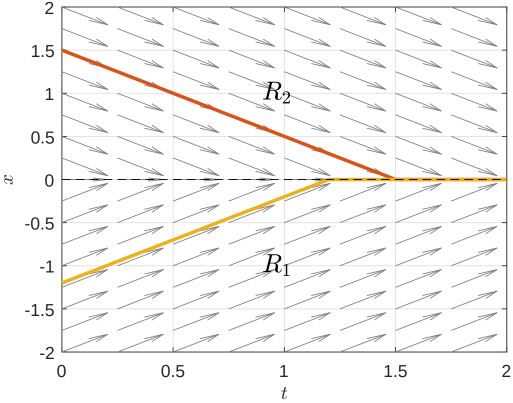

Consider the ODE

$$\dot{x} = -\text{sign}(x) + 0.5\sin(t)$$

And let

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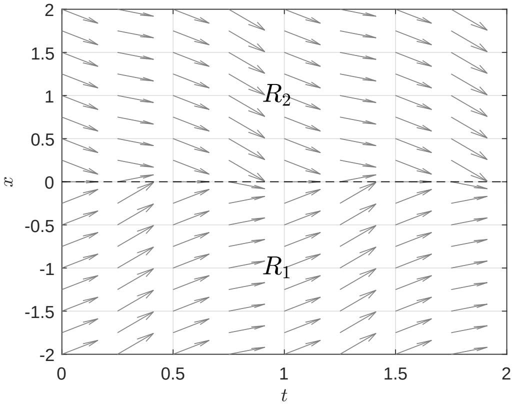

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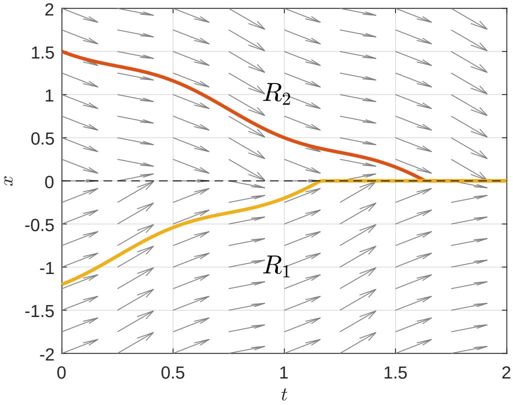


# Motivating examples: ODEs with a discontinuous right-hand side

## Sliding mode

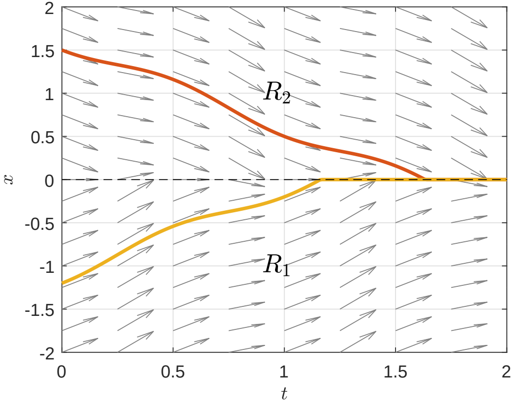
Consider the ODE

$$\dot{x} = -\text{sign}(x) + 0.5\sin(t)$$

And let

$$\text{sign}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$$

We have for some $t > t^*$ that $x(t) = 0$ and $\dot{x}(t) = 0$



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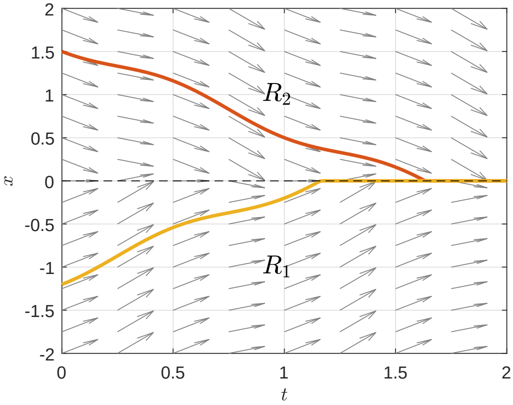

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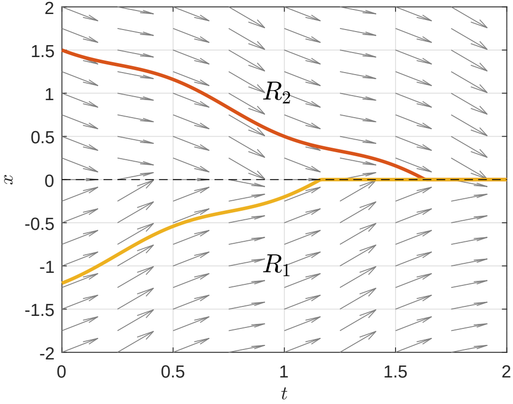
Something went wrong...



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$$\dot{x} \in -\text{sign}(x) + 0.5\sin(t)$$

And let

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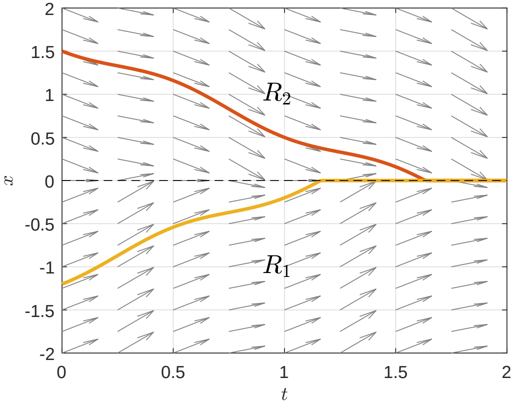


# Motivating examples: ODEs with a discontinuous right-hand side

Sliding mode - fixed

Consider the ODE

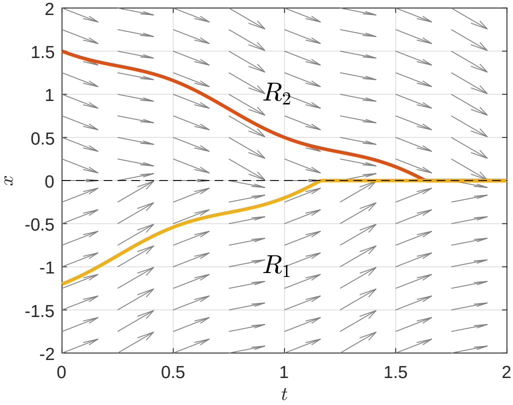
$$\dot{x} \in -\mathrm{sign}(x) + 0.5\sin(t)$$

And let

$$\mathrm{sign}(x) \in \begin{cases} \{-1\}, & \text{if } x < 0 \\ [-1,1], & \text{if } x = 0 \\ \{1\}, & \text{if } x > 0 \end{cases}$$

We have for some $t > t^*$ that $x(t) = 0$ and $\dot{x}(t) = 0$

That is $\mathrm{sign}(0) = [-1,1] \ni 0.5\sin(t)$



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$$\dot{x} \in -\text{sign}(x) + 0.5\sin(t)$$

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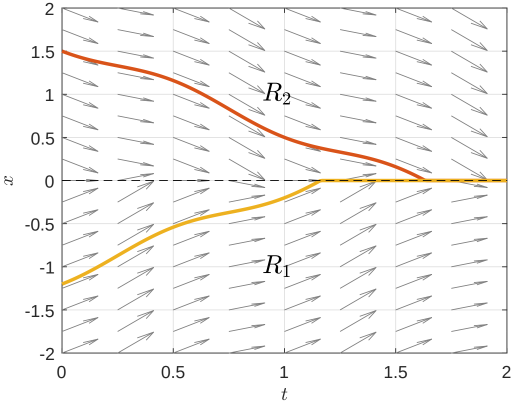
We have for some $t > t^*$ that $x(t) = 0$ and $\dot{x}(t) = 0$

That is $\text{sign}(0) = [-1, 1] \ni 0.5\sin(t)$

It works with set valued extensions.



# Filippov's convexification for ODEs with discontinuous right-hand side

## Filippov differential inclusion

Replace ODE with a discontinuous right-hand side

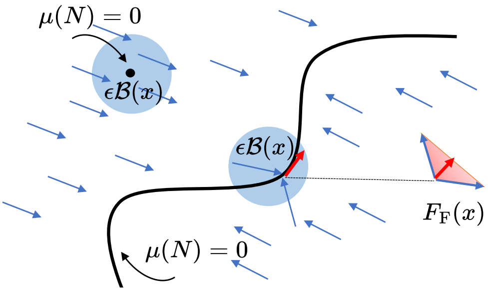
$$\dot{x}(t) = f(x(t))$$

by

$$\dot{x}(t) \in F_{\mathrm{F}}(x(t))$$

where $F_{\mathrm{F}}(x) : \mathbb{R}^{n_x} \to \mathcal{P}(\mathbb{R}^{n_x})$ is defined as:

$$F_{\mathrm{F}}(x) := \bigcap_{\epsilon > 0} \bigcap_{\mu(N)=0} \overline{\mathrm{conv}}\, f(x + \epsilon \mathcal{B}(x) \setminus N)$$




- $f(x)$ continuous at $x$: $F_{\mathrm{F}}(x) = \{f(x)\}$

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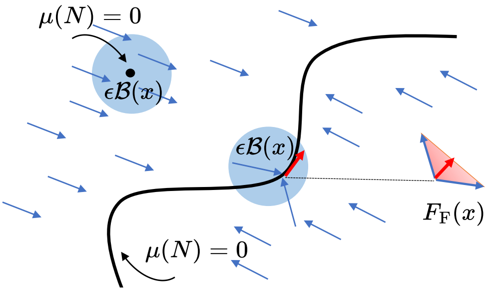



- at discontinuity: convex combination of neighboring vector fields and ignore what is at the discontinuity

# Piecewise smooth systems (PSS)

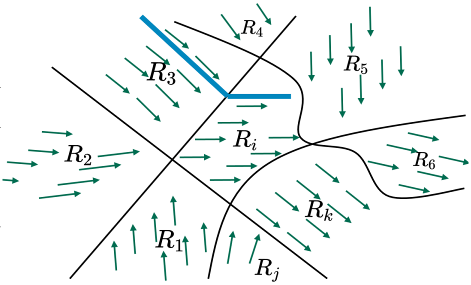
Regard **discontinuous** right-hand side, **piecewise smooth** on disjoint open regions $R_i \subset \mathbb{R}^{n_x}$

## Discontinuous ODE (NSD2)

$$\dot{x} = f_i(x,u), \text{ if } x \in R_i,\ \ i = 1, \dots, n_f$$

$$\begin{aligned}
R_1 &= \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) > 0, \psi_2(x) > 0, \dots \psi_{n_\psi}(x) > 0\} \\
R_2 &= \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) > 0, \psi_2(x) > 0, \dots \psi_{n_\psi}(x) < 0\} \\
&\vdots \\
R_{n_f} &= \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) < 0, \psi_2(x) < 0, \dots \psi_{n_\psi}(x) < 0\}
\end{aligned}$$

- zero level sets of $\psi_i(x) = 0$ - region boundaries
- $n_\psi$ smooth scalar switching functions define $n_f = 2^{n_\psi}$ regions

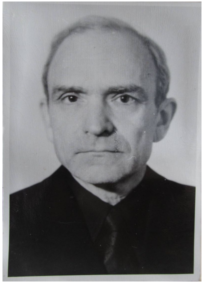


# Filippov convexification for piecewise smooth systems

The "structured" discontinuous right-hand side in PSS enables to define convex multipliers $\theta_i$ to define the convex set $F_{\mathrm{F}}(x,u)$

## Filippov Differential Inclusion

$$\dot{x} \in F_{\mathrm{F}}(x,u) := \left\{ \sum_{i=1}^{n_f} f_i(x,u)\,\theta_i \;\middle|\; \begin{aligned} &\sum_{i=1}^{n_f} \theta_i = 1, \\ &\theta_i \geq 0, \quad i = 1,\dots n_f, \\ &\theta_i = 0, \quad \text{if } x \notin \overline{R_i} \end{aligned} \right\}$$



Aleksei F. Filippov
(1923-2006)
image source: wikipedia

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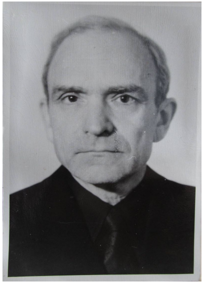

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- for interior points $x \in R_i$ nothing changes: $F_{\mathrm{F}}(x,u) = \{f_i(x,u)\}$
- Provides meaningful generalization on region boundaries
  E.g. on $\overline{R_1} \cap \overline{R_2}$ both $\theta_1$ and $\theta_2$ can be nonzero

# Stewart's representation

Introduced in [Stewart, 1990], used in [Nurkanović et al., 2024]

Assume sets $R_i$ given by $\boxed{R_i = \left\{x \in \mathbb{R}^{n_x} \middle| g_i(x) < \min_{j \neq i} g_j(x) \right\}}$

- How to obtain it from $R_i = \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) > 0, \psi_2(x) > 0, \dots \psi_{n_\psi}(x) > 0\}$?
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## Definition of regions via switching functions

$$\begin{aligned}
R_1 &= \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) > 0, \psi_2(x) > 0, \dots \psi_{n_\psi}(x) > 0\} \\
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\end{aligned}$$

$$\psi(x) := \begin{bmatrix} \psi_1(x) & \psi_2(x) & \dots & \psi_{n_\psi}(x) \end{bmatrix}^\top \in \mathbb{R}^{n_\psi}$$

## Sign matrix

$$S = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & -1 \end{bmatrix}$$

Definition via $i$-th row $S_{i,\bullet}$:

$$R_i = \{x \in \mathbb{R}^{n_x} \mid S_{i,\bullet}\psi(x) > 0\}$$

$$\boxed{g(x) = -S\psi(x)}$$

# Examples for finding switching function

- In Stewart's representation sets $R_i$ given by $R_i = \left\{x \in \mathbb{R}^{n_x} \middle| g_i(x) < \min_{j \neq i} g_j(x)\ \right\}$
- From switching functions $\psi(x) \in \mathbb{R}^{n_\psi}$ obtain *Stewart's indicator functions* $g(x) \in \mathbb{R}^{n_f}$ via $g(x) = -S\psi(x)$

**Example 1 - single switching function**

$$R_1 = \{x \in \mathbb{R}^{n_x} \mid \psi(x) > 0\}$$

$$R_2 = \{x \in \mathbb{R}^{n_x} \mid \psi(x) < 0\}$$

$$S = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$g(x) = \begin{bmatrix} -\psi(x) \\ \psi(x) \end{bmatrix}$$

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## Example 2 - two switching function

$$\psi(x) = (\psi_1(x), \psi_2(x))$$
$$S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix}$$
$$g(x) = \begin{bmatrix} -\psi_1(x) - \psi_2(x) \\ -\psi_1(x) + \psi_2(x) \\ \psi_1(x) - \psi_2(x) \\ \psi_1(x) + \psi_2(x) \end{bmatrix}$$

# Filippov's convexification

Switched ODE not well-defined on region boundaries $\partial R_i$.
Replace ODE by differential inclusion, using convex combination of neighboring vector fields.

## Filippov differential inclusion

$$\dot{x} \in F_{\mathrm{F}}(x,u) := \left\{ \sum_{i=1}^{n_f} f_i(x,u)\,\theta_i \;\middle|\; \sum_{i=1}^{n_f} \theta_i = 1,\ \theta_i \geq 0,\ i = 1,\dots,n_f,\ \ \theta_i = 0,\ \text{if } x \notin R_i \cup \partial R_i \right\}$$

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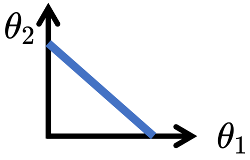


The unit simplex.

# From Filippov to dynamic complementarity systems

Using Stewart's reformulation [Stewart, 1990] and the KKT conditions of the parametric linear program.

$$R_i = \{x \in \mathbb{R}^n \mid g_i(x) < \min_{j \neq i} g_j(x)\}.$$

## Linear programming representation

$$\dot{x} = F(x, u)\, \theta$$

$$\text{with} \quad \theta \in \underset{\tilde{\theta} \in \mathbb{R}^{n_f}}{\operatorname{argmin}} \quad g(x)^\top \tilde{\theta}$$

$$\text{s.t.} \quad 0 \leq \tilde{\theta}$$

$$1 = e^\top \tilde{\theta}$$

$$F(x, u) := [f_1(x, u), \dots, f_{n_f}(x, u)] \in \mathbb{R}^{n_x \times n_f}$$

$$g(x) := [g_1(x), \dots, g_{n_f}(x)]^\top \in \mathbb{R}^{n_f}$$

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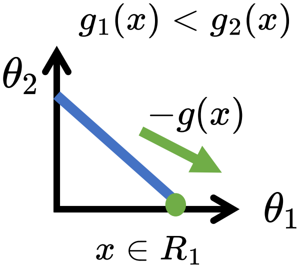

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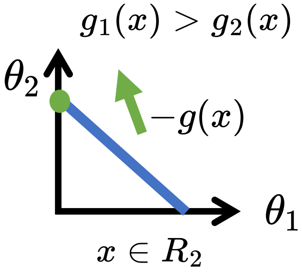

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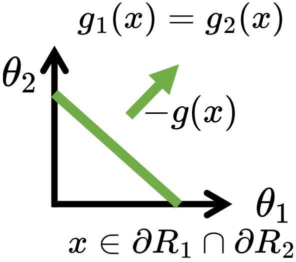
$$\text{with} \quad \theta \in \arg\min_{\tilde{\theta} \in \mathbb{R}^{n_f}} \quad g(x)^\top \tilde{\theta} \quad \text{s.t.} \quad 0 \leq \tilde{\theta}, \quad 1 = e^\top \tilde{\theta}$$

$$F(x,u) := [f_1(x,u), \dots, f_{n_f}(x,u)] \in \mathbb{R}^{n_x \times n_f}$$

$$g(x) := [g_1(x), \dots, g_{n_f}(x)]^\top \in \mathbb{R}^{n_f}$$

$$e := [1, 1, \dots, 1]^\top \in \mathbb{R}^{n_f}$$



# From Filippov to dynamic complementarity systems

Using Stewart's reformulation [Stewart, 1990] and the KKT conditions of the parametric linear program.

$$R_i = \{x \in \mathbb{R}^n \mid g_i(x) < \min_{j \neq i} g_j(x)\}.$$

## Linear programming representation

$$\dot{x} = F(x,u)\,\theta$$

$$\text{with} \quad \theta \in \underset{\tilde{\theta} \in \mathbb{R}^{n_f}}{\operatorname{argmin}} \quad g(x)^\top \tilde{\theta} \quad \text{s.t.} \quad 0 \leq \tilde{\theta}, \quad 1 = e^\top \tilde{\theta}$$

$$F(x,u) := [f_1(x,u), \dots, f_{n_f}(x,u)] \in \mathbb{R}^{n_x \times n_f}$$

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$$e := [1,1,\dots,1]^\top \in \mathbb{R}^{n_f}$$

Replace the LP by its optimality conditions.

## Dynamic complementarity system (DCS)

$$\dot{x} = F(x,u)\,\theta \qquad (1a)$$

$$0 = g(x) - \lambda - e\mu \qquad (1b)$$

$$0 \leq \theta \perp \lambda \geq 0 \qquad (1c)$$

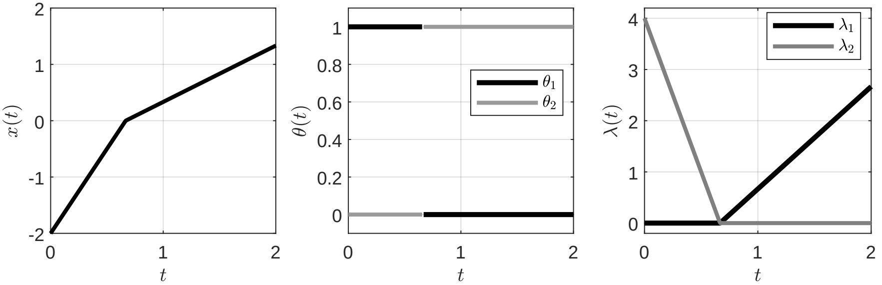
$$1 = e^\top \theta \qquad (1d)$$

- $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}^{n_f}$ Lagrange multipliers.
- (1c) $\Leftrightarrow \min\{\theta, \lambda\} = 0 \in \mathbb{R}^{n_f}$
- Together, (1b), (1c), (1d) determine the $(2n_f + 1)$ variables $(\theta, \lambda, \mu)$ uniquely.

# Example: continuity of multipliers in different switching cases

## Different switching cases

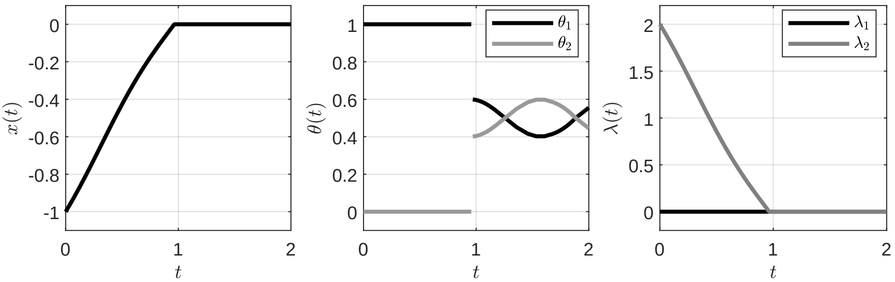
1. Crossing a surface of discontinuity, $\dot{x}(t) \in 2 - \mathrm{sign}(x(t))$,



# Example: continuity of multipliers in different switching cases

## Different switching cases

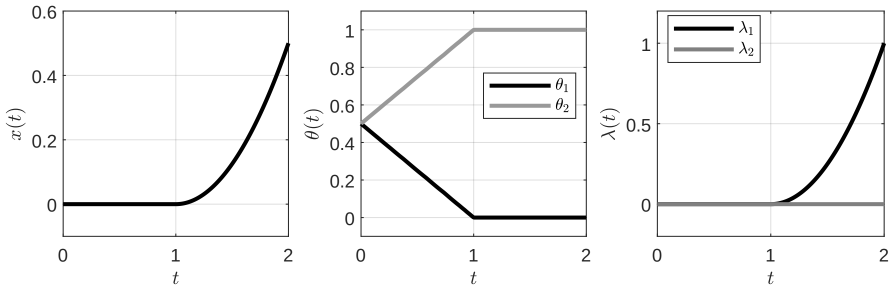
2. Sliding mode, $\dot{x}(t) \in -\mathrm{sign}(x(t)) + 0.2\sin(5t)$,



# Example: continuity of multipliers in different switching cases

## Different switching cases

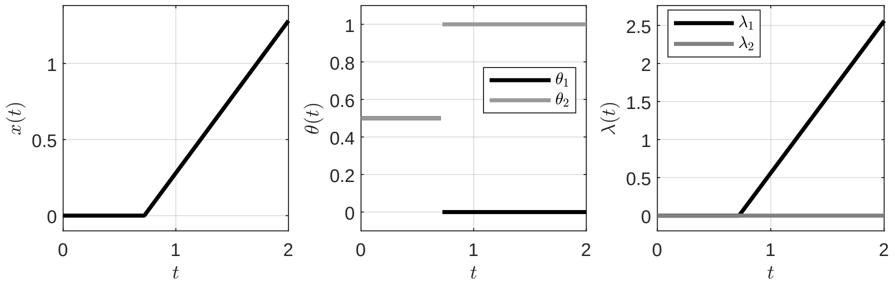
3. Leaving sliding mode $\dot{x}(t) \in -\mathrm{sign}(x(t)) + t$.



# Example: continuity of multipliers in different switching cases

## Different switching cases

4. Spontaneous switch, $\dot{x}(t) \in \mathrm{sign}(x(t))$,



# Example: continuity of multipliers in different switching cases

## Different switching cases

1. Crossing a surface of discontinuity, $\dot{x}(t) \in 2 - \mathrm{sign}(x(t))$,
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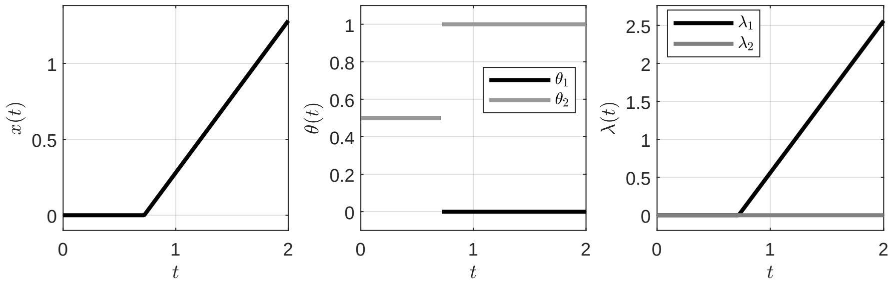
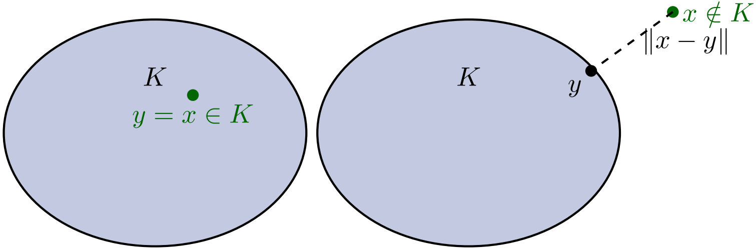
3. Leaving sliding mode $\dot{x}(t) \in -\mathrm{sign}(x(t)) + t$.
4. Spontaneous switch, $\dot{x}(t) \in \mathrm{sign}(x(t))$,



# Projected dynamical systems - the projection operator

Regard convex set $K$. The Euclidean projection is defined as the following convex optimization problem:

$$y = \mathrm{P}_K(x) := \underset{z}{\mathrm{argmin}}\ \frac{1}{2}(x-z)^\top(x-z)$$
$$\text{s.t.} \quad z \in K.$$



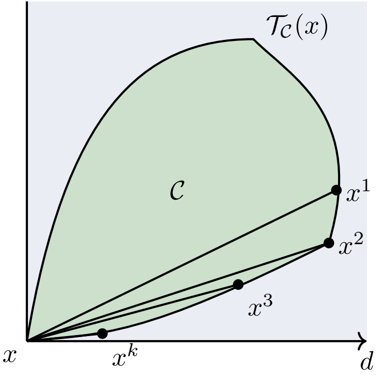
# The tangent cone to a set

**Definition (Tangent cone)**

The tangent cone at $x \in \mathcal{C}$ is defined as the set:

$$\mathcal{T}_{\mathcal{C}}(x) = \{d \in \mathbb{R}^n \mid \exists \{x^k\} \subset \mathcal{C}, \{t^k\} \subset \mathbb{R}_{\geq 0} : \lim_{k\to\infty} t^k = 0, \lim_{k\to\infty} x^k = x, \lim_{k\to\infty} \frac{x^k - x}{t^k} = d\}$$

- assume that the set $\mathcal{C}$ (not necessarily convex) is finely defined, $\mathcal{C} = \{x \in \mathbb{R}^{n_x} \mid c_i(x) \geq 0, i = 1, \dots, m\}$
- assume $\nabla c_i(x), i = 1, \dots, m$ linearly indepedent, LICQ hold.
- then, $\mathcal{T}_{\mathcal{C}}(x) = \{d \in \mathbb{R}^{n_x} \mid \nabla c_i(x)^\top d \geq 0, \forall i \in \mathcal{A}(x)\}$ (convex polyhedral), where $\mathcal{A}(x) = \{i \mid c_i(x) = 0\}$.



# Projected Dynamical Systems (PDS)

Introduced in 1970s by Claude Henry [Henry, 1972, Henry, 1973], Equivalences: [Brogliato et al., 2006, Serea, 2003, Heemels et al., 2000]

**Projected dynamical system (NSD2)**

$$\dot{x}(t) = \mathrm{P}_{\mathcal{T}_{\mathcal{C}}(x(t))} f(x(t), u(t))$$
$$x(0) \in \mathcal{C}$$

Features of PDS:

- state stays within $\mathcal{C} = \{x \in \mathbb{R}^{n_x} \mid c(x) \geq 0\}$ for all time
- derivative may be discontinuous on the boundary of $\mathcal{C}$ (**NSD2**)

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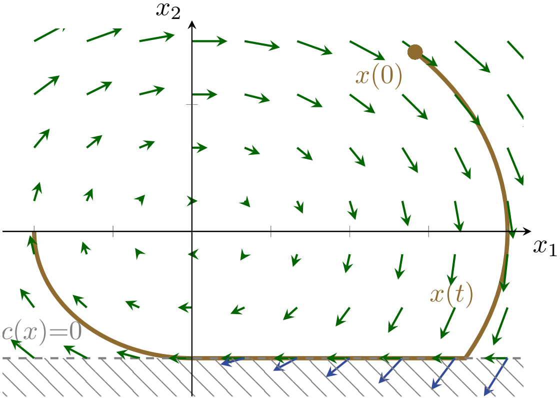
Example trajectory of PDS



# Projected Dynamical Systems (PDS) as DCS

**Projected dynamical system (NSD2)**

$$\dot{x}(t) = \mathrm{P}_{\mathcal{T}_\mathcal{C}(x(t))}(f(x(t), u(t)))$$
$$x(0) \in \mathcal{C}$$

The KKT conditions of $y = \mathrm{P}_{\mathcal{T}_\mathcal{C}(x)}(f(x,u))$:

$$y(t) = f(x(t), u(t)) + \nabla c(x(t))\lambda(t)$$
$$0 \leq \underbrace{\nabla c(x(t))^\top y(t)}_{=\frac{\mathrm{d}}{\mathrm{d}t}c(x(t))} \perp \lambda(t) \geq 0$$

- if $c_i(x) > 0$, then $\lambda_i = 0$ ($\nabla c_i(x)$ does not contribute to tangent cone)
- if $c_i(x) = 0$, and stays active, then $\frac{\mathrm{d}}{\mathrm{d}t}c(x(t)) \geq 0$

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**Projected dynamical system (NSD2)**

$$\dot{x}(t) = \mathrm{P}_{\mathcal{T}_{\mathcal{C}}(x(t))}(f(x(t), u(t)))$$
$$x(0) \in \mathcal{C}$$

**Gradient complementarity system (GCS)**

$$\dot{x}(t) = f(x(t), u(t)) + \nabla c(x(t))\lambda(t)$$
$$0 \leq c(x(t)) \perp \lambda(t) \geq 0$$

The KKT conditions of $y = \mathrm{P}_{\mathcal{T}_{\mathcal{C}}(x)}(f(x,u))$:

$$y(t) = f(x(t), u(t)) + \nabla c(x(t))\lambda(t)$$
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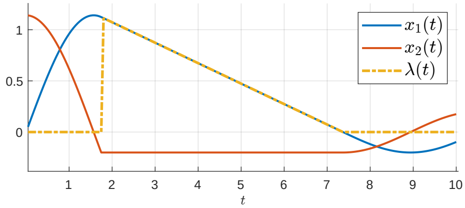


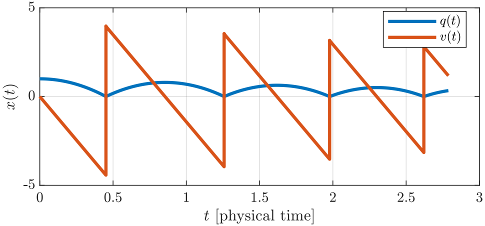

- $\lambda(t)$ - discontinuous w.r.t. time.
- $x(t)$, $c(x(t))$ - continuous w.r.t. time.

# NSD3 state jump example: bouncing ball

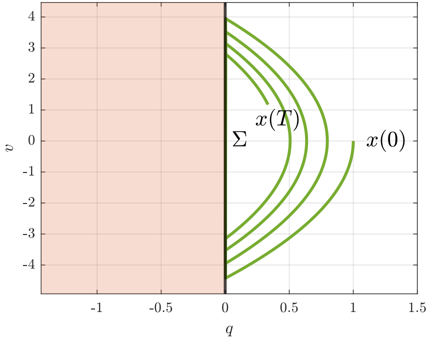
Bouncing ball with state $x = (q, v)$:

$$\dot{q} = v,\ m\dot{v} = -mg, \qquad \text{if } q > 0$$
$$v(t^+) = -0.9\, v(t^-), \qquad \text{if } q(t^-) = 0 \text{ and } v(t^-) < 0$$

Time plot of bouncing ball trajectory:




Phase plot of bouncing ball trajectory:



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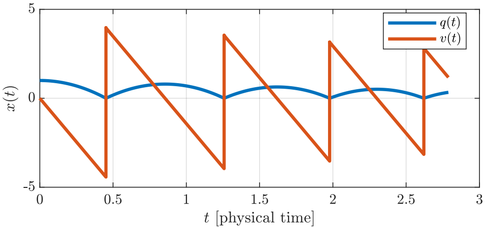
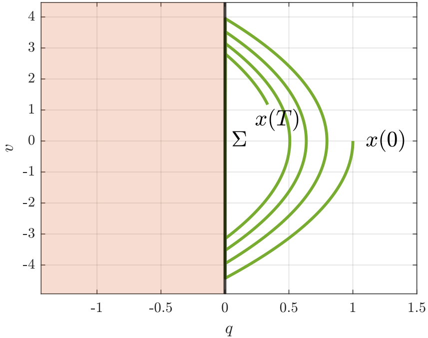



Phase plot of bouncing ball trajectory:




**Question: could we transform NSD3 systems into (easier) NSD2 systems?**

# Three ideas:

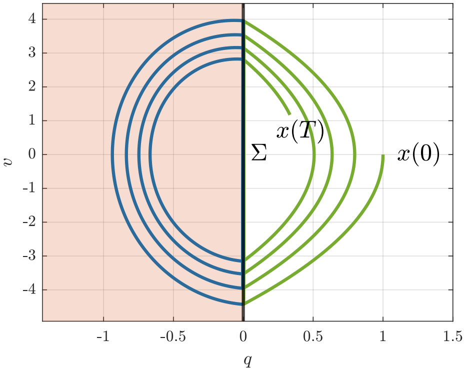
1. mimic state jump by **auxiliary dynamic system** $\dot{x} = f_{\mathrm{aux}}(x)$ on prohibited region
2. introduce a **clock state** $t(\tau)$ that stops counting when the auxiliary system is active
3. adapt speed of time, $\frac{\mathrm{d}t}{\mathrm{d}\tau} = s$ with $s \geq 1$, and **impose terminal constraint** $t(T) = T$

# The time-freezing reformulation

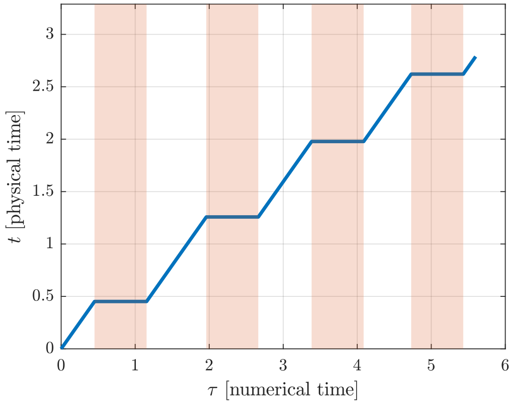
Augmented state $(x, t) \in \mathbb{R}^{n+1}$ evolves in **numerical time** $\tau$. Augmented system is nonsmooth, of NSD2 type:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} x \\ t \end{bmatrix} = \begin{cases} s\begin{bmatrix} f(x) \\ 1 \end{bmatrix}, & \text{if } c(x) \geq 0 \\ \begin{bmatrix} s f_{\mathrm{aux}}(x) \\ 0 \end{bmatrix}, & \text{if } c(x) < 0 \end{cases}$$

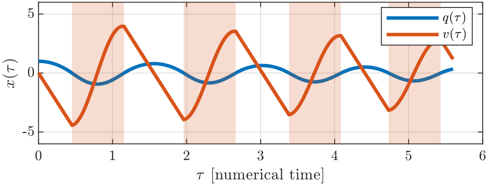
- During normal times, system and clock state evolve with adapted speed $s \geq 1$.
- Auxiliary system $\frac{\mathrm{d}x}{\mathrm{d}\tau} = f_{\mathrm{aux}}(x)$ mimics state jump while time is frozen, $\frac{\mathrm{d}t}{\mathrm{d}\tau} = 0$.

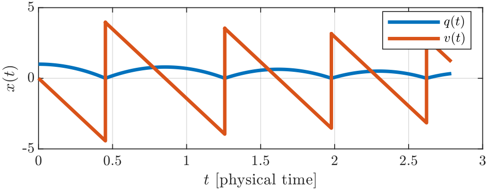


# Time-freezing for bouncing ball example




Evolution of physical time (clock state) during augmented system simulation ($s = 1$).




We can recover the true solution by plotting $x(\tau)$ vs. $t(\tau)$ and disregarding "frozen pieces".

# Example of a time-freezing optimal control problem

Time-freezing tracking

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Time-freezing tracking

# Summary

- State depended switches and jumps (internal) are qualitatively different from integer controls (external).
- Nonsmooth systems exhibit rich behavior not seen in smooth systems.
- Accurate smooth approximation jeopardize the performance of smooth solvers and,
- ... behave numerically as nonsmooth systems
- Different classes of numerical methods for time discretization.
- There are many mathematical formalism to treat nonsmoothness.
- Often, the nonsmooth part is expressed as the solution to a parametric convex problem.

# Recommended reading


- Stewart, D. E. Dynamics with Inequalities: impacts and hard constraints. SIAM, 2011.
- Brogliato, B., and Tanwani, A. Dynamical systems coupled with monotone set-valued operators: Formalisms, applications, well-posedness, and stability. Siam Review 62, 1 (2020), 3–129.
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Heemels, W., Schumacher, J. M., and Weiland, S. (2000).
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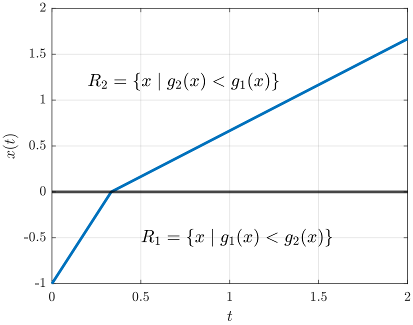
*Numerische Mathematik*, 58(1):299–328.

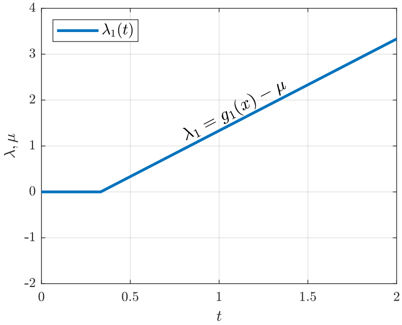
# Interpretation of the DCS multipliers

**Dynamic complementarity system**

$$\dot{x} = F(x,u)\,\theta$$
$$0 = g_i(x) - \lambda_i - \mu, \; i = 1,\ldots,n_f$$
$$0 \leq \theta \perp \lambda \geq 0$$
$$1 = e^\top \theta$$

- If $x \in R_i$, then $\theta_i > 0$, $\lambda_i = 0$ (from complementarity)
- $\lambda_i = g_i(x) - \mu$ (from $\nabla_x \mathcal{L}(x,\lambda,\mu) = 0$)



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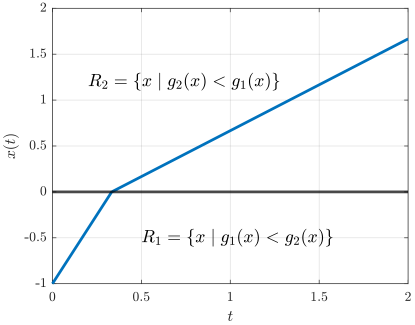


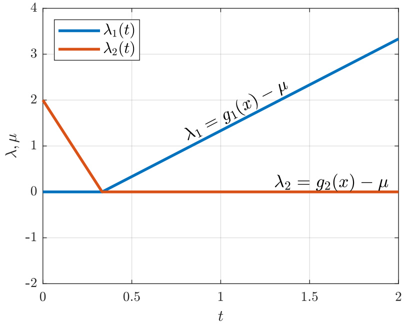

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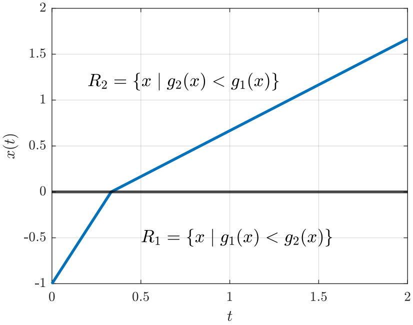
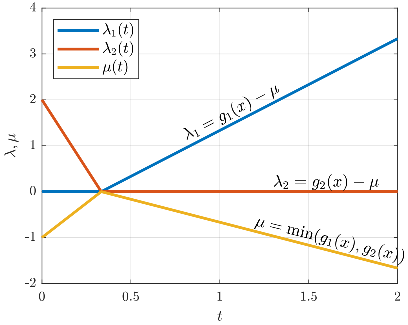
- $\mu = \min_j g_j(x)$ (from definition of $R_i$)
- $\lambda_i = g_i(x) - \min_j g_j(x)$ continuous functions!



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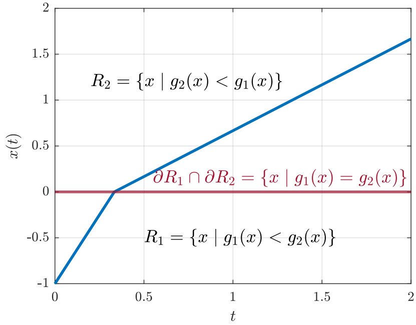
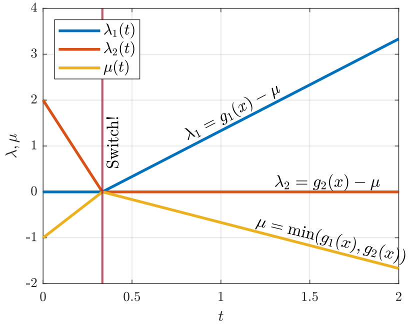
- $\lambda_i = g_i(x) - \min_j g_j(x)$ continuous functions!
- At switch $\lambda_i = \lambda_j = 0 \implies g_i(x) - g_j(x) = 0$ (region boundary)



# The active set of the DCS

## Dynamic complementarity system

$$\dot{x} = F(x,u)\,\theta$$
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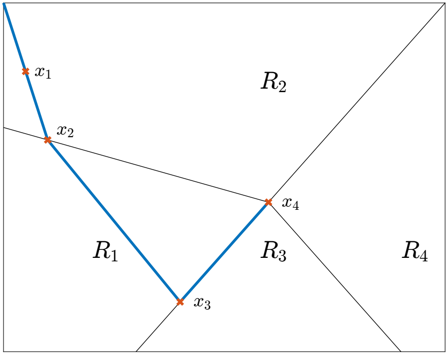
## DAE with fixed $\mathcal{I}$

$$\dot{x} = F_{\mathcal{I}}(x,u)\,\theta_{\mathcal{I}}$$
$$0 = g_{\mathcal{I}}(x) - \mu e,$$
$$1 = e^\top \theta_{\mathcal{I}}$$

- Locally well-behaved smooth ODE or DAE

## Active set

$$\mathcal{I}(x) := \left\{ i \mid g_i(x) = \min_{j \in \mathcal{J}} g_j(x) \right\} = \left\{ i \mid \theta_i > 0 \right\}$$




$$\mathcal{I}(x_1) = \{2\},\ \mathcal{I}(x_2) = \{1,2\},\ \mathcal{I}(x_3) = \{1,3\}$$
$$\mathcal{I}(x_4) = \{1,2,3,4\}$$

# Properties of the DCS

Sufficient conditions for the uniqueness of the solution

## DAE with fixed $\mathcal{I}$

$$\dot{x} = F_{\mathcal{I}}(x, u)\, \theta_{\mathcal{I}} \tag{2a}$$
$$0 = g_{\mathcal{I}}(x) - \mu e, \tag{2b}$$
$$1 = e^\top \theta_{\mathcal{I}} \tag{2c}$$

Given $|\mathcal{I}| \geq 1$, define the matrix

$$M_{\mathcal{I}}(x) = \nabla g_{\mathcal{I}}(x)^\top F_{\mathcal{I}}(x, u) \in \mathbb{R}^{|\mathcal{I}| \times |\mathcal{I}|}.$$

## Proposition

*Suppose that for a fixed active set $\mathcal{I}(x(t)) = \mathcal{I}$ for $t \in [0, T]$, it holds that the matrix $M_{\mathcal{I}}(x(t))$ is invertible and $e^\top M_{\mathcal{I}}(x(t))^{-1} e \neq 0$ for all $t \in [0, T]$. Given the initial value $x(0)$, then the DAE* (2) *has a unique solution for all $t \in [0, T]$.*

*Proof.* Index reduction and implicit function theorem. □

# Differential inclusions

A very general class of nonsmooth dynamical systems is obtained by replacing the right-hand side of a smooth ODE with a set.

## Differential Inclusions (DI)

The following equations is called a differential inclusion:

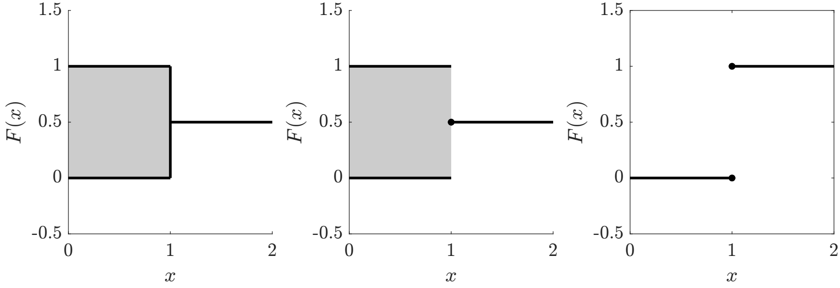
$$\dot{x}(t) \in F(t, x(t)) \text{ for almost all } t \in [0, T], \tag{3}$$

Here $F : \mathbb{R} \times \mathbb{R}^{n_x} \to \mathcal{P}(\mathbb{R}^{n_x})$ is a set-valued map which assigns to any point in time $t$ and $x \in \mathbb{R}^{n_x}$ a set $F(t, x) \subseteq \mathbb{R}^{n_x}$. An element $y \in F(t, x(t))$ for a fixed $(t, x(t))$ is called a *selection*.

# Outer and inner semi-continuous set-valued functions

## Definition (OSC, ISC, continuity)

A set-valued function $F(\cdot)$ is outer-semi continuous (OSC) (resp. inner semi-continuous (ISC)) at $x_0 \in X$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $F(x) \subset F(x_0) + \epsilon\mathcal{B}(0)$ (resp. $F(x_0) \subset F(x) + \epsilon\mathcal{B}(0)$) for all $x \in x_0 + \delta\mathcal{B}(0)$. It is called continuous at $x_0$ if it both OSC and ISC at this point.



# Existence of solutions to differential inclusions

**Theorem (Existence of solution, Theorem 4, p. 101 in Aubin, J. P., and Cellina, A., 1994 )**

*Regard the initial value problem related to the DI* (3) *with the initial value* $x(0) = x_0$. *Suppose that the function* $F : [0, T] \times \mathbb{R}^{n_x} \to \mathcal{P}(\mathbb{R}^{n_x})$ *satisfies the following conditions:.*

i) $\|y\| \leq C(t)(1 + \|x\|)$ *for all* $x$ *and* $y \in F(t, x)$, *where* $C(\cdot)$ *is an integrable function,*

ii) $F(t, \cdot)$ *is outer semi-continuous for all* $t$,

iii) *the set* $F(t, x)$ *is nonempty and closed convex set for all* $t$ *and* $x$,

*Then there exists an absolutely continuous solution* $x(\cdot)$ *to this initial value problem.*

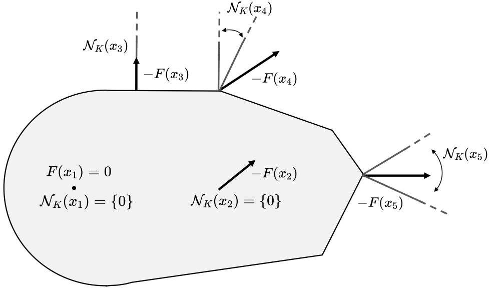
## Definition

Let $K \subseteq \mathbb{R}^n$ be a closed convex set and $F : \mathbb{R}^n \to \mathbb{R}^n$. A variational inequality, denoted by $\mathrm{VI}(K, F)$, is the problem of finding $x \in \mathbb{R}^n$ such that

$$x \in K, \;\; F(x)^\top (y - x) \geq 0, \quad \text{for all } y \in K.$$

The set of solutions to this problem is denoted by $\mathrm{SOL}(K, F)$.

- $x \in K$ is a solution of $\mathrm{VI}(K, F)$ iff either $F(x) = 0$ or $F(x)$ forms a non-obtuse angle with every vector $y - x$ for all $y \in K$
- $\mathcal{N}_K(x) = \{v \in \mathbb{R}^n \mid v^\top (y - x) \leq 0, \; \text{for all } y \in K\}$, $\mathrm{VI}(K, F)$ is the same as: $0 \ni F(x) + \mathcal{N}_K(x)$




$x_1$, $x_3$ and $x_5$ are solutions, $x_2$ and $x_4$ are not

# Differential variational inequalities

**Definition (Differential variational inequalities)**

Given an initial value $x(0) = x_0$, a Differential Variational Inequality (DVI) is the problem of finding functions $x : [0, T] \to \mathbb{R}^{n_x}$ and $z : [0, T] \to \mathbb{R}^{n_z}$ such that

$$\dot{x}(t) = f(t, x(t), z(t)), \tag{4a}$$

$$z(t) \in K, \text{ for almost all } t, \tag{4b}$$

$$0 \leq (\hat{z} - z(t))^\top F(t, x(t), z(t)), \text{ for all } \hat{z} \in K \text{ and for almost all } t. \tag{4c}$$

- DVI can be easily cast into differential inclusions
- Denote the set of all solutions, parameterized by $x(t)$, of the VI (4c) by $\mathrm{SOL}(F(t, x(t), \cdot), K)$.

$$\dot{x}(t) \in f(t, x(t), \mathrm{SOL}(F(t, x(t), \cdot), K)),\ x(0) = x_0.$$

# Dynamic complementarity systems

**Definition (Dynamic complementarity systems)**

Given an initial value $x(0) = x_0$, a dynamic complementarity system is the problem of finding functions $x : [0, T] \to \mathbb{R}^{n_x}$ and $z : [0, T] \to \mathbb{R}^{n_z}$ such that

$$
\begin{aligned}
\dot{x}(t) &= f(t, x(t), z(t)), \ x(0) = x_0, \\
0 \leq z(t) &\perp F(t, x(t), z(t)) \geq 0, \ \text{for almost all } t,
\end{aligned}
$$

- Discrete-time counterpart: nonlinear complementarity problems (e.g. KKT conditions of an NLP)
- Computationally very useful as NCPs can often be solved efficiently
- Found in nonsmooth mechanics: complementarity between gap function and normal contact forces
- Filippov systems can be casted into DCS (next lecture)
- $\text{DI} \supset \text{DVI} \supset \text{DCS} \supset \text{ODE}$.

# Relating CCP to VIs

**Proposition (Proposition 1.1.3. in Facchinei and Pang 2003)**

*Let $K$ be a closed convex cone. A vector $x \in \mathbb{R}^n$ is a solution to $\mathrm{VI}(K, F)$ if and only if it is a solution to the cone complementarity problem:*

$$K \ni x \perp F(x) \in K^*, \tag{5}$$

*where this compact notation means that $x \in K, F(x) \in K^*$ and $F(x)^\top x = 0$.*

*Proof.* Let $x$ be a solution to the $\mathrm{VI}(K, F)$. On one hand, since $K$ is a cone, setting $y = 0 \in K$ we have from $x \in K,\ F(x)^\top (y - x) \geq 0,\ \text{ for all } y \in K$, that $F(x)^\top x \leq 0$. On the other hand, from the definition of a cone $x \in K$ it follows that $2x \in K$. Again, from the VI and setting $y = 2x$ we obtain that $F(x)^\top x \geq 0$. Therefore, $F(x)^\top x = 0$. We further exploit that $F(x)^\top x \geq 0$, i.e., we can see that $F(x)^\top (y - x) \geq 0$ implies that $F(x)^\top y \geq 0$ for all $y \in K$, which is equivalent to $F(x) \in K^*$. Thus we have proven that $x$ solves also (5). Conversely, if $x$ solves (5), we have from the definition that $F(x)^\top y \geq 0$ for all $y \in K$ and $F(x)^\top x = 0$. Subtracting these relations we obtain that the VI holds. □