Sequential Integer Linear Programming with Accelerations by Shortest Path Computations for Integer Optimal Control on One-dimensional Domains

Sven Leyffer[1] Paul Manns[2] Marvin Severitt[2]

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[1]Argonne National Laboratory

[2]TU Dortmund University

$$\underset{y,v\in L^\infty(\Omega)}{\text{minimize}} \ \frac{1}{2}\|y+y_0\|^2_{L^2(D_o)} \ \text{subject to} \ \begin{cases} -\Delta y - k_0^2(1+qv)y = k_0^2 q v y_0 \text{ in } D, \quad \dfrac{\partial y}{\partial n} - ik_0 y = 0 \text{ in } \partial D, \\ v : D_s \to \{0,1,2\}. \end{cases}$$

A Mixed-Integer PDE-Constrained Optimization Problem

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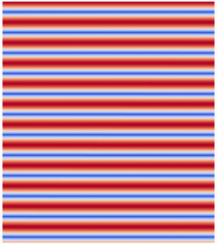
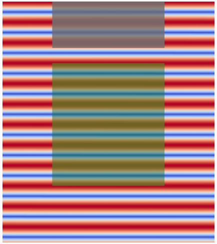


$|\operatorname{Re} y_0|$



Cloak & Scatterer Area

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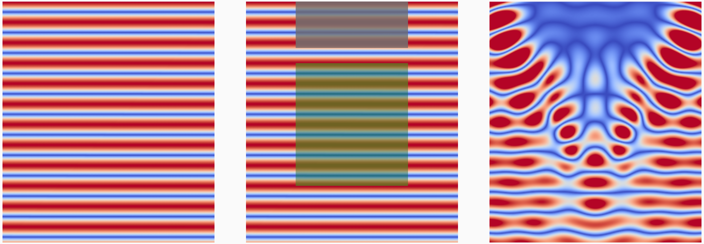

$|\mathrm{Re}\, y_0|$ | Cloak & Scatterer Area | $|\mathrm{Re}\, y^* + y_0|$

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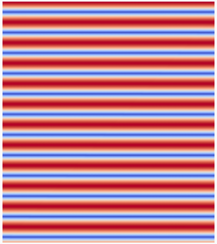

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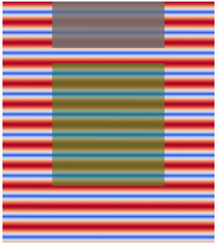

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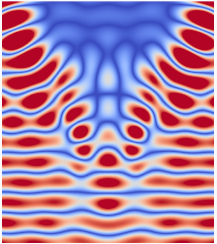

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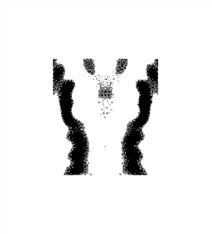

Scatterer Design

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General Problem Class

$$\underset{y,v\in L^\infty(\Omega)}{\text{minimize}}\ J(y)\ \text{subject to}\ \begin{cases} y = S(v), \\ v(x) \in \{\nu_1, \dots, \nu_M\} =: V \subset \mathbb{Z} \text{ for almost all } x \in \Omega. \end{cases} \tag{MIPDECO}$$

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Solution Approaches

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- First-discretize, then-optimize (using MINLP techniques) quickly becomes intractable, see e.g. (Lin, Leyffer, and Munson 2016).

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Solution Approaches

- First-discretize, then-optimize (using MINLP techniques) quickly becomes intractable, see e.g. (Lin, Leyffer, and Munson 2016).
- Relaxation-based solution strategies: bang-bang principles (e.g. Tröltzsch 1979; Kunisch and Wachsmuth 2013), regularization-based methods (e.g. Stadler 2009; Clason and Kunisch 2014), combinatorial integral approximation (e.g. Sager, Bock, and Diehl 2012; Hante and Sager 2013; M. and Kirches 2020b).

$$\underset{v}{\text{minimize}}\ J(y) \quad \text{subject to} \quad y = S(v) \text{ and } \begin{cases} v(x) \in \{\nu_1, \ldots, \nu_M\} \text{ a.e.} & \text{(P)} \\ v(x) \in \operatorname{conv}\{\nu_1, \ldots, \nu_M\} \text{ a.e.} & \text{(R)} \end{cases}$$

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Observation (Sager, Bock, and Diehl 2012; Hante and Sager 2013; Kirches, M., and Ulbrich 2020)
We can prove (and exploit algorithmically) for classes of ODEs, PDEs, and integral equations that

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- Strong assumptions on the PDE.
- Highly oscillating controls.

Unregularized

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Multibang Regularization
(Clason and Kunisch 2014; M. 2021)

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Unregularized

Multibang Regularization (Clason and Kunisch 2014; M. 2021)

The control regularity is *just not enough* to limit oscillations.

Regularization with Total Variation

$$\underset{v}{\textbf{minimize}}\ \ J(y) + \mathrm{TV}(v) \quad \textbf{subject to} \quad y = S(v) \text{ and } v(x) \in \{\nu_1, \ldots, \nu_M\} \text{ a.e.} \tag{Q}$$

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Observation

- Let $J \circ S : L^2(\Omega) \to \mathbb{R}$ be lower semicontinuous, and bounded below. Then (Q)—unlike (P)!—has a minimizer in the space $BV(\Omega)$ (e.g. (Ambrosio, Fusco, and Pallara 2000)).

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Definition 1
Let $r > 0$. Then $v \in BV(\Omega)$ feasible for (Q) is r-optimal (locally optimal) for (Q) if

$$(J \circ S)(v) + \mathrm{TV}(v) \le (J \circ S)(\tilde{v}) + \mathrm{TV}(\tilde{v}) \quad \text{for all } \tilde{v} \in BV(\Omega) \text{ with } \|v - \tilde{v}\|_{L^1} \le r.$$

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Question: can we construct descent algorithms that converge to r-optimal points?

$$
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\underset{v_{\text{next}}}{\text{minimize}} \quad & (\nabla(J \circ S)(v), v_{\text{next}} - v)_{L^2(\Omega)} + \mathrm{TV}(v_{\text{next}}) - \mathrm{TV}(v) \\
\text{subject to} \quad & \|v - v_{\text{next}}\|_{L^1(\Omega)} \leq \Delta \text{ and } v_{\text{next}}(x) \in \{\nu_1, \ldots, \nu_M\} \text{ a.e.}
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Motivation / justification of the subproblem choice

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- (TR) has a minimizer (with objective value ≤ 0).
- L^1 trust region allows for nontrivial changes of the control (formally: v_{next} and v may have level sets that are not homeomorphic).

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- The **TV** term enforces feasibility of weak-* cluster points and suppresses oscillations.
- If $\Omega \subset \mathbb{R}$ is discretized into intervals, then we obtain an integer **linear** program.

$$\begin{aligned} \underset{v_{\text{next}}}{\text{minimize}} \quad & (\nabla(J \circ S)(v), v_{\text{next}} - v)_{L^2(\Omega)} + \mathbf{TV}(v_{\text{next}}) - \mathbf{TV}(v) \\ \text{subject to} \quad & \|v - v_{\text{next}}\|_{L^1(\Omega)} \leq \Delta \text{ and } v_{\text{next}}(x) \in \{\nu_1, \ldots, \nu_M\} \text{ a.e.} \end{aligned} \tag{TR}$$

Motivation / justification of the subproblem choice

- (TR) has a minimizer (with objective value ≤ 0).
- L^1 trust region allows for nontrivial changes of the control (formally: v_{next} and v may have level sets that are not homeomorphic).
- The **TV** term enforces feasibility of weak-* cluster points and suppresses oscillations.
- If $\Omega \subset \mathbb{R}$ is discretized into intervals, then we obtain an integer **linear** program.
- Piecewise constant functions, defined on uniform discretizations, are dense (with respect to the right topologies!) in the set of feasible controls for (Q).

Algorithm 1 Reset trust-region method using (TR)

Input: $\Delta^0 > 0$, v^0 feasible, $\sigma \in (0,1)$.

for $n = 0, \dots$ **do**
  $k \leftarrow 0$
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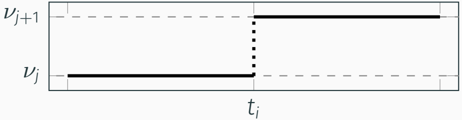

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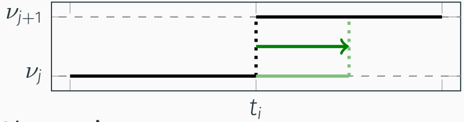

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If $\liminf_{h \downarrow 0} \frac{1}{h} \int_{t_i}^{t_i + h} \nabla(J \circ S)(v) > 0$, then shifting t_i slightly to the right improves the objective of (TR) without changing the **TV** term.

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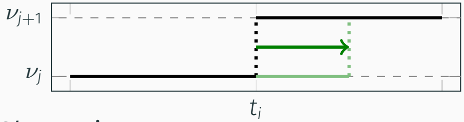

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Sequential Linear Integer Programming Method in Function Space

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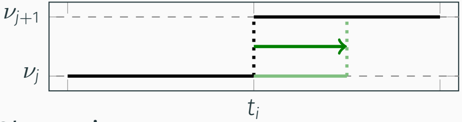

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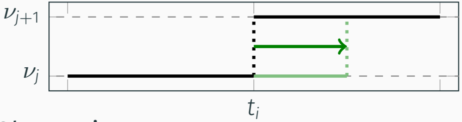

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Results for an LLSQ problem for different (nonadaptive) control discretizations (N) vs MIQP and CIA

	SLIP (with SCIP)		MIQP (with SCIP)		CIA (w./ scipy.optimize, L-BFGS-B, SCARP)	
N	objective	time	objective	time to best objective	objective	time
32	$9.081 \cdot 10^{-3}$	$3.543 \cdot 10^{-1}$ s	$\mathbf{5.079 \cdot 10^{-3}}$	$3.837 \cdot 10^{3}$ s	$1.839 \cdot 10^{-2}$	$1.775 \cdot 10^{1}$ s
64	$9.169 \cdot 10^{-3}$	1.015 s	$\mathbf{4.733 \cdot 10^{-3}}$	$4.064 \cdot 10^{3}$ s*	$6.369 \cdot 10^{-3}$	$1.775 \cdot 10^{1}$ s
128	$7.080 \cdot 10^{-3}$	3.185 s	$\mathbf{5.447 \cdot 10^{-3}}$	$1.434 \cdot 10^{4}$ s*	$5.551 \cdot 10^{-3}$	$1.778 \cdot 10^{1}$ s
256	$5.523 \cdot 10^{-3}$	$2.350 \cdot 10^{1}$ s	$\mathbf{5.513 \cdot 10^{-3}}$	$1.644 \cdot 10^{4}$ s*	$7.741 \cdot 10^{-3}$	$1.777 \cdot 10^{1}$ s
512	$\mathbf{4.426 \cdot 10^{-3}}$	$1.687 \cdot 10^{2}$ s	$6.685 \cdot 10^{-3}$	$1.776 \cdot 10^{4}$ s*	$1.220 \cdot 10^{-2}$	$1.778 \cdot 10^{1}$ s
1024	$\mathbf{4.529 \cdot 10^{-3}}$	$3.303 \cdot 10^{2}$ s	$9.153 \cdot 10^{-3}$	$1.680 \cdot 10^{4}$ s*	$2.350 \cdot 10^{-2}$	$1.784 \cdot 10^{1}$ s
2048	$\mathbf{4.339 \cdot 10^{-3}}$	$1.698 \cdot 10^{4}$ s	$2.727 \cdot 10^{-2}$	$1.746 \cdot 10^{4}$ s*	$4.610 \cdot 10^{-2}$	$1.800 \cdot 10^{1}$ s

* Timeout after $1.8 \cdot 10^4$ s

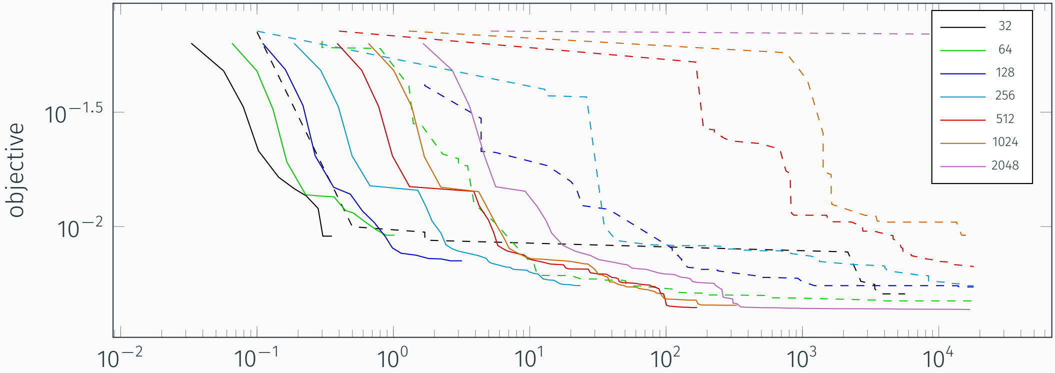

Figure 1: Objective values over running time for SLIP (solid) and MIQP (dashed) for $N = 32, \ldots, 2048$.

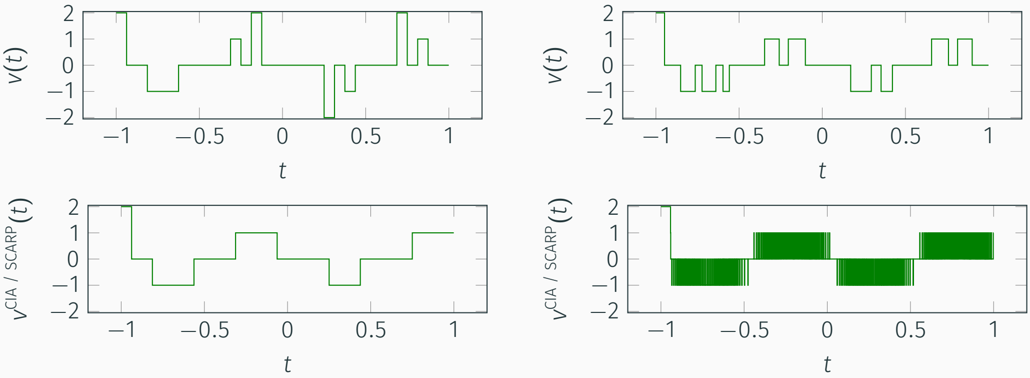

Figure 2: Top row: final control trajectories produced by the SLIP method for $N = 32$ (left, objective value $9.081 \cdot 10^{-3}$) and $N = 2\,048$ (right, objective value $4.339 \cdot 10^{-3}$). Bottom row: control trajectories produced by the combinatorial integral approximation decomposition approach using CIA / SCARP for $N = 32$ (left, objective value $1.839 \cdot 10^{-2}$) and $N = 2\,048$ (right, objective value $4.610 \cdot 10^{-2}$).

Discussion

1. Sufficient regularity condition: $\nabla^2(J \circ S)[v, w] \leq C\|v\|_{L^1}\|w\|_{L^1}$. Why? Need to bound remainder term in Taylor's theorem (because we have $\|v\|_{L^2}^2 \leq C\|v\|_{L^1}$ and Cauchy decrease only in L^1).

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$\Longrightarrow$ Detect this and run second-order methods on switching point optimization NLPs for fast convergence (e.g. (Maurer and Osmolovskii 2004)).

How to solve the trust-region problem (efficiently)?

Discretized trust-region subproblem in 1D as an Integer Linear Program

Let $V \subset \mathbb{Z}$ with $|V| < \infty$. Let $\alpha > 0$, $\Delta > 0$. Let $(v_i)_{i \in \{1,\dots,N\}} \subset V$ be an integer control trajectory.

$$
\mathrm{TR}(v, \Delta) = \begin{cases}
\min\limits_{d,\delta,\xi,\Xi} \; c^T d + \alpha \Xi \\
\quad \text{s.t. } v_i + d_i \in V \text{ for all } i \in \{1, \dots, N\} \\
\qquad -\delta_i \le d_i h_i \le \delta_i \text{ for all } i \in \{1, \dots, N\} \\
\qquad \sum\limits_{i=1}^{N} \delta_i \le \Delta \\
\qquad -\xi_i \le v_{i+1} + d_{i+1} - v_i - d_i \le \xi_i \text{ for all } i \in \{1, \dots, N-1\} \\
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\end{cases}
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Let $V \subset \mathbb{Z}$ with $|V| < \infty$. Let $\alpha > 0$, $\Delta > 0$. Let $(v_i)_{i \in \{1,\ldots,N\}} \subset V$ be an integer control trajectory.

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$v_{\text{next}} = v + d$ is integer

Let $V \subset \mathbb{Z}$ with $|V| < \infty$. Let $\alpha > 0$, $\Delta > 0$. Let $(v_i)_{i \in \{1,\dots,N\}} \subset V$ be an integer control trajectory.

$$\mathrm{TR}(v, \Delta) = \begin{cases} \min\limits_{d,\delta,\xi,\Xi} c^T d + \alpha \Xi \\ \quad \text{s.t. } v_i + d_i \in V \text{ for all } i \in \{1,\dots,N\} \\ \quad\quad -\delta_i \le d_i h_i \le \delta_i \text{ for all } i \in \{1,\dots,N\} \\ \quad\quad \sum_{i=1}^{N} \delta_i \le \Delta \\ \quad\quad -\xi_i \le v_{i+1} + d_{i+1} - v_i - d_i \le \xi_i \text{ for all } i \in \{1,\dots,N-1\} \\ \quad\quad \sum_{i=1}^{N-1} \xi_i \le \Xi \end{cases}$$

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Discretized trust-region subproblem in 1D as an Integer Linear Program

Let $V \subset \mathbb{Z}$ with $|V| < \infty$. Let $\alpha > 0$, $\Delta > 0$. Let $(v_i)_{i\in\{1,\dots,N\}} \subset V$ be an integer control trajectory.

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$c_i = \int_{t_i}^{t_{i+1}} \nabla (J \circ S)(v)(s) \, \mathrm{d}s$

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$\|v_{\text{next}} - v\|_{L^1} \le \Delta$

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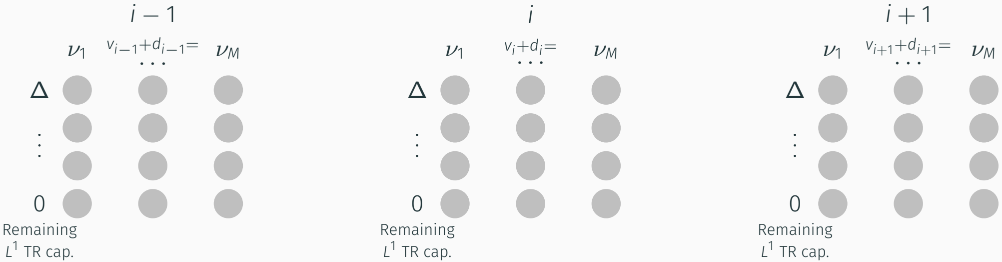
$$

Conjecture / Work in Progress
Let $h_i = 1$ for all i. Then $\mathrm{TR}(v, \Delta)$ is (weakly) NP hard. (Reduce Knapsack to $\mathrm{TR}(v, \Delta)$ in polynomial time – construct V appropriately from weights, capacity, and α.)

Discretized trust-region subproblem in 1D as a shortest path in a layered DAG

Let $h_i = 1$ for all i (generalization to $h_i \in \mathbb{N}$ possible), $\Delta \in \mathbb{N}$.



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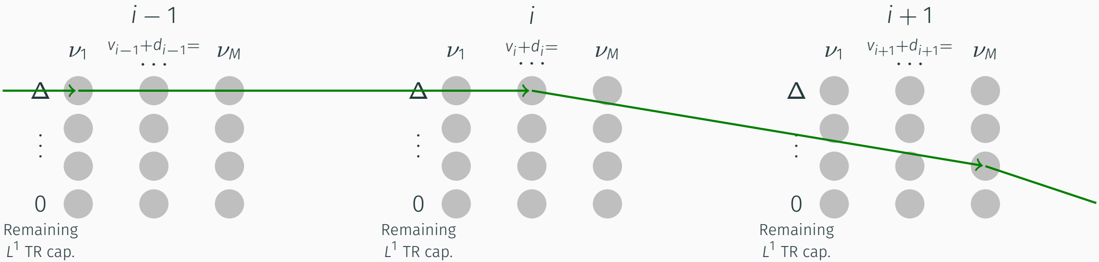

An edge in a feasible path (feasible d) connects different layers. Every feasible path is nonincreasing with respect to the remaining trust region capacity.

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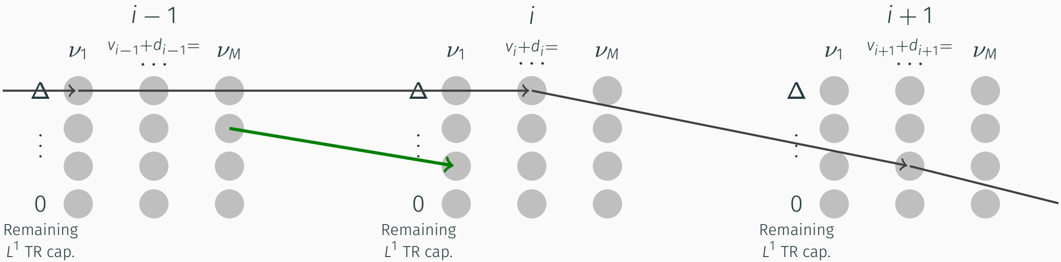

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At edge construction we know $v_{i-1} + d_{i-1}$ and $v_i + d_{i+1}$ $\Rightarrow$ allows to put **TV** term costs on the edges.

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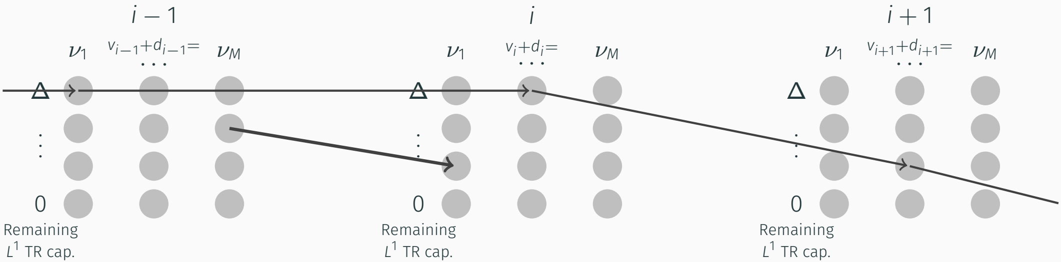

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Work in Progress
The number of nodes is bounded by $\#\text{nodes} = |V|N\Delta$. Using A* with a monotone heuristic we can solve $\text{TR}(v, \Delta)$ in $\#\text{nodes}^2$. The algorithm is pseudo-polynomial.

N	h_1	h_2	h_3	h_4	h_5	Top	SCIP
256	5.726	5.813	5.391	5.79	**1.074**	1.114	37.133
512	38.762	39.6	38.384	40.454	**3.402**	4.926	165.257
1024	244.107	285.039	286.998	286.697	**9.039**	19.906	544.1
2048	1316.92	1600.44	1659.529	1622.13	**15.396**	56.425	492.082

Figure 3: Cumulative running times (seconds) for test instances of a 1D elliptic control problem for simple heuristics (h_1,. . .,h_4), including dual information h_5 vs. topological sorting (Top) and IP solver (SCIP).

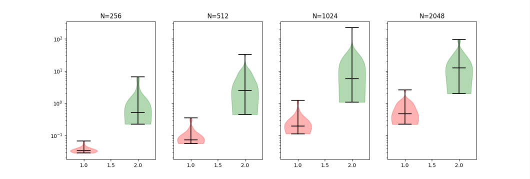

Figure 4: Distribution of running times (seconds) for the subproblem solves for A* with h_5 (red) and SCIP (green).

Discussion

1. Acclerated A* approach beats SCIP and topological sorting (on all tested problems so far) when including dual information.

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2. Several tried improvements and heuristics for A* (path dominance checks (discard paths with lower remaining L^1 TR capacity and equal or worse objective values); inequalities generated from the fact that the value of Ξ is the solution of an LP over an integral polytope; etc.) have not paid off (additional compute time often exceeds running time saves).

Discussion

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2. Several tried improvements and heuristics for A* (path dominance checks (discard paths with lower remaining L^1 TR capacity and equal or worse objective values); inequalities generated from the fact that the value of Ξ is the solution of an LP over an integral polytope; etc.) have not paid off (additional compute time often exceeds running time saves).
3. As for the analysis, the approach does not generalize to the multidimensional case straightforwardly.

We seek to

$$\underset{v}{\textbf{minimize}} \;\; J(y) + \mathrm{TV}(v) \quad \textbf{subject to} \quad y = S(v) \text{ and } v(x) \in \{\nu_1, \ldots, \nu_M\} \text{ a.e.} \tag{Q}$$

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4. yields large-scale integer linear programs that need to be solved.

Thank You for Your Attention!

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