Systems and Control II (SC2)
Albert-Ludwigs-Universität Freiburg – Wintersemester 2015/2016

Exercises 10: Discrete controller design

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1. Compute the discrete equivalents of the controller

$$K(s) = \frac{a}{s+a}\ ,$$

where a is a given parameter, using a) the forward rectangular rule, b) the backward rectangular rule, and c) the trapezoid rule (Tustins method). Consider the sample time T_P also as a given parameter.

SOLUTION:

a) Forward rule: $K_\mathrm{z}(z) = G(s)|_{s=\frac{z-1}{T_\mathrm{P}}}$

$$\Rightarrow K_\mathrm{z}(z) = \frac{a}{\frac{z-1}{T_\mathrm{P}} + a} = \frac{aT_\mathrm{P}}{z-1+aT_\mathrm{P}}$$

b) Backward rule: $K_\mathrm{z}(z) = G(s)|_{s=\frac{z-1}{zT_\mathrm{P}}}$

$$\begin{aligned}\Rightarrow K_\mathrm{z}(z) &= \frac{a}{\frac{z-1}{zT_\mathrm{P}} + a} = \frac{aT_\mathrm{P}z}{z-1+aT_\mathrm{P}z} = \frac{aT_\mathrm{P}z}{z(1+aT_\mathrm{P})-1} \\ &= \frac{aT_\mathrm{P}}{1+aT_\mathrm{P}} \cdot \frac{z}{z - \frac{1}{1+aT_\mathrm{P}}}\end{aligned}$$

c) Tustins method: $K_\mathrm{z}(z) = G(s)|_{s=\frac{2}{T_\mathrm{P}}\frac{z-1}{z+1}}$

$$\begin{aligned}\Rightarrow K_\mathrm{z}(z) &= \frac{a}{\frac{2}{T_\mathrm{P}}\frac{z-1}{z+1} + a} = \frac{aT_\mathrm{P}(z+1)}{2(z-1)+aT_\mathrm{P}(z+1)} = \frac{aT_\mathrm{P}(z+1)}{(2+aT_\mathrm{P})z-2+aT_\mathrm{P}} \\ &= \frac{aT_\mathrm{P}}{2+aT_\mathrm{P}} \cdot \frac{z+1}{z + \frac{-2+aT_\mathrm{P}}{2+aT_\mathrm{P}}}\end{aligned}$$

2. Use the zero-pole matching method to compute the discrete equivalent of the controller

$$K(s) = \frac{a}{s+a}\ ,$$

where a is a given parameter. Consider the sample time T_P also as a given parameter. It is desired that the discrete controller shows no delay in its discrete time response.

SOLUTION:

Procedure:

(a) mapping poles: The pole $s_1 = -a$ of $K(s)$ will map to the pole $z_1 = e^{s_1 T_\mathrm{P}}$ of $K_\mathrm{z}(z)$

(b) mapping zeros: $K(s)$ has no zeros. Hence, there are no zeros that have to be mapped.

(c) additional zeros at -1: to have a discrete time response without a delay, n-q zeros at -1 have to be added, where n is the number of poles of $K(s)$ and q is the number of zeros of $K(s)$. Hence, we have to include a (z+1) term in $K_\mathrm{z}(z)$. (Comment: When a delay of 1 is desired in the discrete response, then n-q-1 zeros are added at -1.

Combining a) to c), this gives the prelimary result:

$$K_\mathrm{z}^*(z) = \frac{z+1}{z - e^{-aT_\mathrm{P}}}$$

(d) match the dc gain: The final controller is $K_\mathrm{z}(z) = k_\mathrm{c} K_\mathrm{z}^*(z)$ where the gain k_c has to be determined such that

$$K(s)|_{s=0} \overset{!}{=} K_\mathrm{z}(z)|_{z=1}$$

Therefore,

$$\begin{aligned} 1 &\overset{!}{=} k_\mathrm{c}\frac{2}{1-e^{-aT_\mathrm{P}}} \\ \Leftrightarrow k_\mathrm{c} &= \frac{1-e^{-aT_\mathrm{P}}}{2}\end{aligned}$$

The final result is

$$K_{\mathrm{z}}(z) = k_{\mathrm{c}} K_{\mathrm{z}}^{*}(z) = \frac{(z+1)(1-e^{-aT_{\mathrm{P}}})}{2(z-e^{-aT_{\mathrm{P}}})}$$

3. Use the ZOH-method to compute the discrete equivalent of the controller

$$K(s) = \frac{a}{s+a}\ ,$$

where a is a given parameter. Consider the sample time T_{p} also as a given parameter.

SOLUTION: The ZOH equivalent is calculated via

$$K_{\mathrm{z}}(z) = (1-z^{-1})\mathcal{Z}\left\{ \mathcal{L}^{-1}\left\{\frac{K(s)}{s}\right\}\Big|_{t=kT_{\mathrm{p}}} \right\}\ .$$

Comment: It is common sense to abbreviate the above equation by

$$K_{\mathrm{z}}(z) = (1-z^{-1})\mathcal{Z}\left\{\frac{K(s)}{s}\right\}\ ,$$

although this does not describe the process precisely.

Using number 6 of the table for z- and laplace-transforms, we find

$$\mathcal{Z}\left\{\frac{K(s)}{s}\right\} = \mathcal{Z}\left\{\frac{a}{s(s+a)}\right\} = \frac{(1-e^{-aT_{\mathrm{P}}})z}{(z-1)(z-e^{-aT_{\mathrm{P}}})}$$

Therefore, the final result is

$$\begin{aligned} K_{\mathrm{z}}(z) &= (1-z^{-1})\frac{(1-e^{-aT_{\mathrm{P}}})z}{(z-1)(z-e^{-aT_{\mathrm{P}}})} \\ &= \frac{z-1}{z}\,\frac{(1-e^{-aT_{\mathrm{P}}})z}{(z-1)(z-e^{-aT_{\mathrm{P}}})} \\ &= \frac{1-e^{-aT_{\mathrm{P}}}}{z-e^{-aT_{\mathrm{P}}}} \end{aligned}$$

4. Design a discrete controller for a DC-motor that is preceeded by a zero-order hold (see 1), so that the closed-loop system has an overshoot of no more than 20%, a rise time $T_{\mathrm{r}} < 0.3$ s, and a settling time of not more than 2s. Use the discrete root locus method to evaluate different controller types and to tune the parameters of the appropriate controller. The sampling time is $T_{\mathrm{P}} = 0.1$ ms . The DC-motor can be approximately described in continuous-time by

$$G(s) = \frac{1}{s(s+1)}\quad .$$

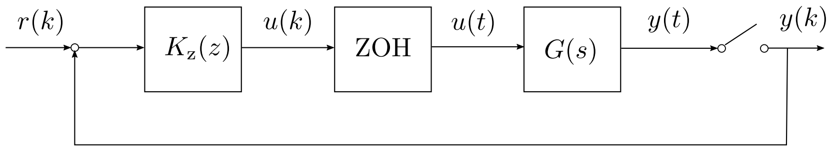

Figure 1: Diagram of a discrete time system

(a) Assume that the closed-loop continuous system can be approximated by a dominant pole pair. Translate the specifications of the continuous closed-loop system into corresponding requirements on the step response, using the dynamic behavior heuristics in Table 1.

(b) Discretize the DC-motor preceeded by the ZOH.

(c) (MATLAB) Consider a proportional controller $K_{\mathrm{z}}(z) = k_{\mathrm{p}}$. Plot the root locus of the closed-loop system with respect to k_{p}. Which value of k_{p} allows us to meet the design objectives?

(d) (MATLAB) Consider a lead compensator $K_{\mathrm{z}}(z) = k_{\mathrm{p}}\frac{z-z_{01}}{z-z_{1}}$. Choose the parameters z_{01}, z_{1}, and k_{p} so that the design objectives are met.
Hint: Use the MATLAB-function `rltool` and activate the grid over the context menu.

(e) What is the steady state error for a step input of the closed-loop system using the lead-compensator from d).

Peak time T_m	$\frac{\pi}{\omega_0\sqrt{1-\zeta^2}}$
Rise time T_r	$\frac{1.8}{\omega_0}$
Settling time $T_{5\%}$	$\frac{3}{\zeta\omega_0}$
Settling time $T_{2\%}$	$\frac{4.5}{\zeta\omega_0}$

ϕ_ζ	ζ	Δh
$66°$	0.4	25%
$54°$	0.58	10%
$45°$	0.7	5%
$37°$	0.8	2%

Table 1: Dynamic behavior heuristics of a second order system with complex conjugate poles $\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2}$ for $\zeta < 0.8$.

SOLUTION:

(a) overshoot: $\Delta h \overset{!}{<} 20\% \implies \zeta \geq 0.5$.

rise time: $T_\mathrm{r} \overset{!}{<} 0.3\mathrm{s} \implies \omega_0 \geq 6\,\frac{\mathrm{rad}}{\mathrm{s}}$.

settling time: $T_{5\%} = \frac{3\ \mathrm{rad}}{\omega_0\zeta} \overset{!}{<} 2\mathrm{s}$. ζ and ω_0 are already defined by the overshoot and rise time requirements.
We check if the requirements are consistent: $T_{5\%} < \frac{3\mathrm{rad}}{0.5\cdot 6\frac{\mathrm{rad}}{\mathrm{s}}} = 1\mathrm{s}$.
Hence, the requirements are realizable, under the assumption of a dominant pole pair.

(b) The discretized plant can be computed via

$$G_\mathrm{z}(z) = (1-z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}\Big|_{t=kT_\mathrm{p}}\right\}.$$

Using number 7 of the table with Lapclace- and z-transforms, we find

$$\begin{aligned} G_\mathrm{z}(z) &= (1-z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}\Big|_{t=kT_\mathrm{P}}\right\} = (1-z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\}\Big|_{t=kT_\mathrm{P}}\right\} \\ &= (1-z^{-1})\frac{z\left[\left(T_\mathrm{P}-1+e^{-T_\mathrm{P}}\right)z + \left(1-e^{-T_\mathrm{P}}-T_\mathrm{P}e^{-T_\mathrm{P}}\right)\right]}{(z-1)^2(z-e^{-T_\mathrm{P}})} \\ &= \frac{\left(T_\mathrm{P}-1+e^{-T_\mathrm{P}}\right)z + \left(1-e^{-T_\mathrm{P}}-T_\mathrm{P}e^{-T_\mathrm{P}}\right)}{(z-1)(z-e^{-T_\mathrm{P}})} \\ &= \frac{0.004837z + 0.004679}{(z-1)(z-0.9048)} \end{aligned}$$

(c) Plot the root locus of the open loop:

```
s = tf('s');
G = 1/(s^2+s);
Tp = 0.1;
Gz = c2d(G,Tp,'zoh');
rlocus(Gz);
```

Conclusion: the frequency $\omega_0 = 6\frac{\mathrm{rad}}{\mathrm{s}}$ is only reached for unstable systems. Therefore the objectives cannot be met with a proportional controller.

(d) The idea: Use the lead compensator to push the pole at 0.9048 to the left. This will bring the disembarkation point of the root locus more to the left so that the root locus will bend down and path the point of $\zeta = 0.5$ and $\omega_0 = 6$.
Moving the pole to the left can be accomplished by placing the zero z_{01} at the pole at 0.9048. Then, place the pole z_1 somewhere left of the compensated pole and move it around until the root locus passes the desired region.
Comment: It is not an alternative to compensate the plant zero at -0.9672 with the controller pole and then using the controller zero to move the rootlocus to the inner of the unit circle in order to satisfy the requirements. This is due to the fact that slightly damped controller poles on the negative real axis easily introduce oscillations. In particular, the step response from command input to controller output can show high oscillations. Hence, it is usually good to avoid controller poles on the negative real axis.

i. As a starting point, we guess the pole location at z = 0.5.

```
z = tf('z', Tp)
Kz = tf([1 -0.9048], [1 -0.5], Tp)
rltool(Gz, Kz);
```

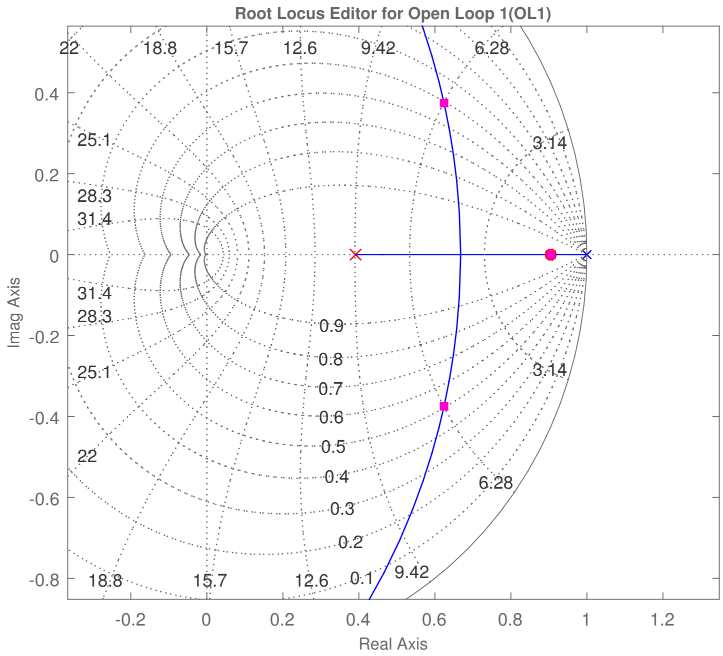

Figure 2: Direct compensation of plant pole

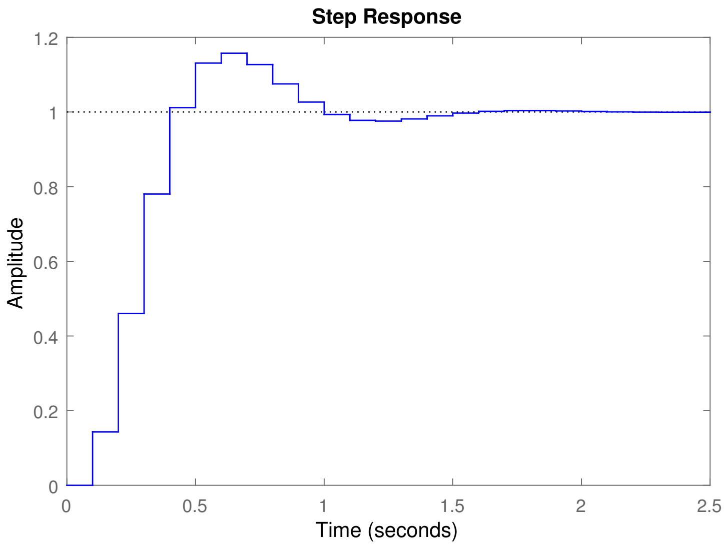

Figure 3: Step response

ii. Activate *grid* from the context menu and play around with the compensator pole so that the root locus crosses $\zeta = 0.5$ and $\omega_0 = 6$. This is possible for $z_1 = 0.391$ (see Fig. **??**)

iii. The closed-loop poles take the desired locations for an overall controller gain of 4.63 (seen by rltool). Hence, the gain k_p is calculated to

$$K_\mathrm{z}(1) = k_\mathrm{p}\frac{1-0.9048}{1-0.391} \overset{!}{=} 4.63$$
$$\Leftrightarrow k_\mathrm{p} = 4.63 \cdot \frac{0.578}{0.0952} = 29.49$$

This gives the controller

$$K_\mathrm{z}(z) = 29.49\frac{z-0.9048}{z-0.391}\,.$$

The resulting command step response of the closed-loop is shown in Fig. **??**. *Comment: The command response can be shown directly in the rltool by using the menu Analysis and then marking step und Other loop response.*
It can be seen from the step response that all design requirements are satisfied.

iv. *Comment: the step response shows an oscillating behavior. Playing with the rltool, it can be found that the controller can improved by moving the controller zero slightlty to the left to 0.885*

Nr.	function $f(t)$ with f(t) = for t¡0	$F(s)=\mathcal{L}\{f(t)\}$	$F(z)=\mathcal{Z}\{f(k)\}$ with $f(k)=f(kT_\mathrm{p})$	
1	$\sigma(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$	
2	t	$\frac{1}{s^2}$	$\frac{T_\mathrm{p}z}{(z-1)^2}$	
3	t^2	$\frac{2}{s^3}$	$\frac{T_\mathrm{p}^2 z(z+1)}{(z-1)^3}$	
4	e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT_\mathrm{p}}}$	
5	te^{-at}	$\frac{1}{(s+a)^2}$	$\frac{\alpha T_\mathrm{p}z}{(z-\alpha)^2}$	$\alpha=e^{-aT_\mathrm{p}}$
6	$1-e^{at}$	$\frac{a}{s(s+a)}$	$\frac{(1-\alpha)z}{(z-1)(z-\alpha)}$	$\alpha=e^{-aT_\mathrm{p}}$
7	$\frac{1}{a}\left(at-1-e^{-at}\right)$	$\frac{a}{s^2(s+a)}$	$\frac{z[(aT_\mathrm{p}-1+\alpha)z+(1-\alpha-aT_\mathrm{p}\alpha)]}{a(z-1)^2(z-\alpha)}$	$\alpha=e^{-aT_\mathrm{p}}$
8	$(1-at)\,e^{-at}$	$\frac{s}{(s+a)^2}$	$\frac{z[z-\alpha(1+aT_\mathrm{p})]}{(z-\alpha)^2}$	$\alpha=e^{-aT_\mathrm{p}}$
9	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\frac{\beta z}{z^2-2\gamma z+1}$	$\beta=\sin(\omega T_\mathrm{p})$ $\gamma=\cos(\omega T_\mathrm{p})$
10	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\frac{z^2-\gamma z}{z^2-2\gamma z+1}$	$\gamma=\cos(\omega T_\mathrm{p})$
11	$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{\alpha\beta z}{z^2-2\alpha\gamma z+\alpha^2}$	$\alpha=e^{-aT_\mathrm{p}}$ $\beta=\sin(\omega T_\mathrm{p})$ $\gamma=\cos(\omega T_\mathrm{p})$

Table 2: Table of Laplace- and z-Transforms