Systems and Control II (SC2)
Albert-Ludwigs-Universität Freiburg – Wintersemester 2015/2016

Exercises 14: Course revision and exam preparation

(Thursday 11.02.2016 at 15:00 in Room SR 00 014)

Dr. Jörg Fischer, Prof. Dr. Moritz Diehl and Jochem De Schutter

3. A process plant $G(s)$ can be divided into two partial processes $G_1(s)$ and $G_2(s)$ in series. The transfer functions of these partial processes are given by

$$G_1(s) = \frac{3}{(2s+1)(4s+1)} \quad \text{and} \quad G_2(s) = \frac{4}{15s+1} \ .$$

Design a cascaded controller that realizes a closed-loop system with a bandwidth $\omega_{\mathrm{BW,cl}} \approx 0.04 \frac{\mathrm{rad}}{\mathrm{s}}$.

(a) Design a causal controller $K_1(s)$ with internal model control (IMC) for the inner loop process $G_1(s)$. Minimize the Integral Square Error (ISE). Make sure that the inner closed-loop bandwidth is approximately $\omega_{\mathrm{BW},1} \approx 0.25 \frac{\mathrm{rad}}{\mathrm{s}}$.

(b) Approximate the obtained inner closed-loop system by a first order system $G_{\mathrm{a}}(s) = \frac{k_{\mathrm{a}}}{T_{\mathrm{a}}s+1}$.

(c) Design a realizable controller $K_2(s)$ with internal model control (and with minimal ISE) for the overall process $G_{\mathrm{a}}(s)G_2(s)$. Make sure that the bandwidth requirement is met.

(d) Evaluate the validity of the approximation of the inner closed-loop system, computed in 3b.

SOLUTION:

(a) The inner loop process $G_1(s)$ is minimum-phase. Therefore we can invert it directly and thus obtain an ideal IMC-controller

$$K^*_{\mathrm{IMC},1}(s) = \frac{(2s+1)(4s+1)}{3} \ .$$

In order to have a causal controller we add a second order filter $V_{\mathrm{f},1}(s) = \frac{1}{(T_1 s+1)^2}$, so that

$$\begin{aligned} K_{\mathrm{IMC},1}(s) &= K^*_{\mathrm{IMC},1}(s) V_{\mathrm{f},1}(s) \\ &= \frac{(2s+1)(4s+1)}{3(T_1 s+1)^2} \end{aligned}$$

Due to the internal model control, the closed-loop transfer function boils down to the open-loop transfer function (under the assumption of a perfect model):

$$G_{\mathrm{r},1}(s) = K_{\mathrm{IMC},1}(s)G(s) = V_{\mathrm{f},1}(s) \ .$$

We can therefore approximate the bandwidth of the inner closed-loop as $\omega_{\mathrm{BW},1} \approx \frac{1}{T_1}$. This gives us $T_1 = 4$ sec.

(b) The first order approximation of the inner closed-loop system should have the same gain as the inner closed-loop transfer function:

$$k_{\mathrm{a}} = G_{\mathrm{r},1}(0) = V_{\mathrm{f},1}(0) = 1 \ .$$

The time constant T_{a} is found as

$$T_{\mathrm{a}} = \sum_{i=1}^{n} T_i - \sum_{i=1}^{q} T_{0i} + T_{\mathrm{d}} = T_1 + T_1 = 8 \ \mathrm{s} \ .$$

This gives us as an approximation

$$G_{\mathrm{a}}(s) = \frac{1}{8s+1} \ .$$

(c) The outer open loop process can now be written as

$$G_{\mathrm{a}}(s)G_2(s) = \frac{4}{(8s+1)(15s+1)} \ ,$$

which is also minimum-phase. The ideal IMC-controller is found as

$$K^*_{\mathrm{IMC},2}(s) = \frac{(8s+1)(15s+1)}{4} \ .$$

Again, we need a second-order filter $V_{\mathrm{f},2}(s) = \frac{1}{(T_2 s+1)^2}$ in order to make the IMC-controller causal. Similarly to the inner loop, we can estimate the outer closed-loop bandwidth as $\omega_{\mathrm{BW,cl}} \approx \frac{1}{T_2}$. This gives us $T_2 = 25$ sec.

(d) The bandwidth of the first-order approximation of the inner closed-loop should be larger than the bandwidth of the closed-loop system, so that it can be used in a valid way. Since

$$\frac{1}{T_\mathrm{a}} = 0.125\frac{\mathrm{rad}}{\mathrm{s}} > 0.04\frac{\mathrm{rad}}{\mathrm{s}} = \frac{1}{T_2} \,,$$

the approximation used here is valid.

4. Consider the following system, of which the state space model is decomposed into Kalman form:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \begin{bmatrix} 2 & -3 & 0 \\ 0 & -1 & 0 \\ 2.8 & 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \sqrt{2} \\ 0 \\ -2 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \mathbf{x}(t)\,.\end{aligned}$$

(a) Evaluate the controllability and observability of the different states.

(b) Is the system controllable? If not, is it stabilizable? Is the system observable? (If not, is it detectable?)

(c) Compute the transfer function of this system. Also, determine that state space model that is a minimal realization of this transfer function.

(d) Given the state space model obtained in 4c, determine the desired pole locations $\lambda_{\mathrm{cl,i}}$, so that the closed-loop system has a settling time $T_\mathrm{s} \leq 0.3$ sec.

(e) If the system is *at least* stabilizable and detectable, determine by 'comparison of coefficients' the feedback controller matrix $\mathbf{K}$ that places the closed-loop poles at the desired locations $\lambda_{\mathrm{cl,i}}$.

SOLUTION:

(a) First analyze the different states:

- The state $x_1(t)$ is controllable, since the first element in the $\mathbf{b}$-matrix is non-zero. It is also observable since the first element of the $\mathbf{c}$-matrix is non-zero.
- The state $x_2(t)$ is not controllable, since the second element in the $\mathbf{b}$-matrix is zero, and since the other states do not appear in its state equation. It is observable however, since the second element in the $\mathbf{c}$-matrix is non-zero.
- The state $x_3(t)$ is controllable, since the third element in the $\mathbf{b}$-matrix is non-zero. It is not observable since the third element of the $\mathbf{c}$-matrix is zero, and since this state does not appear in state equations of the other (observable) states.

(b) Since the system has a non-controllable state ($x_2(t)$), is it not controllable. The eigenvalue corresponding to this state ($\lambda_2 = -1$) is negative, therefore the system is stabilizable. Given the non-observable state ($x_3(t)$), the system is not observable. The eigenvalue corresponding to this state ($\lambda_3 = -2$) is negative, so that the system is detectable.

(c) The transfer function of this system can be easily found by using the fact that

$$\begin{aligned}G(s) &= \mathbf{c}(sI - \mathbf{A})^{-1})\mathbf{b} + d = \mathbf{c}_\mathrm{co}(sI - \mathbf{A}_\mathrm{co})^{-1})\mathbf{b}_\mathrm{co} + d \\ &= \frac{1}{\sqrt{2}}(s-2)^{-1}\sqrt{2} = \frac{1}{s-2}\,.\end{aligned}$$

The state space model that is a minimal realization of this transfer function is then exactly the controllable and observable part of the system:

$$\begin{aligned}\dot{x}(t) &= 2x(t) + \sqrt{2}u(t) \\ y(t) &= \frac{1}{\sqrt{2}}x(t)\,.\end{aligned}$$

(d) Since this minimal realization is a first-order system, full state feedback with pole placement will give us a first-order closed-loop system as well. There is thus only one pole location to be chosen. For a first-order system $\frac{1}{\frac{s}{s_1}+1}$, we find that the settling time T_s can be found as

$$T_\mathrm{s} = \frac{3}{|s_1|}\,,$$

which gives us the desired pole location $\lambda_{\mathrm{cl},1} = -\frac{3}{T_\mathrm{s}} = -10\ \frac{\mathrm{rad}}{\mathrm{s}}$.

(e) Via pole placement we want to place the eigenvalue of the expression $A - bK$ at $\lambda_1 = \lambda_{\mathrm{cl},1}$. This requirement gives us the linear equation

$$A - bK = 2 - \sqrt{2}K = -10\,,$$

which gives us the controller $K = 8.48$.