4. Direct methods for nonsmooth nonlinear optimal control

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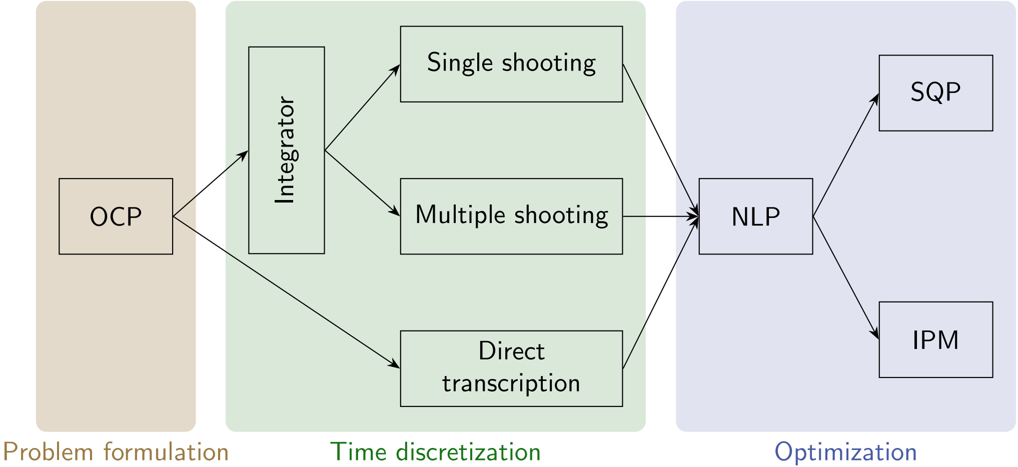
Outline of the lecture

Work flow in smooth direct optimal control

First discretize, then optimize.



OCP = Optimal Control Problem

NLP = Nonlinear Program

SQP = Sequential Quadratic Programming

IPM = Interior-Point Method

Figure inspired by Lecture 1, Numerical Methods for Optimal Control: Introduction, 2022, by Mario Zanon and Sébastien Gros.

The obvious way to tackle nonsmooth optimal control problems

Let us follow the path that worked so far:

1. Apply fixed-step size integration methods within direct single shooting, direct multiple shooting or direct description to the **nonsmooth** optimal control problem.

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The obvious way to tackle nonsmooth optimal control problems

Let us follow the path that worked so far:

1. Apply fixed-step size integration methods within direct single shooting, direct multiple shooting or direct description to the **nonsmooth** optimal control problem.
2. Smooth the nonsmooth model, and apply standard direct methods.

Due to nonsmooth dynamics, the resulting optimization problem is nonsmooth only in a few points.

What can go wrong?

Tutorial nonsmooth optimal control problem

Tutorial example inspired by [Stewart and Anitescu, 2010]. Further studied in [Nurkanović et al., 2020].

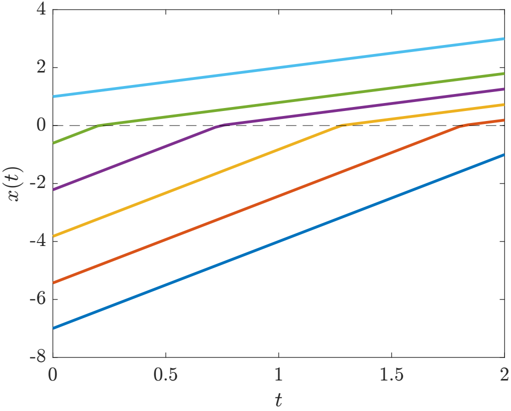
Continuous-time OCP

$$\min_{x(\cdot)\in\mathcal{C}^0([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2)-5/3)^2$$
$$\text{s.t.} \quad \dot{x}(t) \in 2-\operatorname{sign}(x(t)), \quad t\in[0,2]$$

Free initial value $x(0)$ is the effective degree of freedom.

Equivalent reduced problem

$$\min_{x_0\in\mathbb{R}} V(x_0)$$



Example trajectories.

Tutorial nonsmooth optimal control problem

Tutorial example inspired by [Stewart and Anitescu, 2010]. Further studied in [Nurkanović et al., 2020].

Continuous-time OCP

$$\min_{x(\cdot),\lambda(\cdot),s(\cdot)} \int_0^2 x(t)^2 \mathrm{d}t + (x(2) - 5/3)^2$$
$$\text{s.t.} \quad \dot{x}(t) = 2 - s(t)$$
$$0 \leq \lambda(t) - x(t) \perp 1 + s(t) \geq 0$$
$$0 \leq \lambda(t) \perp 1 - s(t) \geq 0, \ t \in [0, 2]$$

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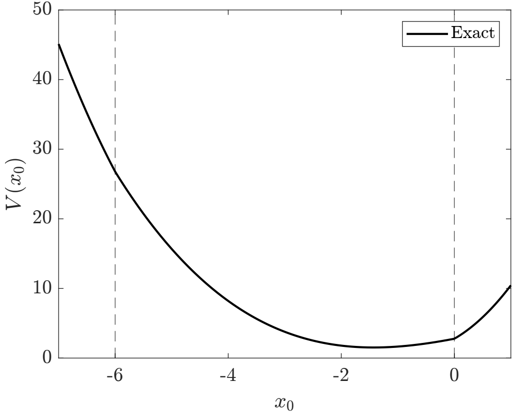

Denote by $V(x_0)$ the nonsmooth objective value for the unique feasible trajectory starting at $x(0) = x_0$.

Tutorial nonsmooth optimal control problem

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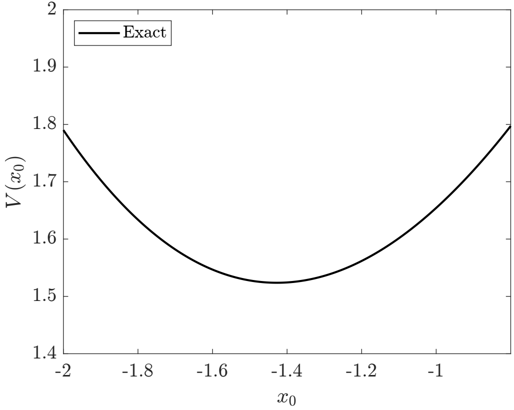

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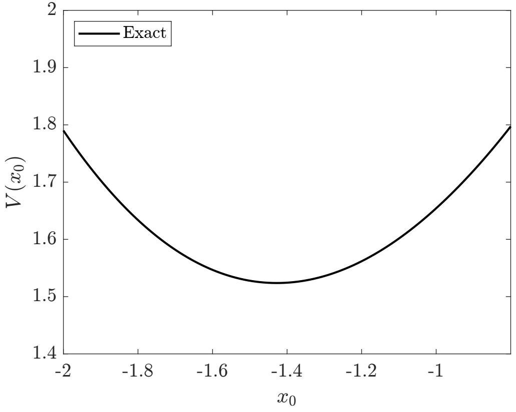
1. Direct optimal control with a time stepping IRK discretization

Tutorial example inspired [Stewart and Anitescu, 2010]. Further studied in [Nurkanović et al., 2020].

Continuous-time OCP

$$\begin{aligned} \min_{x(\cdot),\lambda(\cdot),s(\cdot)} \quad & \int_0^2 x(t)^2 \mathrm{d}t + (x(2) - 5/3)^2 \\ \text{s.t.} \quad & \dot{x}(t) = 2 - s(t) \\ & 0 \leq \lambda(t) - x(t) \perp 1 + s(t) \geq 0 \\ & 0 \leq \lambda(t) \perp 1 - s(t) \geq 0, \ t \in [0,2] \end{aligned}$$

- discretize the DCS with fixed step size IRK methods
- e.g., midpoint rule, Gauss-Legendre IRK with $n_{\mathrm{s}} = 1$, accuracy $O(h^2)$



Locally quadratic objective.

1. Direct optimal control with a time stepping IRK discretization

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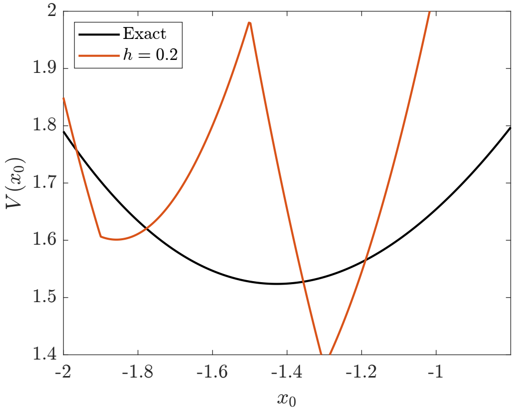
Discrete-time OCP

$$\min_{\mathbf{x},\mathbf{z}} \quad \sum_{n=0}^{N-1} \ell_n(x_n) + (x_N - 5/3)^2$$

$$\text{s.t.} \quad x_{n+1} = \phi_f(x_n, z_n)$$

$$0 = \phi_{\text{int}}(x_n, z_n), \; n = 0, \dots N-1$$

- discretize the DCS with fixed step size IRK methods
- e.g., midpoint rule, Gauss-Legendre IRK with $n_{\mathrm{s}} = 1$, accuracy $O(h^2)$
- step size $h = 0.2$, i.e., $N = 10$ integration steps



Many artificial local minima and wrong derivatives.

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- discretize the DCS with fixed step size IRK methods
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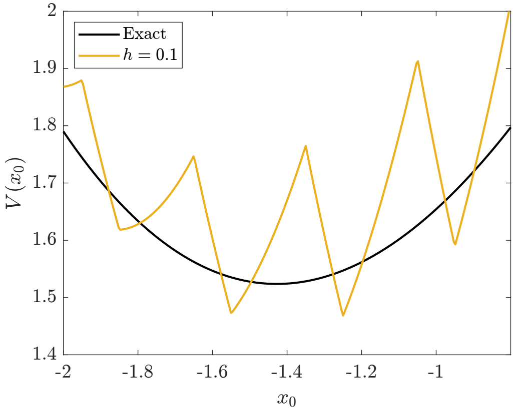
- step size $h = 0.1$, i.e., $N = 20$ integration steps



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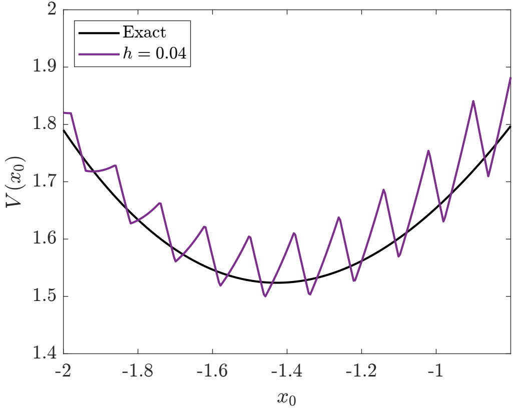
- discretize the DCS with fixed step size IRK methods
- e.g., midpoint rule, Gauss-Legendre IRK with $n_{\mathrm{s}} = 1$, accuracy $O(h^2)$
- step size $h = 0.04$, i.e., $N = 50$ integration steps



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- discretize the DCS with fixed step size IRK methods
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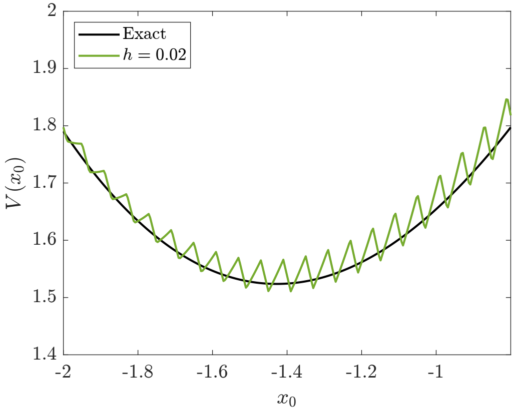
- step size $h = 0.02$, i.e., $N = 100$ integration steps



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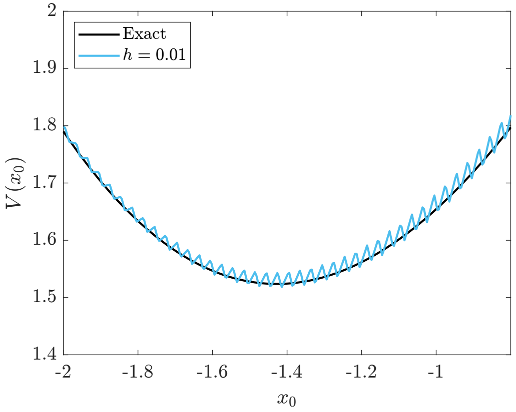
- e.g., midpoint rule, Gauss-Legendre IRK with $n_{\text{s}} = 1$, accuracy $O(h^2)$
- step size $h = 0.01$, i.e., $N = 200$ integration steps



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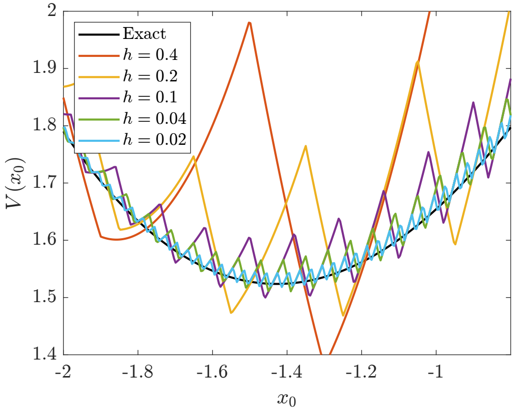
$$\text{s.t.} \quad x_{n+1} = \phi_f(x_n, z_n)$$

$$0 = \phi_{\text{int}}(x_n, z_n), \; n = 0, \ldots N-1$$

- discretize the DCS with fixed step size IRK methods
- e.g., midpoint rule, Gauss-Legendre IRK with $n_\text{s} = 1$, accuracy $O(h^2)$
- decreasing the step size might worsen the situation



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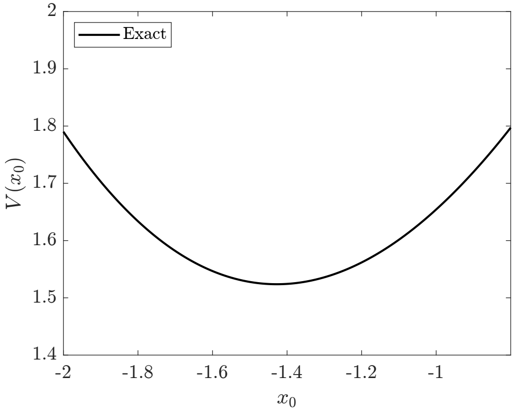
Continuous-time OCP

$$\min_{x(\cdot)\in\mathcal{C}^0([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2) - 5/3)^2$$
$$\text{s.t.} \quad \dot{x}(t) = 2 - \operatorname{sign}(x(t)), \quad t \in [0,2]$$

- midpoint rule, with $h = 0.05$; $N = 40$



2. Direct optimal control with a standard IRK discretization - smoothing

Tutorial example inspired [Stewart and Anitescu, 2010]. Further studied in [Nurkanović et al., 2020].

Smoothed continuous-time OCP

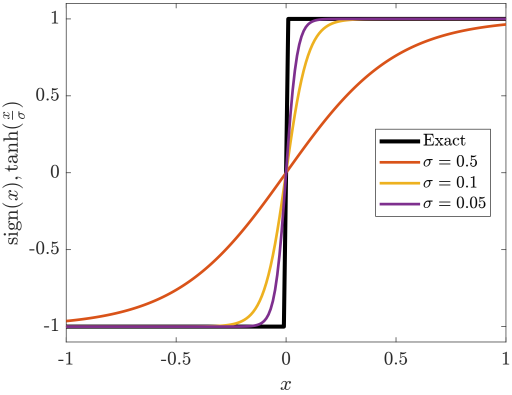
$$\min_{x(\cdot)\in\mathcal{C}^\infty([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2)-5/3)^2$$

$$\text{s.t.} \quad \dot{x}(t) = 2 - \tanh\left(\frac{x(t)}{\sigma}\right), \quad t \in [0,2]$$

Equivalent reduced problem

$$\min_{x_0\in\mathbb{R}} V_\sigma(x_0)$$

- midpoint rule, with $h = 0.05$; $N = 40$
- solve smoothed OCP for different σ



Tutorial example inspired [Stewart and Anitescu, 2010]. Further studied in [Nurkanović et al., 2020].

Smoothed continuous-time OCP

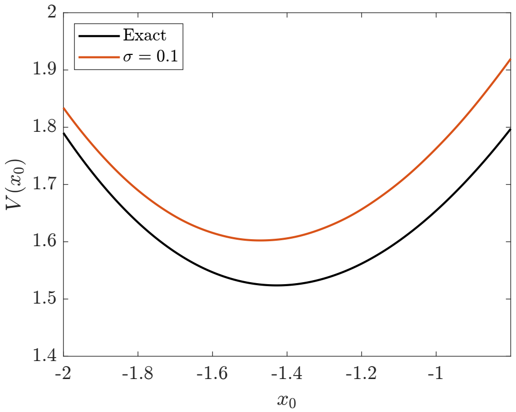
$$\min_{x(\cdot)\in\mathcal{C}^\infty([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2) - 5/3)^2$$

$$\text{s.t.} \quad \dot{x}(t) = 2 - \tanh\left(\frac{x(t)}{\sigma}\right), \quad t \in [0,2]$$

Equivalent reduced problem

$$\min_{x_0\in\mathbb{R}} V_\sigma(x_0)$$

- midpoint rule, with $h = 0.05;\ N = 40$
- solve smoothed OCP with $\sigma = 0.1$



2. Direct optimal control with a standard IRK discretization - smoothing

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$$\min_{x(\cdot)\in\mathcal{C}^\infty([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2)-5/3)^2$$

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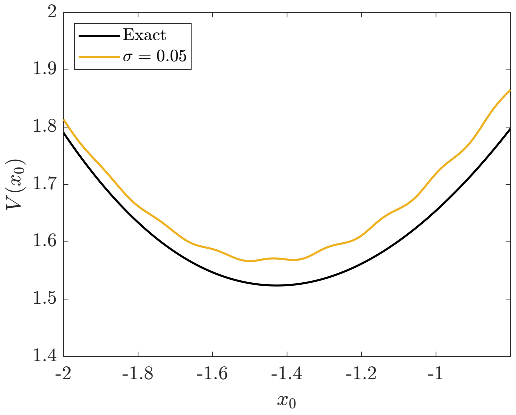
Equivalent reduced problem

$$\min_{x_0\in\mathbb{R}} V_\sigma(x_0)$$

- midpoint rule, with $h = 0.05$; $N = 40$
- solve smoothed OCP with $\sigma = 0.05$



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Smoothed continuous-time OCP

$$\min_{x(\cdot)\in\mathcal{C}^{\infty}([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2) - 5/3)^2$$

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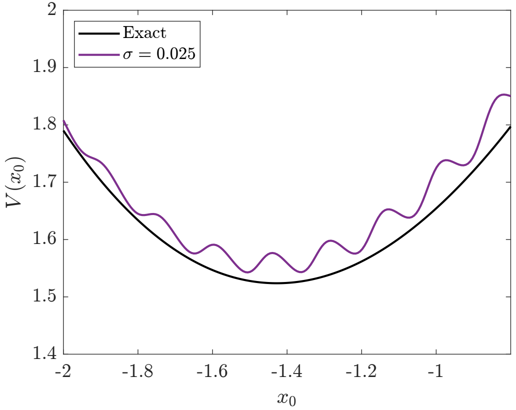
Equivalent reduced problem

$$\min_{x_0 \in \mathbb{R}} V_{\sigma}(x_0)$$

- midpoint rule, with $h = 0.05;\ N = 40$
- solve smoothed OCP with $\sigma = 0.025$



Tutorial example inspired [Stewart and Anitescu, 2010]. Further studied in [Nurkanović et al., 2020].

Smoothed continuous-time OCP

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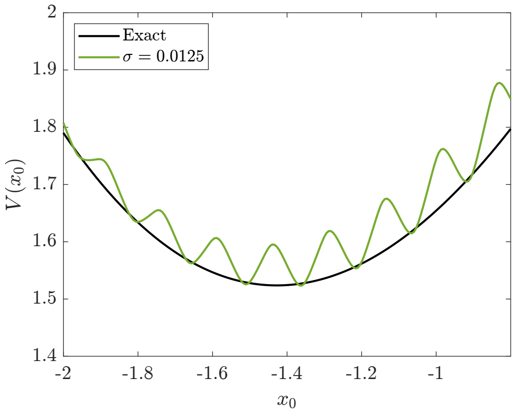
$$\text{s.t.} \quad \dot{x}(t) = 2 - \tanh\left(\frac{x(t)}{\sigma}\right), \quad t \in [0,2]$$

Equivalent reduced problem

$$\min_{x_0\in\mathbb{R}} V_\sigma(x_0)$$

- midpoint rule, with $h = 0.05$; $N = 40$
- solve smoothed OCP with $\sigma = 0.0125$



Tutorial example inspired [Stewart and Anitescu, 2010]. Further studied in [Nurkanović et al., 2020].

Smoothed continuous-time OCP

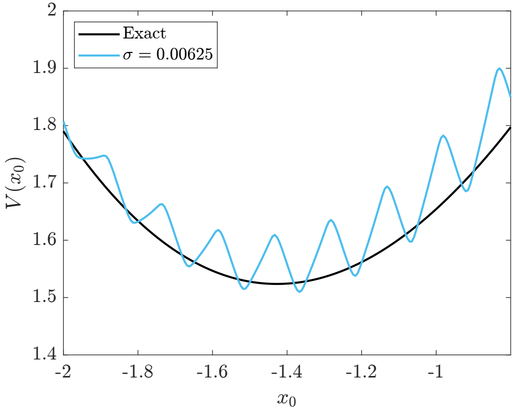
$$\min_{x(\cdot)\in\mathcal{C}^\infty([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2)-5/3)^2$$

$$\text{s.t.} \quad \dot{x}(t) = 2 - \tanh\left(\frac{x(t)}{\sigma}\right), \quad t \in [0,2]$$

Equivalent reduced problem

$$\min_{x_0\in\mathbb{R}} V_\sigma(x_0)$$

- midpoint rule, with $h = 0.05$; $N = 40$
- solve smoothed OCP with $\sigma = 0.00625$



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Smoothed continuous-time OCP

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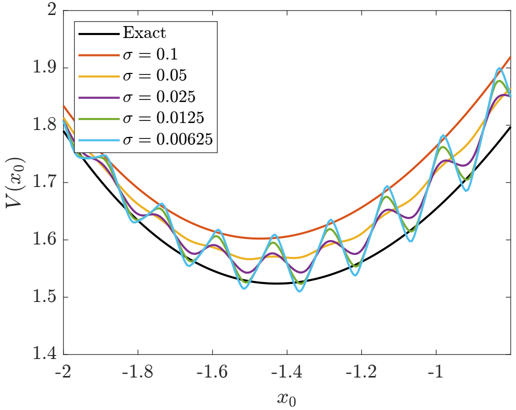

If $h \gg \sigma$, then the smooth approximation behaves **the same as the nonsmooth problem!**

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Smoothed continuous-time OCP

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Equivalent reduced problem

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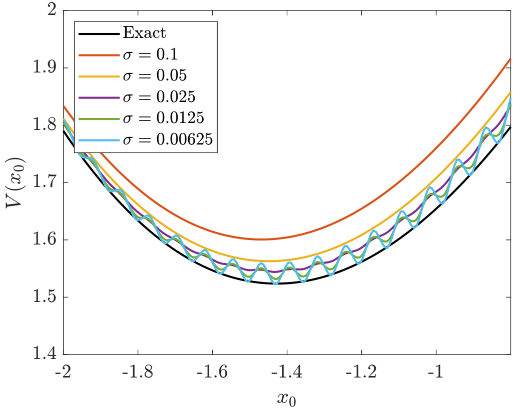
- midpoint rule, with $h = 0.025$; $N = 80$



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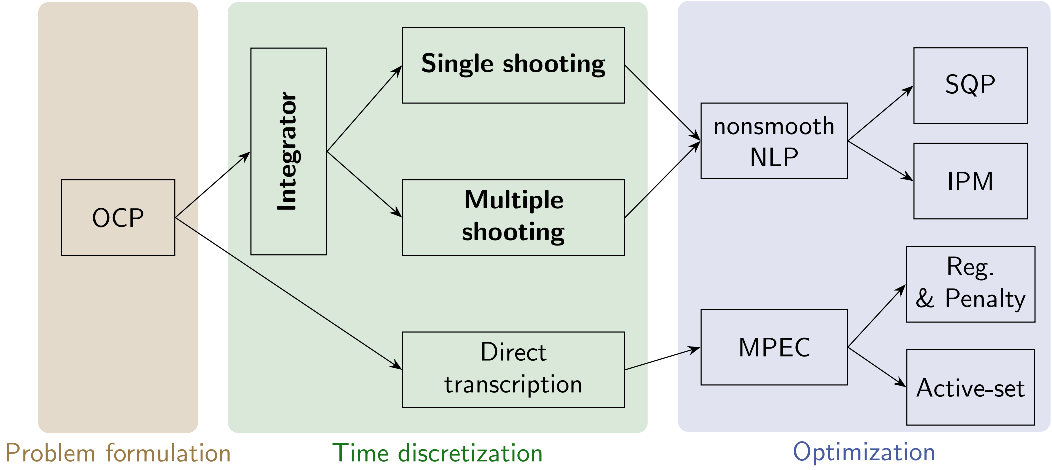
Work flow in nonsmooth direct optimal control

First discretize, then optimize.



OCP = Optimal Control Problem

NLP = Nonlinear Program

MPEC = Mathematical Program with Equilibrium Constraints

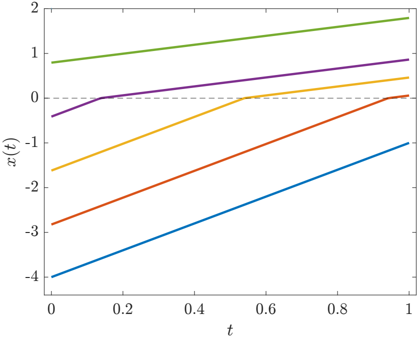
SQP = Sequential Quadratic Programming

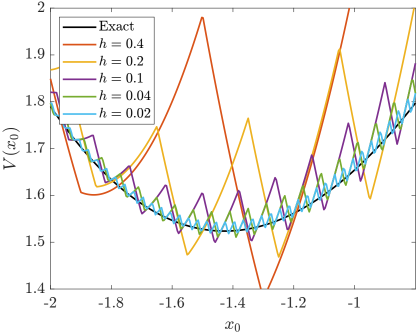
IPM = Interior-Point Method

Reg. = Regularization

We need to:

1. Use a switch detecting integration method: restore accuracy of integration method.
2. Compute derivatives correctly.





Computing derivatives of a discrete time system

Regard an ODE, for simplicity without a control:

$$x(t) = f(x(t)),\ x(t_k) = x_k,\ t \in [t_k, t_{k+1}]$$

In direct optimal control, with the use of an integrator we regard:

$$x_{k+1} = \psi(x_k)$$

Computing derivatives of a discrete time system

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In Newton-type optimization we need to linearize this equation, e.g., at a feasible point $(\bar{x}_{k+1}, \bar{x}_k)$:

$$0 = \psi(\bar{x}_k) - \bar{x}_{k+1} + \frac{\partial \psi(\bar{x}_k)}{\partial x}(\hat{x}_k - \bar{x}_k) - (\hat{x}_{k+1} - \bar{x}_{k+1})$$

$$0 = \underbrace{\psi(\bar{x}_k) - \bar{x}_{k+1}}_{=0} + \frac{\partial \psi(\bar{x}_k)}{\partial x}\Delta x_k - \Delta x_{k+1}$$

Computing derivatives of a discrete time system

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$$0 = \psi(\bar{x}_k) - \bar{x}_{k+1} + \frac{\partial \psi(\bar{x}_k)}{\partial x}(\hat{x}_k - \bar{x}_k) - (\hat{x}_{k+1} - \bar{x}_{k+1})$$

$$0 = \underbrace{\psi(\bar{x}_k) - \bar{x}_{k+1}}_{=0} + \frac{\partial \psi(\bar{x}_k)}{\partial x}\Delta x_k - \Delta x_{k+1}$$

Change in final state by change initial state described by:

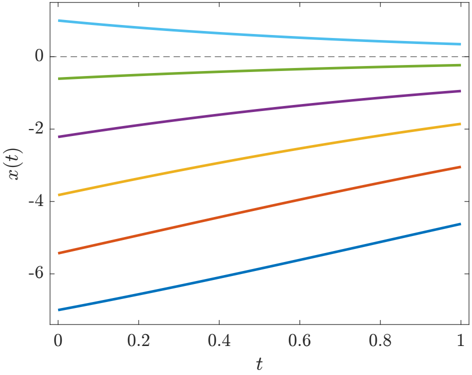
$$\Delta x_{k+1} = \frac{\partial \psi(\bar{x}_k)}{\partial x}\Delta x_k$$

Here $S(t) = \frac{\mathrm{d}x(t;x_k)}{\mathrm{d}x_k}$, i.e., $S(t_{k+1}) = \frac{\mathrm{d}x(t_{k+1};x_k)}{\mathrm{d}x_k} = \frac{\mathrm{d}\phi(x_k)}{\mathrm{d}x_k}$ is the **sensitivity matrix**.

Geometric interpretation of the sensitivity matrix $S(t)$ - smooth case

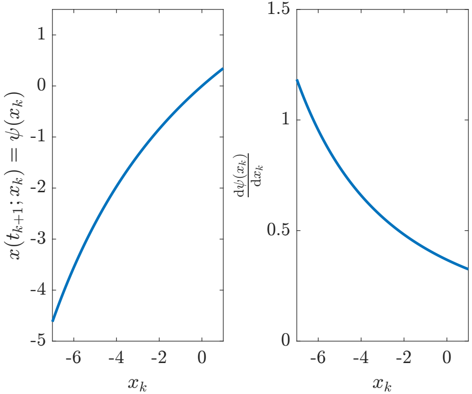
Consider the smooth ODE:

$$\dot{x} = -x - 0.2x^2$$



Trajectory examples.

$$\Delta x_{k+1} = \frac{\partial \psi(\bar{x}_k)}{\partial x} \Delta x_k$$



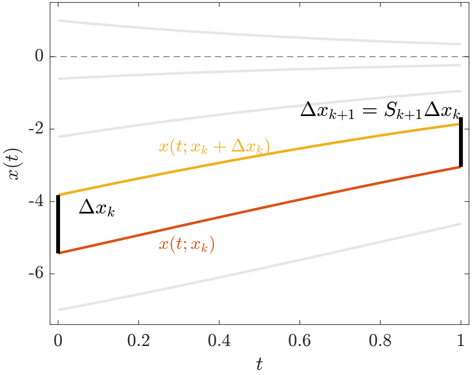
Solution and sensitivity maps.

Geometric interpretation of the sensitivity matrix $S(t)$ - smooth case

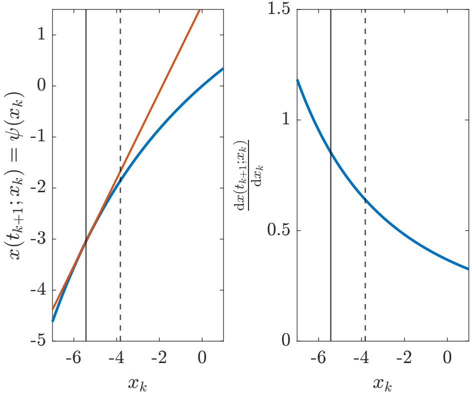
Consider the smooth ODE:

$$\dot{x} = -x - 0.2x^2$$

$$\Delta x_{k+1} = \frac{\partial \psi(\bar{x}_k)}{\partial x} \Delta x_k$$



Linear approximation at t_{k+1}.



Linear approximation of solution map.

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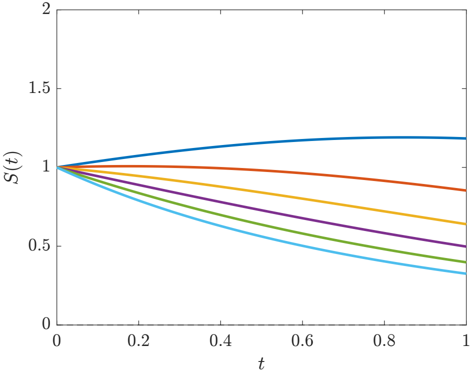
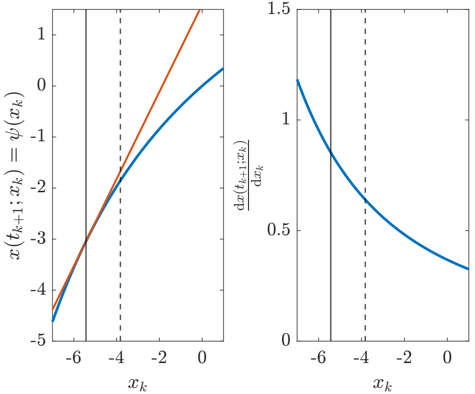


Sensitivities in time.



Linear approximation of solution map.

Computation of $S(t)$ - smooth case

An excellent reference on sensitivity computation is the PhD Thesis [Quirynen, 2017].

$$S(t) = \frac{\mathrm{d}x(t;x_k)}{\mathrm{d}x_k}, t > t_k, \quad S(t_k) = \frac{\mathrm{d}x(t_k;x_k)}{\mathrm{d}x_k} = \frac{\mathrm{d}x_k}{\mathrm{d}x_k} = I \text{ (initial value)}$$

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We take the first differentiate, then integrate:

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}x(t;x_k)}{\mathrm{d}x_k} = \frac{\mathrm{d}\dot{x}(t;x_k)}{\mathrm{d}x_k}$$

$$= \frac{\mathrm{d}f(x(t;x_k))}{\mathrm{d}x_k} = \frac{\partial f(x(t;x_k))}{\partial x_k}\frac{\mathrm{d}x(t;x_k)}{\mathrm{d}x_k} = \frac{\partial f(x(t;x_k))}{\partial x_k}S(t)$$

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Then, jointly integrate:

$$\begin{pmatrix}\dot{x}(t) \\ \dot{S}(t)\end{pmatrix} = \begin{pmatrix} f(x(t)) \\ \frac{\partial f(x(t))}{\partial x}S(t)\end{pmatrix}, \quad x(t_k) = x_k, \; S(t_k) = S_k$$

Computation of $S(t)$ - smooth case

An excellent reference on sensitivity computation is the PhD Thesis [Quirynen, 2017].

$$S(t) = \frac{\mathrm{d}x(t;x_k)}{\mathrm{d}x_k}, t > t_k, \quad S(t_k) = \frac{\mathrm{d}x(t_k;x_k)}{\mathrm{d}x_k} = \frac{\mathrm{d}x_k}{\mathrm{d}x_k} = I \text{ (initial value)}$$

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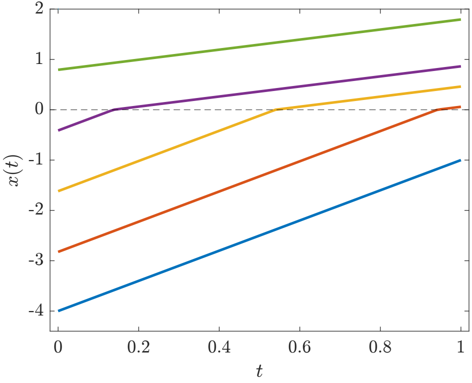
But if the function $f(x)$ is not differentiable in x?

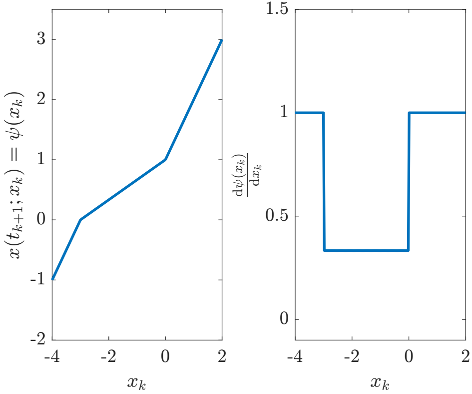
Consider the nonsmooth ODE:

$$\dot{x}(t) = \begin{cases} 3, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}, \quad t \in [t_k, t_{k+1}]$$

$$\Delta x_{k+1} = \frac{\partial \psi(\bar{x}_k)}{\partial x} \Delta x_k$$



Solution has kinks in time.



Solution map has kinks.

Consider the nonsmooth ODE:

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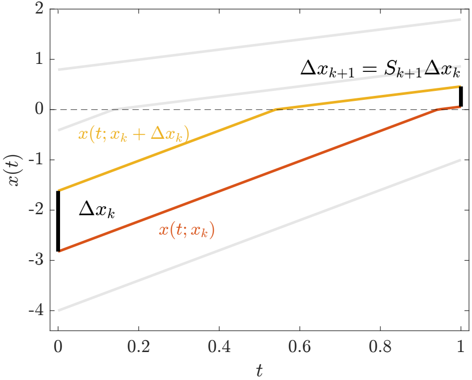

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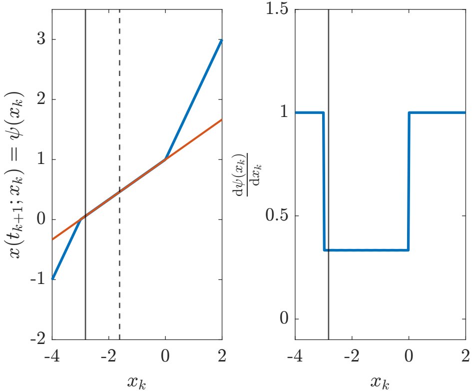

Sensitivity discontinuous.

Geometric interpretation of the sensitivity matrix $S(t)$ - **nonsmooth** case

Consider the nonsmooth ODE:

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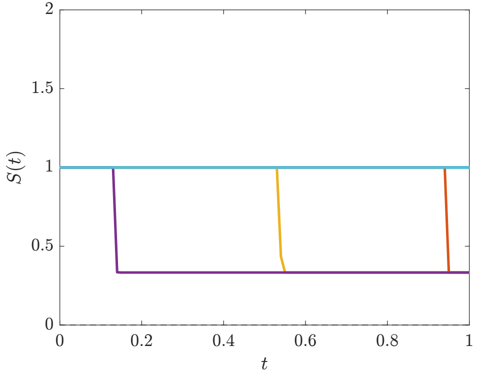
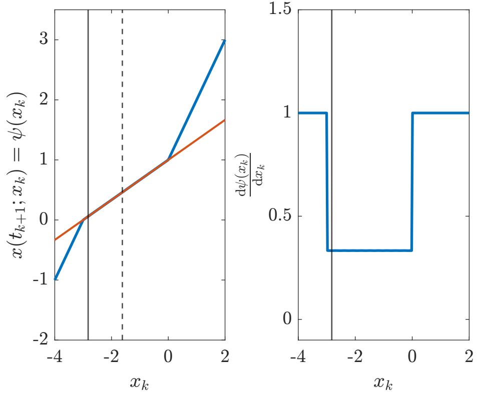


Sensitivities jump in time.

$$\Delta x_{k+1} = \frac{\partial \psi(\bar{x}_k)}{\partial x} \Delta x_k$$



Linearization wrong across switches.

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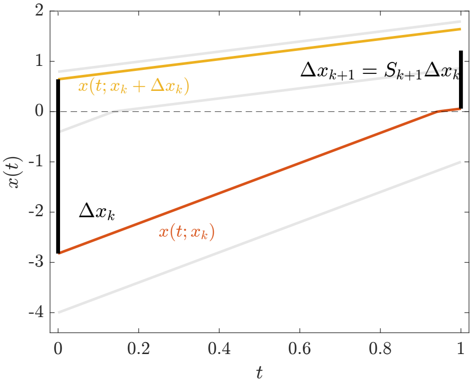
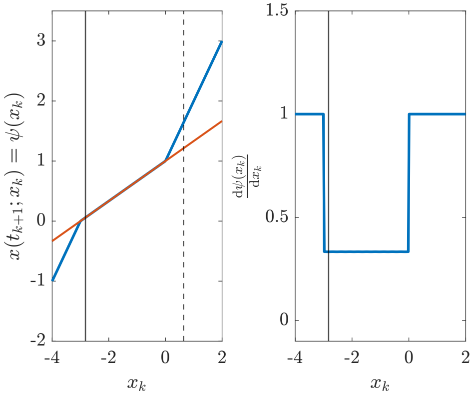


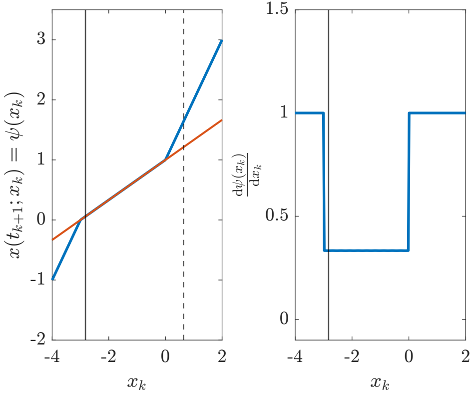
Large error.



Linearization wrong across switches.

Summary on sensitivities of nonsmooth systems

- Solution map $\phi(x_k)$ has kinks, sensitivity $\frac{\mathrm{d}\phi(x_k)}{\mathrm{d}x_k}$ jumps
- The discontinuity of $f(x(t))$ introduces also jumps in $S(t)$ in time
- Linearization can be arbitrarily wrong, if there are changes of switches
- Correct computation of $S(t)$ requires switch detection and updates



Computation of $S(t)$ - nonsmooth case

Regard a bimodal system:

$$\dot{x}(t) = \begin{cases} f_1(x(t)), & \psi(x(t)) < 0, \\ f_2(x(t)), & \psi(x(t)) \geq 0. \end{cases} \tag{1}$$

At some t_{s} trajectory $x(t)$ crosses switching surface $\psi(x) = 0$, e.g.:

- before crossing: $\dot{x} = f_1(x)$ for $t \in [0, t_{\mathrm{s}})$, with solution $x_1(t; x_0)$
- after crossing: t_{s} we have $\dot{x} = f_2(x)$ for $t \in (t_{\mathrm{s}}, T]$ with solution $x_2(t; x_1(t_{\mathrm{s}}; x_0))$

Trajectory pieces $x_1(t)$ and $x_2(t)$ glued together by condition:

$$\psi(x_1(t_{\mathrm{s}}(x_0); x_0)) = 0.$$

Computing sensitivity for:

- $t < t_{\mathrm{s}}$ - just like in the smooth case;
- $t > t_{\mathrm{s}}$ - everything depends implicitly on the switching times $t_{\mathrm{s}}(x_0)$

Computation of $S(t)$ via the Saltation matrix

Before and after the switch the $S(t)$ obey linear variational differential equation (VDE)

$$\dot{S}_i(t) = \frac{\partial f_i(x)}{\partial x} S_i(t), \ \ i = 1, 2$$

The function $S(t)$ obeys smooth VDEs, on both sides of t_s, but exhibits a jump at t_s.

Proposition

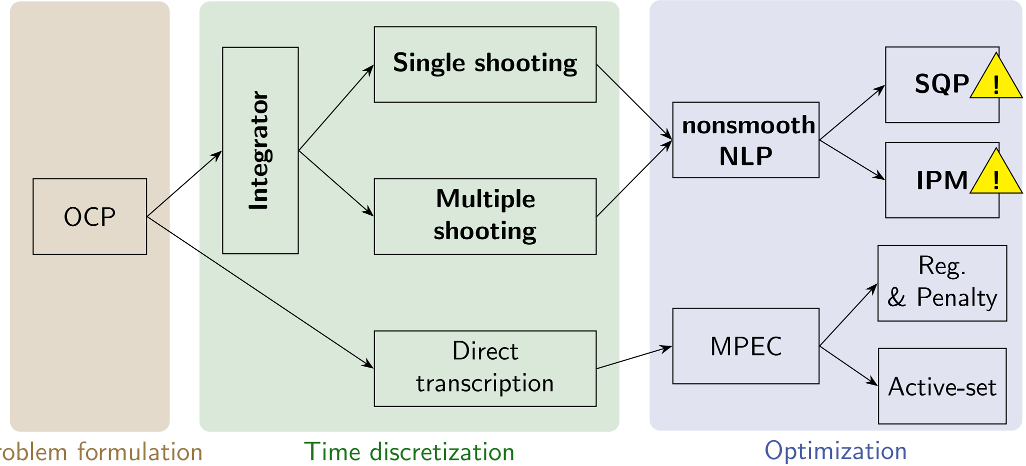
Regard the system (1) *with* $x(0) = x_0 \in R_i$ *on an interval* $[0, T]$ *with a switch at* $t_\mathrm{s} \in (0, T)$. *Assume that the functions* $f_1(x), \ f_2(x), \psi_{i,j}(x)$ *are continuously differentiable along* $x(t), t \in [0, T]$. *Assume the solution* $x(t)$ *reaches the surface of discontinuity transversally, i.e.,* $\nabla \psi(x(t_\mathrm{s}))^\top f_1(x(t_\mathrm{s})) > 0$. *Then the sensitivity* $S(T; 0)$ *of a solution* $x(t; x_0)$ *of the system described by the ODE* (1) *is given by*

$$S(T; 0) = S(T; t_\mathrm{s}^+) J(x(t_\mathrm{s}; x_0)) S(t_\mathrm{s}^-; 0) \ \textit{with}$$

$$J(x(t_\mathrm{s}; x_0)) \coloneqq I + \frac{(f_2(x(t_\mathrm{s}; x_0)) - f_1(x(t_\mathrm{s}; x_0))) \nabla \psi(x(t_\mathrm{s}; x_0))^\top}{\nabla \psi(x(t_\mathrm{s}; x_0))^\top f_1(x(t_\mathrm{s}; x_0))}.$$

Work flow in nonsmooth direct optimal control

First discretize, then optimize.



OCP = Optimal Control Problem

NLP = Nonlinear Program

MPEC = Mathematical Program with Equilibrium Constraints

SQP = Sequential Quadratic Programming

IPM = Interior-Point Method

Reg. = Regularization

Switch detecting integrators in shooting methods is an old idea

Some historical references:

- 1950s - First derivations of the Saltation matrix [Aizerman and Gantmakher, 1958]
- 1980s - Multiple shooting, switch detection, first described in PhD thesis of Hans Georg Bock [Bock, 1987] (multiple shooting introduce by Bock and Plitt [Bock and Plitt, 1984])
- 2000s - Other attempts with multiple shooting and SQP: with single step methods (RK4) [Kirches, 2006], with multi-step methods (BDF) [Brandt-Pollmann, 2004]
- ...
- 2020s: More recent in robotics, hybrid iLQR [Kong et al., 2021], [Kong et al., 2024]

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Can work sometimes quite good, but why not established yet?

Newton's method applied to nonsmooth optimization problems

For simplicity we consider

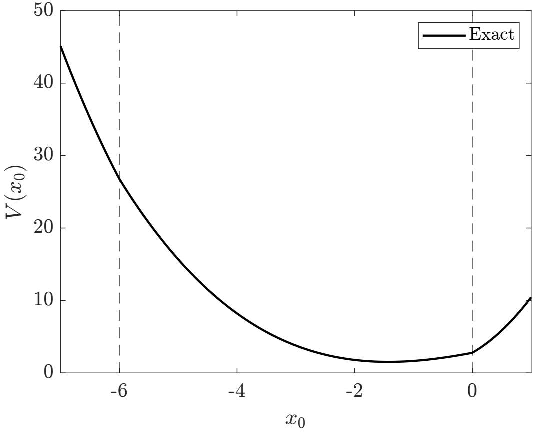
$$\min_{w} F(w)$$

where $F : \mathbb{R}^n \to \mathbb{R}$ is a piecewise smooth function. Without constraints, KKT conditions reduce to

$$\nabla F(w) = 0$$

SQP and IPM reduce to Newton's method and read as

$$w^{k+1} = w^k - [\nabla^2 F(w^k)]^{-1} \nabla F(w^k)$$

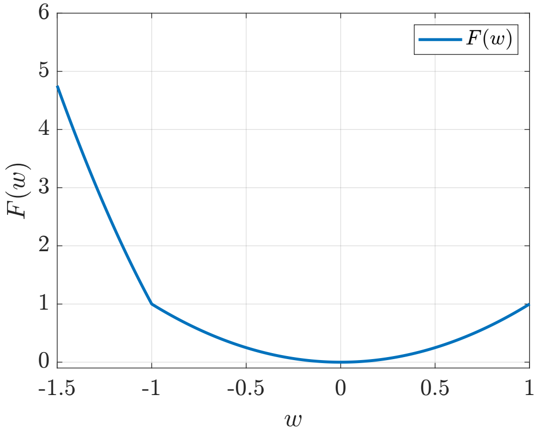


Nonsmooth optimization examples: convex kink not at solution

$$\min_{w} F(w)$$

where

$$F(w) = \begin{cases} 3w^2 - 2, & w < -1 \\ w^2, & w > -1 \end{cases}$$



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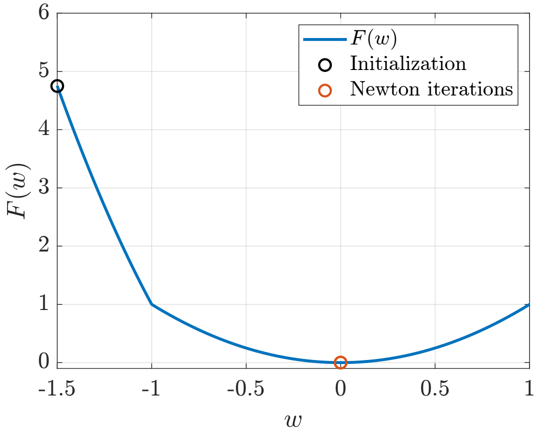

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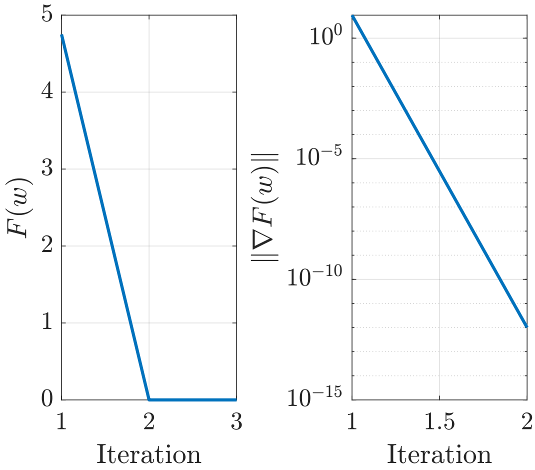

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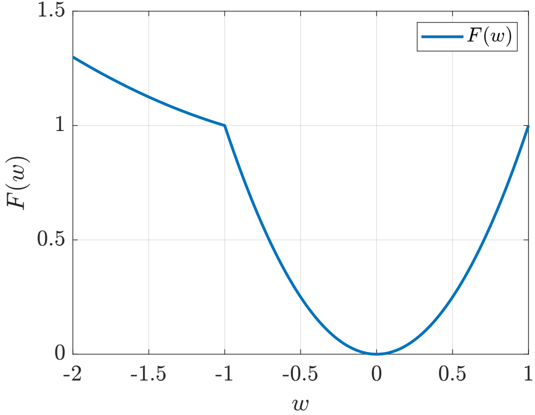

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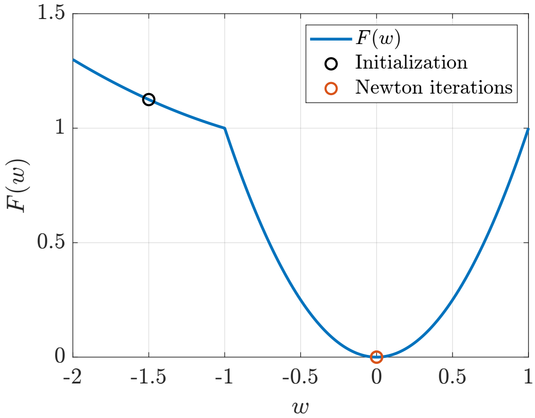

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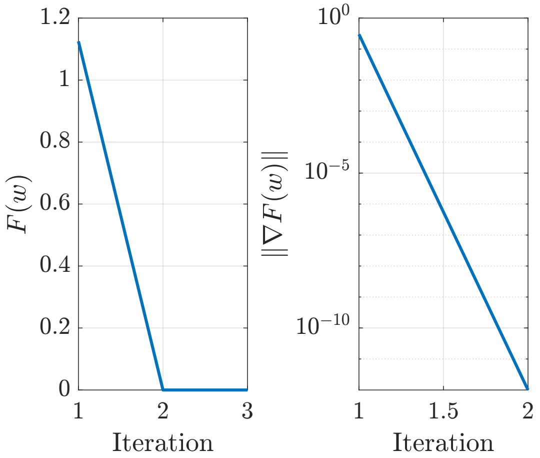

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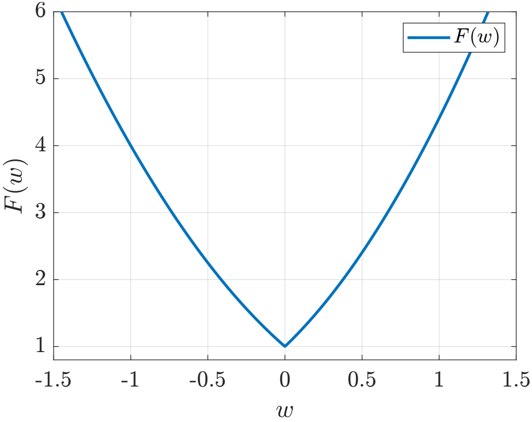

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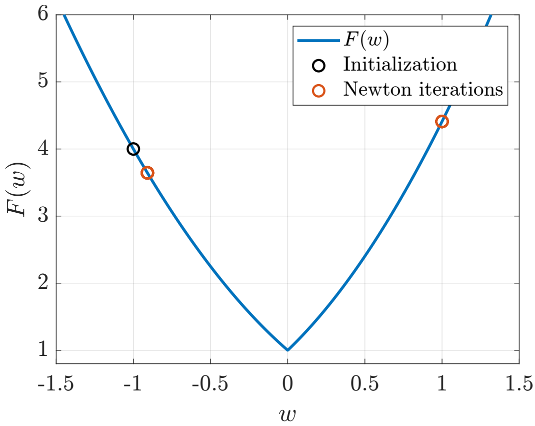

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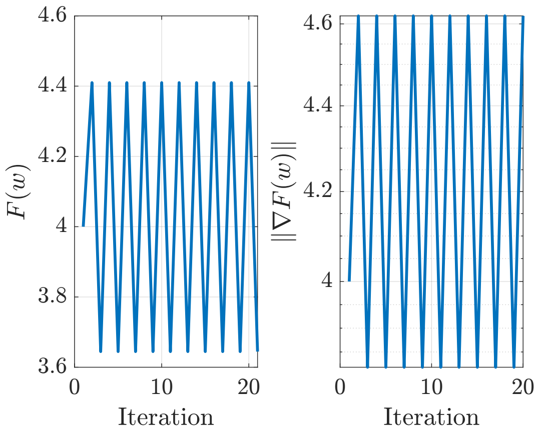
- It may be difficult to converge to a convex kink or to verify it
- At convex kink, we may need to compute a subgradient, and check if $0 \in \partial F(w)$
- More complicated to get "KKT conditions" for nonsmooth problems, may not work at other kinks



line search may help, but still need to verify $0 \in \partial F(w)$

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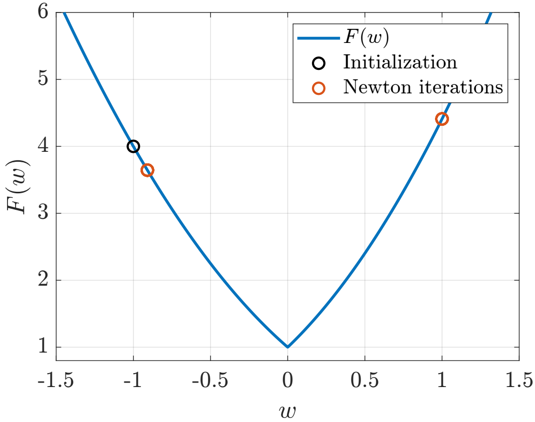
- More complicated to get "KKT conditions" for nonsmooth problems, may not work at other kinks
- Even more difficult: generic solver that solves these "KKT conditions"



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Stopping despite having a descent direction

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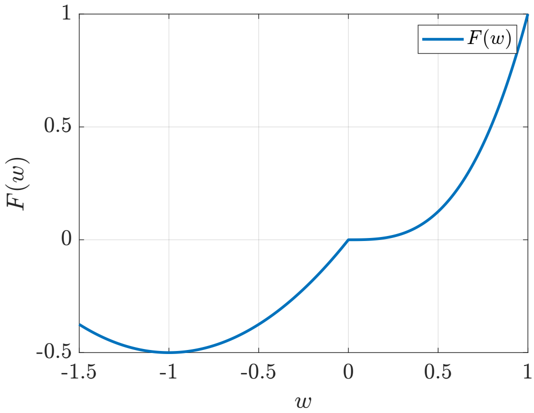
$$F(w) = \begin{cases} w + 0.5w^2 & w < 0 \\ w^3, & w > 0 \end{cases}$$



Nonsmooth optimization examples: concave kink lead to stalling

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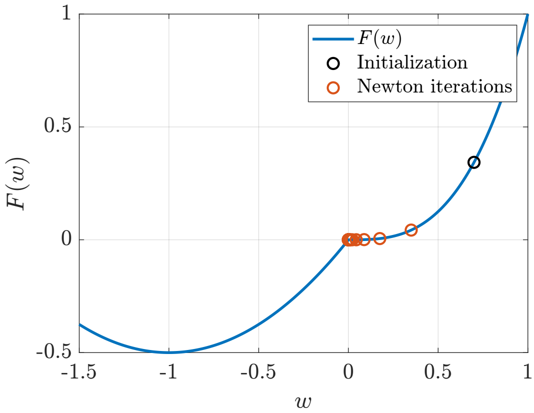

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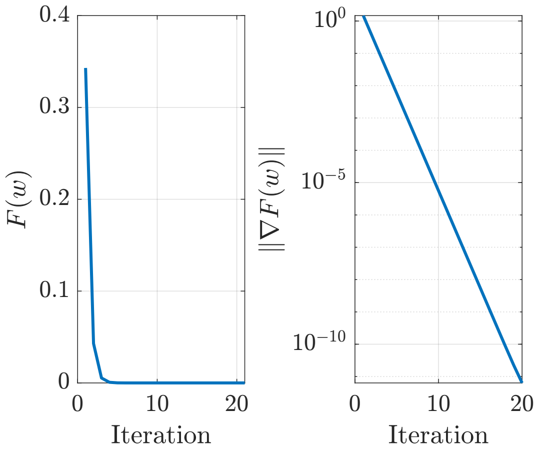

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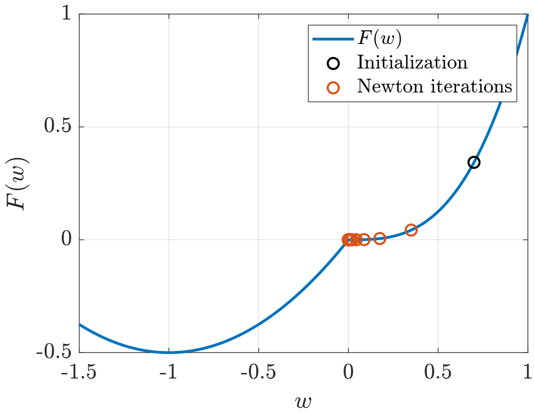
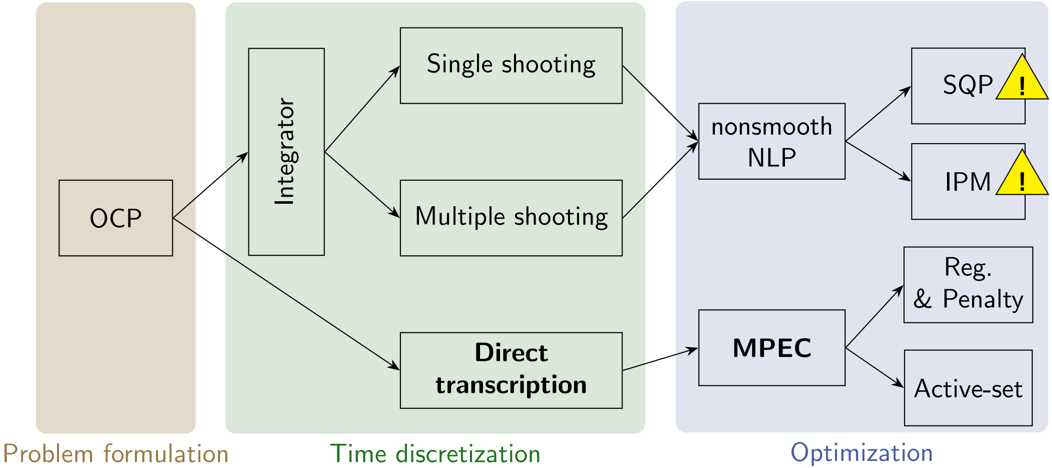
- **Conclusion**: we use the wrong optimality conditions and step computation
- We resolve these problems in the Lectures 5 and 6



Outline

Work flow in nonsmooth direct optimal control

First discretize, then optimize.



OCP = Optimal Control Problem

NLP = Nonlinear Program

MPEC = Mathematical Program with Equilibrium Constraints

SQP = Sequential Quadratic Programming

IPM = Interior-Point Method

Reg. = Regularization

Main ideas of FESD

Based on [Baumrucker and Biegler, 2009, Nurkanović et al., 2024, Nurkanović and Diehl, 2022]

FESD overview

1. Transform nonsmooth system into dynamic complementarity system (Lecture 3)

$$\dot{x} \in F_{\mathrm{F}}(x,u) \quad \iff \quad \begin{aligned} \dot{x} &= F(x,u)\,\theta \\ 0 &= g(x) - \lambda - e\mu \\ 0 &\leq \theta \perp \lambda \geq 0 \\ 1 &= e^\top \theta \end{aligned}$$

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1. Transform nonsmooth system into dynamic complementarity system (Lecture 3)
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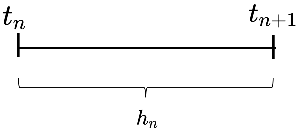

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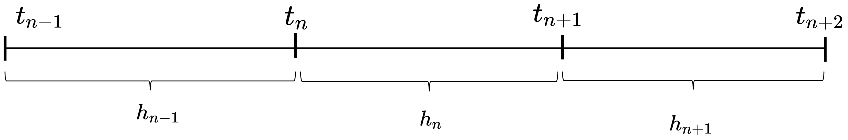

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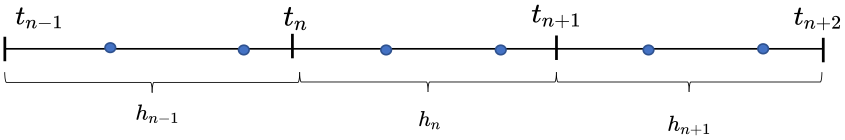
3. Use general implicit Runge-Kutta methods (Lectures 2)



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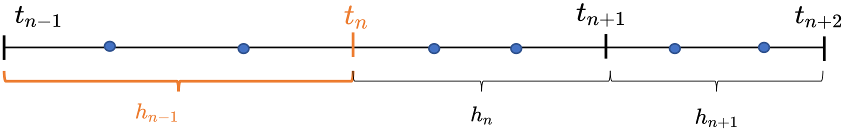
1. Transform nonsmooth system into dynamic complementarity system (Lecture 3)
2. Consider at least two integration intervals = finite elements
3. Use general implicit Runge-Kutta methods (Lectures 2)
4. Let step sizes h_n be degrees of freedom (under-determined system)



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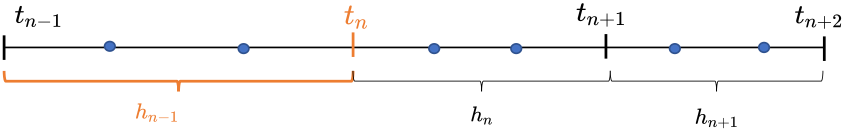

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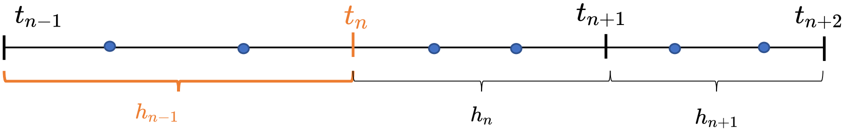
4. Let step sizes h_n be degrees of freedom
5. Cross complementarity conditions - adapt h_n for switch detection
6. Step equilibration - remove degrees of freedom if no switch



From Filippov to dynamic complementarity systems

Using the KKT conditions of the parametric LP

LP representation

$$\dot{x} = F(x,u)\ \theta$$

$$\text{with} \quad \theta \in \underset{\tilde{\theta} \in \mathbb{R}^{n_f}}{\operatorname{argmin}} \quad g(x)^\top \tilde{\theta}$$

$$\text{s.t.} \quad 0 \leq \tilde{\theta}$$

$$1 = e^\top \tilde{\theta}$$

where

$$F(x,u) := [f_1(x,u), \dots, f_{n_f}(x,u)] \in \mathbb{R}^{n_x \times n_f}$$

$$g(x) := [g_1(x), \dots, g_{n_f}(x)]^\top \in \mathbb{R}^{n_f}$$

$$e := [1, 1, \dots, 1]^\top \in \mathbb{R}^{n_f}$$

From Filippov to dynamic complementarity systems

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$$F(x,u) := [f_1(x,u), \dots, f_{n_f}(x,u)] \in \mathbb{R}^{n_x \times n_f}$$
$$g(x) := [g_1(x), \dots, g_{n_f}(x)]^\top \in \mathbb{R}^{n_f}$$
$$e := [1,1,\dots,1]^\top \in \mathbb{R}^{n_f}$$

Express equivalently by optimality conditions:

Dynamic Complementarity System (DCS)

$$\dot{x} = F(x,u)\,\theta \tag{2a}$$
$$0 = g(x) - \lambda - e\mu \tag{2b}$$
$$0 \le \theta \perp \lambda \ge 0 \tag{2c}$$
$$1 = e^\top \theta \tag{2d}$$

Compact notation

$$\dot{x} = F(x,u)\,\theta$$
$$0 = G_{\mathrm{LP}}(x,\theta,\lambda,\mu),$$

- $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}^{n_f}$ are Lagrange multipliers
- (1c) $\Leftrightarrow \min\{\theta,\lambda\} = 0 \in \mathbb{R}^{n_f}$
- Together, (1b), (1c), (1d) determine the $(2n_f+1)$ variables (θ,λ,μ) uniquely

Conventional discretization by Implicit Runge Kutta (IRK) method

Continuous time DCS

$$
\begin{aligned}
x(0) &= \bar{x}_0, \\
\dot{x}(t) &= v(t) \\
v(t) &= F(x(t), u(t))\, \theta(t) \\
0 &= g(x(t)) - \lambda(t) - e\mu(t) \\
0 &\leq \theta(t) \perp \lambda(t) \geq 0 \\
1 &= e^\top \theta(t), \quad t \in [0, T]
\end{aligned}
$$

Conventional discretization by Implicit Runge Kutta (IRK) method

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x(0) &= \bar{x}_0, \\
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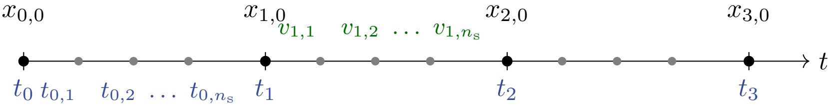
0 &\leq \theta(t) \perp \lambda(t) \geq 0 \\
1 &= e^\top \theta(t), \quad t \in [0, T]
\end{aligned}
$$

Discrete time IRK-DCS equation

$$
\begin{aligned}
x_{0,0} &= \bar{x}_0, \quad x_{n+1,0} = x_{n,0} + h \textstyle\sum_{i=1}^{n_\mathrm{s}} b_i v_{n,i} \\
x_{n,i} &= x_{n,0} + h \textstyle\sum_{j=1}^{n_\mathrm{s}} a_{i,j} v_{n,j} \\
v_{n,i} &= F(x_{n,i}, u_{n,i})\,\theta_{n,i} \\
0 &= g(x_{n,i}) - \lambda_{n,i} - e\mu_{n,i} \\
0 &\leq \theta_{n,i} \perp \lambda_{n,i} \geq 0 \\
1 &= e^\top \theta_{n,i}, \quad i = 1, \ldots, n_\mathrm{s}, \quad n = 0, \ldots, N-1
\end{aligned}
$$

Notation: $x_{n,i} \in \mathbb{R}^{n_x}, \theta_{n,i} \in \mathbb{R}^m$ etc. RK stage values with:

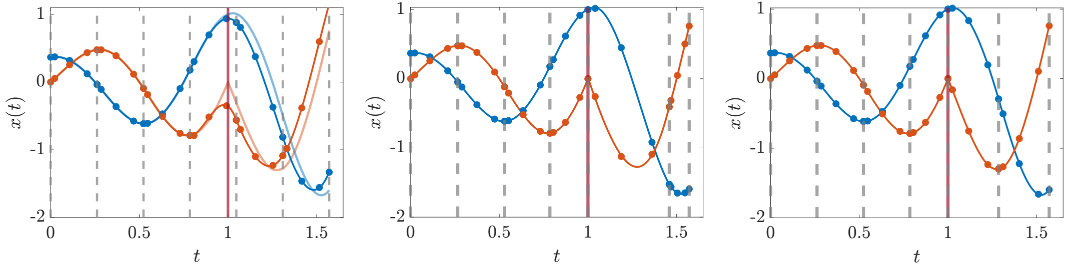
- $n \in \{0, 1, \ldots, N\}$ - index of integration step; step length $h := T/N$
- $i, j \in \{0, 1, \ldots, n_\mathrm{s}\}$ - index of intermediate IRK stage / collocation point
- $a_{i,j}$ and b_i - Butcher tableau entries of Implicit Runge Kutta method



Finite Elements with Switch Detection (FESD)

FESD is a novel DCS discretization method based on three ideas:

- make step sizes h_n free, ensure $\sum_{n=0}^{N-1} h_n = T$ (cf. [Baumrucker and Biegler, 2009])
- allow switches only at element boundaries, enforce via **cross-complementarities**,
- remove spurious degrees of freedom via **step equilibration**.



conventional discretization | variable step sizes and cross-complementarities | FESD discretization with step equilibration

Conventional DCS and FESD discretization without step equilibration

Time-stepping discretization

$$
\begin{aligned}
x_{0,0} &= \bar{x}_0, \quad h = T/N \\
x_{n+1,0} &= x_{n,0} + h \sum_{i=1}^{n_s} b_i v_{n,i} \\
x_{n,i} &= x_{n,0} + h \sum_{j=1}^{n_s} a_{i,j} v_{n,j} \\
v_{n,i} &= F(x_{n,i}, u_{n,i})\, \theta_{n,i} \\
0 &= g(x_{n,i}) - \lambda_{n,i} - e\mu_{n,i} \\
0 &\leq \theta_{n,i} \perp \lambda_{n,i} \geq 0 \\
1 &= e^\top \theta_{n,i}
\end{aligned}
$$

for $i = 1, \dots, n_s$
and $n = 0, \dots, N-1$

FESD discretization without step equilibration

$$
\begin{aligned}
x_{0,0} &= \bar{x}_0, \ \sum_{n=0}^{N-1} h_n = T \\
x_{n+1,0} &= x_{n,0} + h_n \sum_{i=1}^{n_s} b_i v_{n,i} \\
x_{n,i} &= x_{n,0} + h_n \sum_{j=1}^{n_s} a_{i,j} v_{n,j} \\
v_{n,i} &= F(x_{n,i}, u_{n,i})\, \theta_{n,i} \\
0 &= g(x_{n,i'}) - \lambda_{n,i'} - e\mu_{n,i'} \\
0 &\leq \theta_{n,i} \perp \lambda_{n,i'} \geq 0 \quad \text{(cross-complementarities)} \\
1 &= e^\top \theta_{n,i}
\end{aligned}
$$

for $i = 1, \dots, n_s$ and $n = 0, \dots, N-1$
and $i' = 0, 1, \dots, n_s$

- N extra variables $(h_0, \dots, h_{N-1})$ restricted by one extra equality
- Additional multipliers $\lambda_{n,0}, \mu_{n,0}$ are uniquely determined

Conventional DCS and FESD discretization with step equilibration

Time-stepping discretization

$$
\begin{aligned}
x_{0,0} &= \bar{x}_0, \quad h = T/N \\
x_{n+1,0} &= x_{n,0} + h \textstyle\sum_{i=1}^{n_{\mathrm{s}}} b_i v_{n,i} \\
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0 &= g(x_{n,i}) - \lambda_{n,i} - e\mu_{n,i} \\
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for $i = 1, \dots, n_{\mathrm{s}}$
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FESD discretization with step equilibration

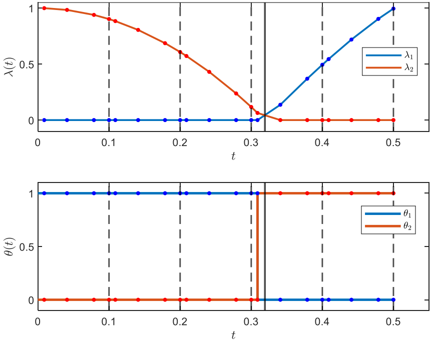
$$
\begin{aligned}
x_{0,0} &= \bar{x}_0, \ \textstyle\sum_{n=0}^{N-1} h_n = T \\
x_{n+1,0} &= x_{n,0} + h_n \textstyle\sum_{i=1}^{n_{\mathrm{s}}} b_i v_{n,i} \\
x_{n,i} &= x_{n,0} + h_n \textstyle\sum_{j=1}^{n_{\mathrm{s}}} a_{i,j} v_{n,j} \\
v_{n,i} &= F(x_{n,i}, u_{n,i})\, \theta_{n,i} \\
0 &= g(x_{n,i'}) - \lambda_{n,ii'} - e\mu_{n,i'} \\
0 &\leq \theta_{n,i} \perp \lambda_{n,i'} \geq 0 \quad \text{(cross-complementarities)} \\
1 &= e^\top \theta_{n,i} \\
0 &= \nu(\theta_{n'}, \theta_{n'+1}, \lambda_{n'}, \lambda_{n'+1}) \cdot (h_{n'} - h_{n'+1})
\end{aligned}
$$

for $i = 1, \dots, n_{\mathrm{s}}$ and $n = 0, \dots, N-1$
and $i' = 0, 1, \dots, n_{\mathrm{s}}$ and $n' = 0, \dots, N-2$

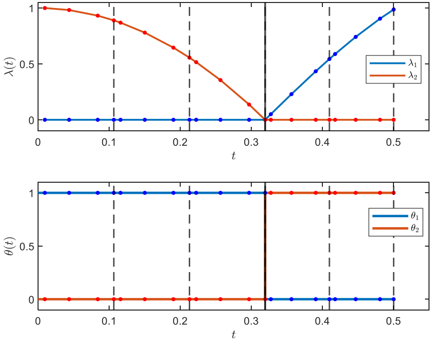
- N extra variables $(h_0, \dots, h_{N-1})$ restricted by one extra equality
- Additional multipliers $\lambda_{n,0}, \mu_{n,0}$ are uniquely determined
- Indicator function $\nu(\theta_{n'}, \theta_{n'+1}, \lambda_{k'}, \lambda_{k'+1})$ only zero if a switch occurs

Multipliers in conventional and FESD discretization

Time stepping discretization:



FESD discretization:



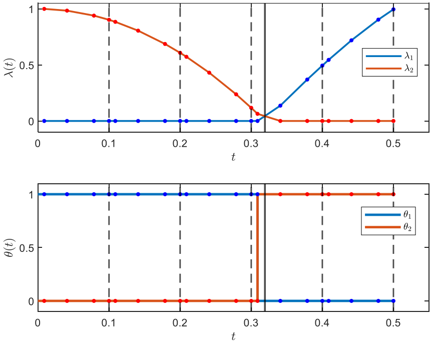
Lemma (Cross complementarity)

If any $\theta_{n,j,i}$ with $j = 1, \ldots, n_{\mathrm{s}}$ is positive, then all $\lambda_{n,j',i}$ with $j' = 0, \ldots, n_{\mathrm{s}}$ must be zero. Conversely, if any $\lambda_{n,j',i}$ is positive, then all $\theta_{n,j,i}$ are zero.

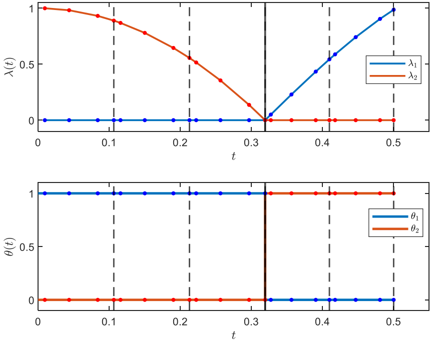
Notation $\lambda_{n,j,i}$ - n - finite element, j - RK stage, i - component of vector

Multipliers in conventional and FESD discretization

Time stepping discretization:



FESD discretization:



FESD's cross-complementarities exploit the fact that the multiplier $\lambda_i(t)$ is continuous in time.
On boundary, $\lambda_i(t_n)$ must be zero if $\theta_i(t) > 0$ for any $t \in [t_{n-1}, t_{n+1}]$ on the adjacent intervals.
This implicitly imposes the constraint $g_i(x_n) - g_j(x_n) = 0$.
$\implies$ h_n adapts for exact switch detection

Step equilibration

- if no switches happen, cross complementarity implied by standard complementarity
- spurious degrees of freedom in h_n: more degrees of freedom than equations

Step equilibration

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- spurious degrees of freedom in h_n: more degrees of freedom than equations
- exploit complementarity of θ_n, λ_n to encode switching logic
- define (very complicated) switch indicator function ν (cf. [Nurkanović, 2023]:

$$\nu(\theta_n, \theta_{n+1}, \lambda_n, \lambda_{n+1}) := \begin{cases} \text{positive}, & \text{if no switch at } t_{n+1} \\ 0, & \text{if switch at } t_{n+1} \end{cases}$$

Step equilibration

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- step equilibration:

$$0 = \nu(\theta_n, \theta_{n+1}, \lambda_n, \lambda_{n+1}) \cdot (h_n - h_{n+1}), \quad n = 0, \dots, N-2$$

Step equilibration

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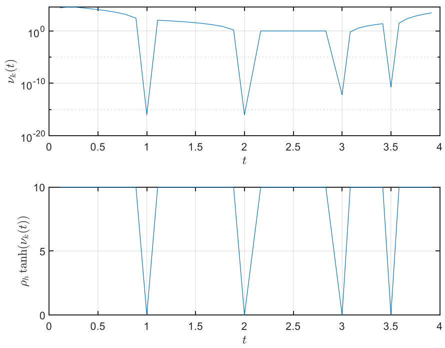
- step equilibration:

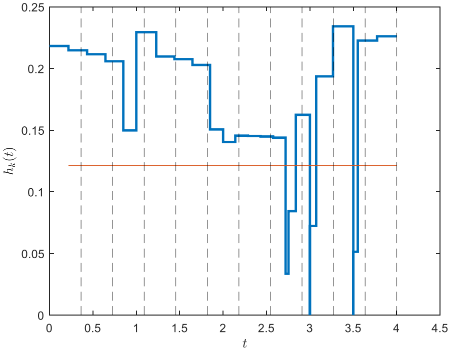
$$0 = \nu(\theta_n, \theta_{n+1}, \lambda_n, \lambda_{n+1}) \cdot (h_n - h_{n+1}), \qquad n = 0, \dots, N-2$$

- Summary:
  - If switch happens, then h_n is determined by cross complementarity.
  - If no switch happens, then h_n is determined by step equilibration.

Indicator function over time:



Step size over time:

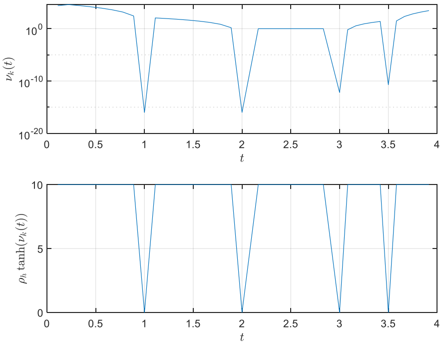


Optimizer varies step size randomly, potentially playing with integration errors.

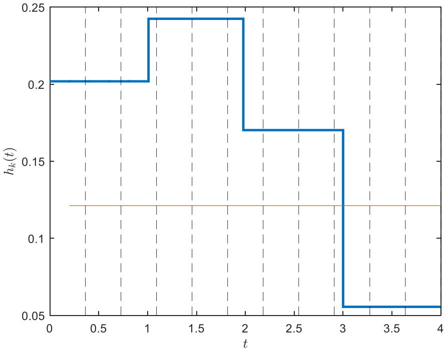
Numerical solution with equilibration

Example with four switches

Indicator function over time:



Step size over time:



Equidistant grid on each "switching stage". Jumps exactly at switching times.

Integration order plots for FESD and IRK time stepping

Revisit example from Lecture 3

Tutorial example

$$\dot{x} = \begin{cases} A_1 x, & \|x\|_2^2 < 1, \\ A_2 x, & \|x\|_2^2 > 1, \end{cases}$$

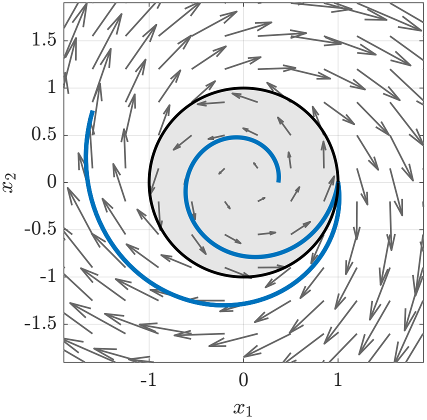
with $A_1 = \begin{bmatrix} 1 & 2\pi \\ -2\pi & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & -2\pi \\ 2\pi & 1 \end{bmatrix}$

$x(0) = (e^{-1}, 0)$ for $t \in [0, \frac{\pi}{2}]$.

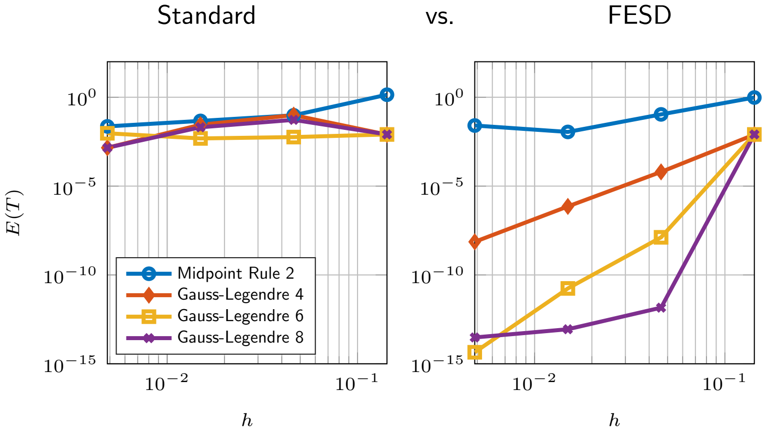
Compute global integration error $E(T)$ using different strategies.

Compute solution approximation:

1. With fixed step size IRK methods (time-stepping).
2. FESD with same underlying IRK methods.



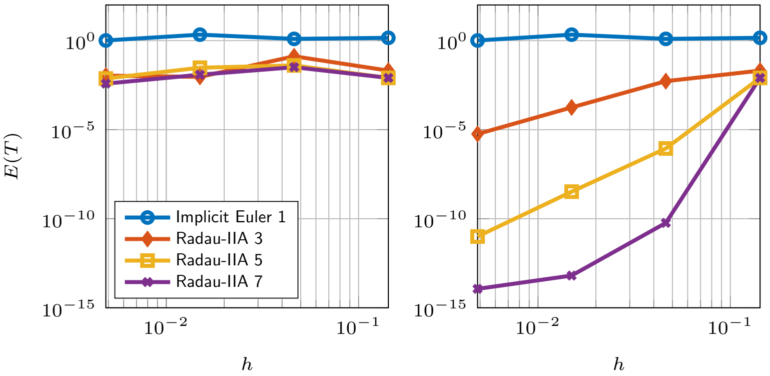
FESD recovers high integration order for switched systems



Integration error $E(T)$ at time $T = \pi/2$ vs. step-size h, for different IRK methods.
FESD discretization recovers high integration order

FESD recovers high integration order for switched systems

Standard vs. FESD



Integration error $E(T)$ at time $T = \pi/2$ vs. step-size h, for different IRK methods.
FESD discretization recovers high integration order

Outline

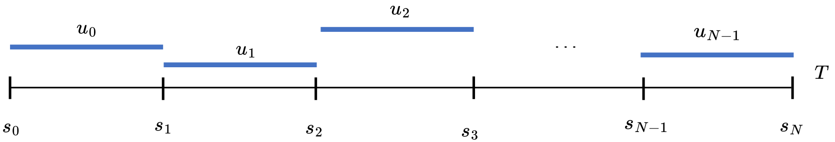
Discretizing optimal control problems with FESD

Discretized optimal control problem

$$\min_{s,z,u} \sum_{k=0}^{N-1} \Phi_L(s_k, z_k, u_k) + E(s_N)$$

$$\begin{aligned} \text{s.t.} \quad s_0 &= \bar{x}_0 \\ s_{k+1} &= \Phi_f(s_k, z_k, u_k) \\ 0 &= \Phi_{\text{int}}(s_k, z_k, u_k) \\ 0 &\geq h(s_k, u_k), \; k = 0, \dots, N-1 \\ 0 &\geq r(s_N) \end{aligned}$$

Control horizon $[0, T]$ with N control stages



- States at control grid points $s = (s_0, \dots, s_N)$
- Piecewise controls $u = (u_0, \dots, u_{N-1})$
- FESD with N_{FE} finite elements applied on every control interval

Discretizing optimal control problems with FESD

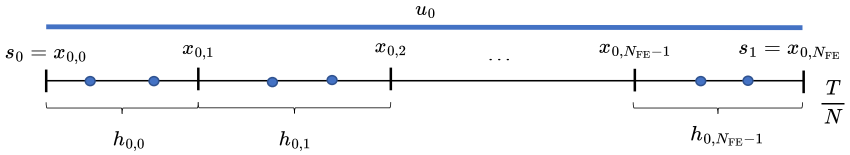
Discretized optimal control problem

$$\min_{s,z,u} \sum_{k=0}^{N-1} \Phi_L(s_k, z_k, u_k) + E(s_N)$$

$$\begin{aligned}
\text{s.t.} \quad s_0 &= \bar{x}_0 \\
s_{k+1} &= \Phi_f(s_k, z_k, u_k) \\
0 &= \Phi_{\text{int}}(s_k, z_k, u_k) \\
0 &\geq h(s_k, u_k), \; k = 0, \dots, N-1 \\
0 &\geq r(s_N)
\end{aligned}$$

- States at control grid points $s = (s_0, \dots, s_N)$
- Piecewise controls $u = (u_0, \dots, u_{N-1})$
- FESD with N_{FE} finite elements applied on every control interval
- Φ_{int} summarizes all internal FESD equations: RK, cross complementarity, step equilibration,...

Control horizon $[0, T]$ with N control stages



Discretizing optimal control problems with FESD

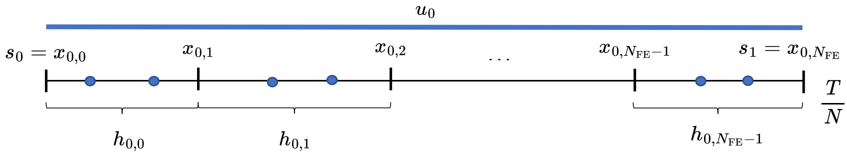
Discretized optimal control problem

$$\min_{s,z,u} \sum_{k=0}^{N-1} \Phi_L(s_k, z_k, u_k) + E(s_N)$$

$$\begin{aligned}
\text{s.t.} \quad s_0 &= \bar{x}_0 \\
s_{k+1} &= \Phi_f(s_k, z_k, u_k) \\
0 &= \Phi_{\text{int}}(s_k, z_k, u_k) \\
0 &\geq h(s_k, u_k), \; k = 0, \dots, N-1 \\
0 &\geq r(s_N)
\end{aligned}$$

- States at control grid points $s = (s_0, \dots, s_N)$
- Piecewise controls $u = (u_0, \dots, u_{N-1})$
- FESD with N_{FE} finite elements applied on every control interval
- Φ_{int} summarizes all internal FESD equations: RK, cross complementarity, step equilibration,...
- $z = (z_0, \dots, z_{N-1})$ - all interval variables: internal states, stage values of states and multipliers, step sizes, ...

Control horizon $[0, T]$ with N control stages



FESD-discretized optimal control problems are MPCC

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$$0 \geq h(s_k, u_k), \; k = 0, \dots, N-1$$
$$0 \geq r(s_N)$$

Collect $w = (s, z, u) \in \mathbb{R}^{n_w}$
Mathematical programs with complementarity constraints (MPCC) are more difficult than standard NLPs

NLP with Complementarity Constraints

$$\min_{w \in \mathbb{R}^{n_w}} F(w)$$
$$\text{s.t.} \quad 0 = G(w)$$
$$0 \geq H(w)$$
$$0 \leq G_1(w) \perp G_2(w) \geq 0$$

Standard and cross complementarity constraints summarized in

$$0 \leq G_1(w) \perp G_2(w) \geq 0$$

Revisiting the OCP example - now with FESD

Tutorial example inspired by [Stewart & Anitescu, 2010]

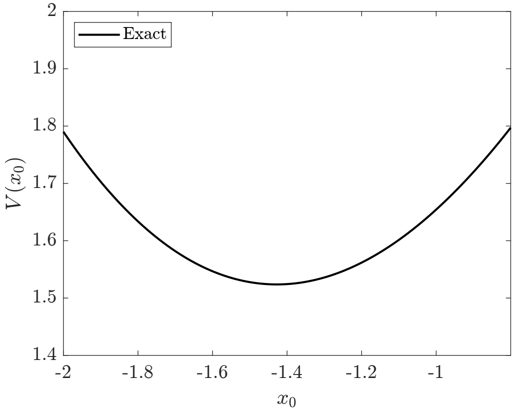
Continuous-time OCP

$$\min_{x(\cdot)\in\mathcal{C}^0([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2) - 5/3)^2$$
$$\text{s.t.} \quad \dot{x}(t) = 2 - \mathrm{sign}(x(t)), \quad t \in [0,2]$$

Free initial value $x(0)$ is the effective degree of freedom.

Equivalent reduced problem

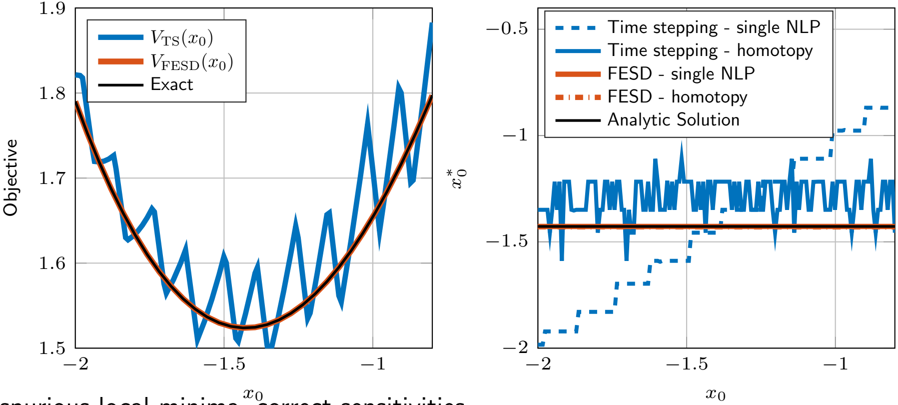
$$\min_{x_0\in\mathbb{R}} V(x_0)$$



- Denote by $V(x_0)$ the nonsmooth objective value for the unique feasible trajectory starting at $x(0) = x_0$.

Revisiting the OCP example - now with FESD

Tutorial example inspired by [Stewart & Anitescu, 2010]



- no spurious local minima, correct sensitivities
- convergence to the "true" local minimum, both with homotopy and without it
- accuracy of order $O(h^p)$, in contrast to standard approach with only $O(h)$

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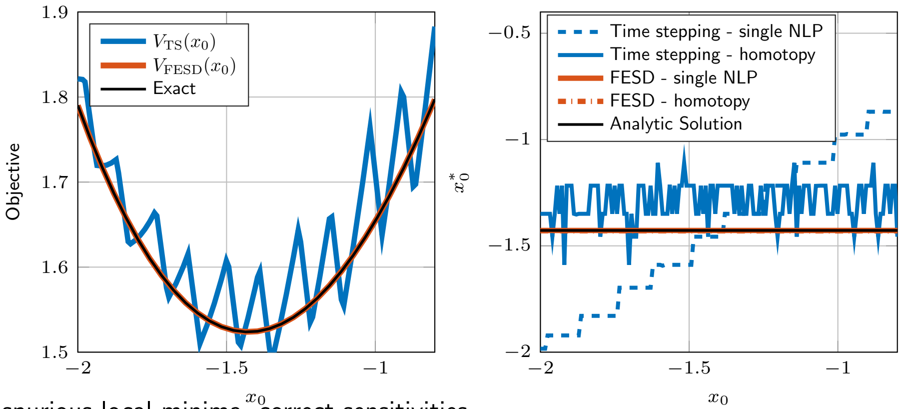

- no spurious local minima, correct sensitivities
- convergence to the "true" local minimum, both with homotopy and without it
- accuracy of order $O(h^p)$, in contrast to standard approach with only $O(h)$
- FESD solves the accuracy and convergence issues

nosnoc

An open source tool for optimal control of nonsmooth systems

Overview

- All reformulations are automated, simply provide problem and model data.
- A wide variety of features: DAEs, nonlinear and quadratic costs, general (including complementarity) path constraints, and terminal constraints.
- C++ code generation for embedded MPC
- Developed and Maintained by **Anton Pozharskiy**, Jonathan Frey, and Armin Nurkanović.

github.com/nosnoc/nosnoc

github.com/nosnoc/nosnoc_py

Available Systems

nosnoc supports various systems such as:

- Piecewise Smooth Systems (via Stewart or Heaviside Step reformulations).
- Heaviside Step Differential Inclusions [Nurkanović et al., 2024].
- Complementarity Lagrangian systems (via FESD-J [Nurkanović et al., 2024] or time-freezing [Nurkanović et al., 2023]).
- Projected Dynamical Systems [Pozharskiy et al., 2024].

Conclusions and summary

- Finite Elements with Switch Detection (FESD) allow highly accurate simulation and optimal control for nonsmooth systems of level NSD2
- Following similar lines, FESD can be derived for the Heaviside step reformulation

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- Key ideas: make step sizes degrees of freedom and introduce implicit relations that locate the switches
- Switch detection not only essential for high accuracy, **but also for correct sensitivities** (no spurious solutions)

Conclusions and summary

- Finite Elements with Switch Detection (FESD) allow highly accurate simulation and optimal control for nonsmooth systems of level NSD2
- Following similar lines, FESD can be derived for the Heaviside step reformulation
- Key ideas: make step sizes degrees of freedom and introduce implicit relations that locate the switches
- Switch detection not only essential for high accuracy, **but also for correct sensitivities** (no spurious solutions)
- FESD solves many of the issues that standard methods have: integration accuracy, convergence of sensitivities
- Main practical difficulty: solving Mathematical Programs with Complementarity Constraints (MPCC)

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Derivation of the Saltation matrix

Before and after the switch the $S(t)$ obey linear variational differential equation (VDE)

$$\dot{S}_i(t) = \frac{\partial f_i(x)}{\partial x} S_i(t),\ \ i = 1,2$$

The function $S(t)$ obeys smooth VDEs, on both sides of t_{s}, but exhibits a jump at t_{s}.

Proposition

Regard the system (1) *with* $x(0) = x_0 \in R_i$ *on an interval* $[0,T]$ *with a switch at* $t_{\mathrm{s}} \in (0,T)$*. Assume that the functions* $f_1(x),\ f_2(x), \psi_{i,j}(x)$ *are continuously differentiable along* $x(t), t \in [0,T]$*. Assume the solution* $x(t)$ *reaches the surface of discontinuity transversally, i.e.,* $\nabla\psi(x(t_{\mathrm{s}}))^\top f_1(x(t_{\mathrm{s}})) > 0$*. Then the sensitivity* $S(T;0)$ *of a solution* $x(t;x_0)$ *of the system described by the ODE* (1) *is given by*

$$\begin{aligned} S(T;0) &= S(T;t_{\mathrm{s}}^+)J(x(t_{\mathrm{s}};x_0))S(t_{\mathrm{s}}^-;0) \ \text{with} \\ J(x(t_{\mathrm{s}};x_0)) &\coloneqq I + \frac{(f_2(x(t_{\mathrm{s}};x_0)) - f_1(x(t_{\mathrm{s}};x_0)))\nabla\psi(x(t_{\mathrm{s}};x_0))^\top}{\nabla\psi(x(t_{\mathrm{s}};x_0))^\top f_1(x(t_{\mathrm{s}};x_0))}. \end{aligned} \tag{3}$$

Proof follows similar lines as in [Stewart and Anitescu, 2010]

For $t < t_{\mathrm{s}}$, the solution $x(t; x_0)$ satisfies the ODE $\dot{x} = f_i(x(t; x_0))$ and the sensitivity matrix $S^x(t, 0; x_0) = \frac{\partial x(t;x_0)}{\partial x_0}$ obeys:

$$\dot{S^x}(t) = \frac{\partial f(x)}{\partial x} S^x(t), \;\; S^x(0) = I.$$

At $t = t_{\mathrm{s}}$ the solution reaches the surface of discontinuity:

$$\psi_{i,j}(x(t_{\mathrm{s}}(x_0; x_0))) = 0. \tag{4}$$

For $t > t_{\mathrm{s}}$, one has $y(t) = f_*(y(t; y_0))$ which is related to the solution via

$$y(t; y_0) = x(t + t_{\mathrm{s}}(x_0); x_0), \;\; y_0(x_0) = x(t_{\mathrm{s}}(x_0); x_0). \tag{5}$$

Proof of the proposition (2/3)

Note that $y(t - t_s(x_0); x_0) = x(t; x_0)$. Therefore, the sensitivity, for $t > t_s$ can be computed via

$$\begin{aligned} S^x(t, 0; x_0) = \frac{\partial x(t, x_0)}{\partial x_0} &= \frac{\partial y(t - t_s(x_0)); y_0(x_0))}{\partial x_0} \\ &= \frac{\partial y(t - t_s}{\partial t} \frac{\partial t_s(x_0)}{\partial x_0}^\top + S^y(t - t_s; y_0) \frac{\partial y_0(x_0)}{\partial x_0}, \\ &= -f_*(x(t)) \frac{\partial t_s(x_0)}{\partial x_0}^\top + S^y(t - t_s; y_0) \frac{\partial y_0(x_0)}{\partial x_0}, \end{aligned} \tag{6}$$

We can compute $\frac{\partial y_0(x_0)}{\partial x_0}$ at $t = t_s^-$ using (5)

$$\frac{\partial y_0(x_0)}{\partial x_0} = \frac{\partial x(t_s(x_0); x_0)}{\partial x_0} = f_i(x) \frac{\partial t_s(x_0)}{\partial x_0}^\top + S^x(t_s^-, 0; x_0). \tag{7}$$

Using the implicit function theorem (cf. [Dontchev and Rockafellar, 2014, Theorem 1B.1]) for (4), again at $t = t_s^-$ we obtain

$$\frac{\partial t_s(x_0)}{\partial x_0}^\top = -\frac{\nabla \psi_{i,j}(x(t_s(x_0; x_0)))^\top S^x(t_s^-, 0; x_0)}{\nabla \psi_{i,j}(x(t_s(x_0; x_0)))^\top f_i(x)}. \tag{8}$$

Proof of the proposition (3/3)

Let $t \to t_{\mathrm{s}}^{+}$ in (6), then $S^{y}(t - t_{\mathrm{s}}; y_0) \to I$. Plugging (7) and (8) into (6) for the remaining unknown terms we obtain

$$\begin{aligned} S^{x}(t_{\mathrm{s}}^{+}, 0; x_0) &= f_{*}(x(t)) \frac{\nabla \psi_{i,j}(x(t_{\mathrm{s}}(x_0; x_0)))^{\top} S^{x}(t_{\mathrm{s}}^{-}, 0; x_0)}{\nabla \psi_{i,j}(x(t_{\mathrm{s}}(x_0; x_0)))^{\top} f_i(x)} \\ &- f_i(x) \frac{\nabla \psi_{i,j}(x(t_{\mathrm{s}}(x_0; x_0)))^{\top} S^{x}(t_{\mathrm{s}}^{-}, 0; x_0)}{\nabla \psi_{i,j}(x(t_{\mathrm{s}}(x_0; x_0)))^{\top} f_i(x)} S^{x}(t_{\mathrm{s}}^{-}, 0; x_0) + S^{x}(t_{\mathrm{s}}^{-}, 0; x_0) \end{aligned} \tag{9}$$

Finally, from the chain rule we have $S^{x}(T, 0, x_0) = S^{x}(T, t_{\mathrm{s}}^{+}, x_0) S^{x}(t_{\mathrm{s}}^{+}, 0, x_0)$ and (9) we obtain (3).

Switch detection - example

Suppose that $x(t)$ crosses from R_1 to R_2 and recall that $\mu = \min_j g_j(x)$

Continuous time:

- Before switch: $\theta_1(t) > 0, \lambda_1(t) = 0$, and $\theta_2(t) = 0, \lambda_2 \geq 0$
- After switch: $\theta_1(t) = 0, \lambda_1(t) \geq 0$, and $\theta_2(t) > 0, \lambda_2 = 0$

Switch detection - example

Suppose that $x(t)$ crosses from R_1 to R_2 and recall that $\mu = \min_j g_j(x)$
Discrete time (switch between the n-th and $n+1$-st finite element):

- Before switch: $\theta_{n,j,1} > 0, \lambda_{n,j,1} = 0$, and $\theta_{n,j,2} = 0, \lambda_{n,j,2} \geq 0$
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From Lemma 1 it follows that $\lambda_{n,n_\mathrm{s},1} = \lambda_{n,n_\mathrm{s},2} = 0$

Switch detection conditions

$$g_1(x_{n+1}) = \lambda_{n,n_\mathrm{s},1} - \mu_{n,n_\mathrm{s}}$$

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Switch detection condition

$$g_1(x_{n+1}) = 0 - g_2(x_{n+1})$$

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From Lemma 1 it follows that $\lambda_{n,n_{\mathrm{s}},1} = \lambda_{n,n_{\mathrm{s}},2} = 0$

Switch detection conditions

$$0 = g_1(x_{n+1}) - g_2(x_{n+1}) = \psi_{12}(x_{n+1})$$

Implies constraint such that h_n must adapt for exact switch detection!

Summary of FESD theoretical results

1. An FESD problem needs to solve a nonlinear complementarity problem (NCP) to advance the integration. The solutions of these NCP are locally unique.
   - For a given point determine which constraint cross comp. and step eq. are binding, and which implicitly satisfied.
   - Obtain square system and apply implicit function theorem.

2. Convergence of the FESD method to a Filippov solution of the underlying system with accuracy $O(h^p)$ is proven. Here, p is the order of the underlying smooth IRK method.
   - Solution approximation and true solution predict same active set.
   - Switching time accuracy also $O(h^p)$.

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    - Solution approximation and true solution predict same active set.
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3. Convergence of numerical sensitivities to the true value with $O(h^p)$ is given.
    - Cross. comp. implicitly enforce switching condition and lead to correct sensitivities.
    - The Stewart & Anitescu problem is solved.

Optimal control benchmark with FESD

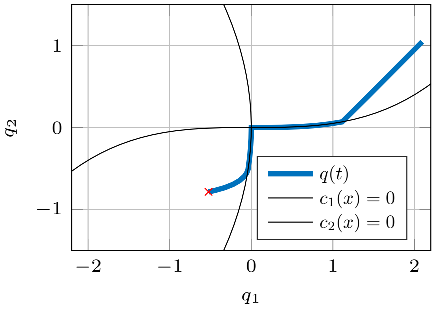
Benchmark example with entering/leaving sliding mode

OCP with sliding modes

$$
\begin{aligned}
\min_{x(\cdot),u(\cdot)} \quad & \int_0^4 u(t)^\top u(t) + v(t)^\top v(t)\,\mathrm{d}t \\
\text{s.t.} \quad & x(0) = \left(\frac{2\pi}{3}, \frac{\pi}{3}, 0, 0\right) \\
& \dot{x}(t) = \begin{bmatrix} -\mathrm{sign}(c(x(t))) + v(t) \\ u(t) \end{bmatrix} \\
& -2e \leq v(t) \leq 2e \\
& -10e \leq u(t) \leq 10e \quad t \in [0,4], \\
& q(T) = \left(-\frac{\pi}{6}, -\frac{\pi}{4}\right)
\end{aligned}
$$

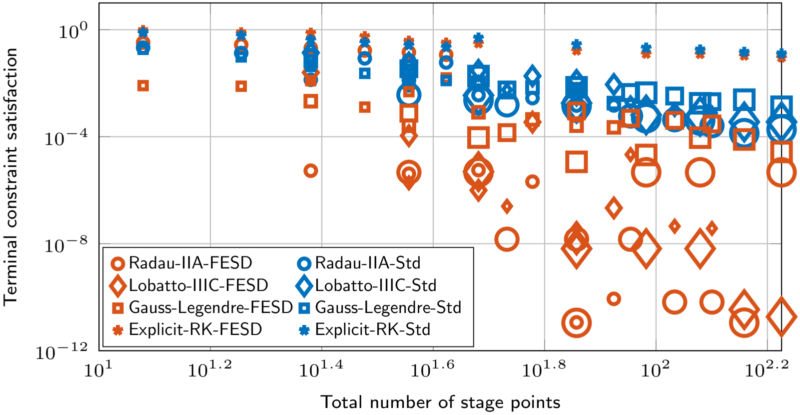
States $q, v \in \mathbb{R}^2$ and control $u \in \mathbb{R}^2$, $x = (q, v)$

Switching functions $c(x) = \begin{bmatrix} q_1 + 0.15q_2^2 \\ 0.05q_1^3 + q_2 \end{bmatrix}$



FESD vs standard IRK - number of function evaluations

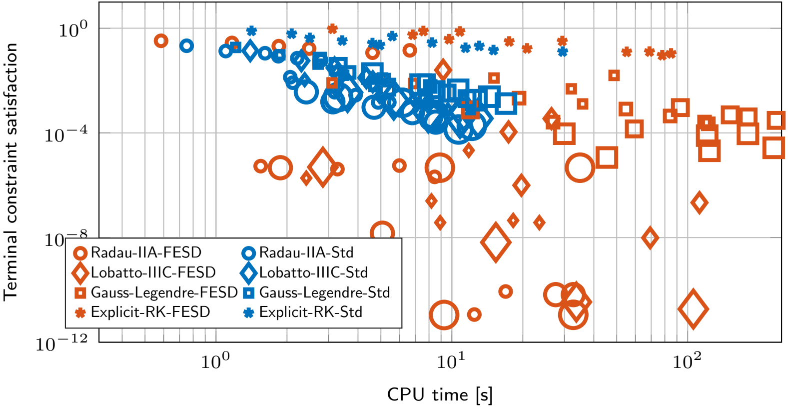
Benchmark on an optimal control problem with nonlinear sliding modes



Terminal constraint satisfaction vs. number of stage points

FESD vs standard IRK - CPU Time

Benchmark on an optimal control problem with nonlinear sliding modes



Terminal constraint satisfaction vs. CPU time

FESD one million times more accurate than Std. for CPU time of ≈ 2 s